Note

On avoiding odd partial Latin squares and $r$-multi Latin squares

Jaromy Scott Kuhl$^a$, Tristan Denley$^b$

$^a$Department of Mathematics and Statistics, University of West Florida, Pensacola, FL 32514, USA
$^b$Department of Mathematics, University of Mississippi, Oxford, MI 38677, USA

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Abstract

We show that for any positive integer $k \geq 4$, if $\mathcal{R}$ is a $(2k - 1) \times (2k - 1)$ partial Latin square, then $\mathcal{R}$ is avoidable given that $\mathcal{R}$ contains an empty row, thus extending a theorem of Chetwynd and Rhodes. We also present the idea of avoidability in the setting of partial $r$-multi Latin squares, and give some partial fillings which are avoidable. In particular, we show that if $\mathcal{R}$ contains at most $nr/2$ symbols and if there is an $n \times n$ Latin square $\mathcal{L}$ such that $\delta n$ of the symbols in $\mathcal{L}$ cover the filled cells in $\mathcal{R}$ where $0 < \delta < 1$, then $\mathcal{R}$ is avoidable provided $r$ is large enough.

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1. Introduction

An $n \times n$ partial Latin square is an $n \times n$ array of at most $n$ distinct symbols so that no symbol appears more than once in each row and column of the array. If a partial Latin square has no empty cells then it is called a Latin square. We will always assume that the $n$ distinct symbols are $[n] = \{1, 2, \ldots, n\}$, unless otherwise stated. Perhaps the most natural question to ask concerning partial Latin squares is that of completing partial Latin squares. A partial Latin square $\mathcal{R}$ is completable if there is a Latin square containing $\mathcal{R}$. Avoiding partial Latin squares is the converse of this. A partial Latin square $\mathcal{R}$ is avoidable if there is a Latin square containing no part of $\mathcal{R}$. Hence, the Latin square $\mathcal{L}$ avoids $\mathcal{R}$ if for symbol $k$ in cell $(i, j)$ of $\mathcal{R}$ and symbol $l$ in cell $(i, j)$ of $\mathcal{L}$, $l \neq k$. Determining if a partial Latin square is avoidable was first introduced by Häggkvist in [7]. In fact, Häggkvist introduced this in a more general setting. He considered not only partial Latin squares but $n \times n$ arrays in general. Chetwynd and Rhodes in [5] continued this idea of avoidability for partial Latin squares and showed that every even sized partial Latin square is avoidable and that every odd sized partial Latin square is avoidable given that it contains an empty row and column. Trivially, they also showed that every completable partial Latin square is avoidable. In Section 2, we present a proof extending the same techniques used by Chetwynd and Rhodes in [5] and show that every partial $(2k - 1) \times (2k - 1)$ Latin square $\mathcal{R}$ is avoidable given that $\mathcal{R}$ contains an empty row (or column) for $k \geq 4$.
Consider the following generalization of a partial Latin square. An \( n \times n \) partial \( r \)-multi Latin square is an \( n \times n \) array of at most \( nr \) symbols so that each cell of the array contains \( r \) symbols and each symbol appears once in each row and column. If the \( n \times n \) partial \( r \)-multi Latin square contains no empty cells then it is called an \( r \)-multi Latin square. Naturally, questions concerning partial Latin squares are appropriate for partial \( r \)-multi Latin squares. For example, in [6], we give a class of \( r \)-multi Latin squares that are completable. Therefore, continuing in the theme of avoidability, a partial \( r \)-multi Latin square \( \mathcal{R} \) is avoidable if there is an \( r \)-multi Latin square which does not contain any part of \( \mathcal{R} \). If \( \mathcal{L} \) avoids \( \mathcal{R} \) and \( L \) and \( R \) are the set of symbols in cell \((i,j)\) of \( \mathcal{L} \) and \( \mathcal{R} \), respectively, then \( L \cap R = \emptyset \). In Section 3, we present some results on avoiding partial \( r \)-multi Latin squares. We prove an analogue of the Evans conjecture, if \( \mathcal{R} \) is a partial \( n \times n \) \( r \)-multi Latin square with at most \( (n-1) \) cells filled, then \( \mathcal{R} \) is avoidable. We also show that if \( \mathcal{R} \) uses \( \frac{1}{2} \) of the \( nr \) symbols and if each symbol used appears at most \( \frac{n}{2} \) times, then \( \mathcal{R} \) is avoidable. Finally, we provide an asymptotic result. We show that if \( \mathcal{R} \) contains at most \( \frac{nr}{2} \) symbols and if there is an \( n \times n \) Latin square \( \mathcal{L} \) such that \( \partial n \) of the symbols in \( \mathcal{L} \) cover the filled cells in \( \mathcal{R} \) where \( 0 < \delta < 1 \), then \( \mathcal{R} \) is avoidable provided \( r \) is large enough.

2. Avoiding odd partial Latin squares

The following lemmas and theorem will be crucial for the proof of the main theorem. The first lemma can be found in [5], the second in [9], and Ryser’s theorem in [8].

Lemma 1. For any \( n \geq 4 \), there is an \( n \times n \) Latin square which has symbol \( n \) in each cell of the leading diagonal, and has the entries of the last row in the same order as the last column.

Lemma 2. Let \( \mathcal{R} \) be a partial \( n \times n \) L.S. with \( n-1 \) filled cells and let \( z \) be a specified entry. Then the rows and columns of \( \mathcal{R} \) can be permuted so that \( z \) lies on the leading diagonal, and all the other filled cells lie above it.

Theorem 1 (Ryser). Let \( \mathcal{R} \) be an \( r \times s \) rectangular array so that \( r, s \leq n \). Then \( \mathcal{R} \) can be extended to an \( n \times n \) Latin square if and only if each of the \( n \) symbols appears at least \( r + s - n \) times in \( \mathcal{R} \).

We also will use the following notation. For an \( n \times n \) Latin square, the set of cells \( \{(i,1), (i+1,2), \ldots, (n,n-i+1), (1,n-i+2), (2,n-i+3), \ldots, (i-1,n)\} \) are referred to as the \( i \)th diagonal. Hence the first diagonal would be the leading diagonal. In addition to this notation, since the term Latin square and Latin rectangle is used often we abbreviate these with L.S. and L.R., respectively.

Theorem 2. Let \( k \geq 4 \) and let \( \mathcal{R} \) be a partial \( (2k-1) \times (2k-1) \) L.S. with one row (or one column) being empty. Then \( \mathcal{R} \) is avoidable.

Proof. We may assume that the empty row is the last row of \( \mathcal{R} \). Let \( \mathcal{T} \) be a \( k \times k \) L.S. filled according to Lemma 1 with symbols \( X_1, X_2, \ldots, X_k \). We will consider \( \mathcal{R} \) also as a \( k \times k \) partial L.S. where cell \((i, j)\) of this \( k \times k \) array is the following:

\[
(i,j) = \begin{cases} 
2 \times 2 \text{ (partial) L.S.} & \text{if } i < k \text{ and } j < k, \\
2 \times 1 \text{ (partial) L.R.} & \text{if } i < k \text{ and } j = k, \\
1 \times 2 \text{ (partial) L.R.} & \text{if } i = k \text{ and } j < k, \\
1 \times 1 \text{ (partial) L.S.} & \text{if } i = k \text{ and } j = k.
\end{cases}
\]

Furthermore, for \( i < k \) and \( j < k \), cell \((i, j)\) will be the (partial) \( 2 \times 2 \) L.S. consisting of cells \( \{(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1), (2i, 2j)\} \). Note that \((i,j)\) is a cell in \( \mathcal{R} \) as a partial \( (2k-1) \times (2k-1) \) L.S. and \((i,j)\) is a cell in \( \mathcal{R} \) as a \( k \times k \) partial L.S.

For \( i < k \), we identify \( X_i \) with the set of \( 2 \times 2 \) L.S.s appearing in the corresponding cells of \( \mathcal{R} \). For \( X_i \) in cell \((i,k)\) of \( \mathcal{T} \), we identify it with the arrangement of cells from \((i,k)\) and the two back diagonal cells in \((i,k)\) of \( \mathcal{R} \). This arrangement of cells is not a \( 2 \times 2 \) square but is analogous to the cells \( (1,2), (2,1), (1,3), (2,3) \) in a \( 2 \times 3 \) square. Likewise, for cell \((k,i)\) of \( \mathcal{T} \), we identify it with the arrangement of cells from \((k,i)\) and the two back diagonal cells in \((i,k)\) of \( \mathcal{R} \). To avoid an arrangement of filled cells is defined in the same manner as avoiding a L.S.
Let $S$ be a set of $2k - 2$ symbols. Furthermore, let $S_1, S_2, \ldots, S_{k-1}$ be an ordered partition of $S$ such that each $S_i$ is a pair of symbols. There are
\[
\frac{(2k - 2)!}{2^{k-1}}
\]
such ordered partitions of $S$. For $1 \leq i \leq k - 1$ and $X_i$ not appearing in the last row or column of $\mathcal{T}$, we say that $S_i$ is suitable for $X_i$ if $S_i$ can form a $2 \times 2$ L.S. such that it avoids all $2 \times 2$ (partial) L.S. identified by $X_i$. For $X_i$ appearing in the last row and column of $\mathcal{T}$, $S_i$ is suitable for $X_i$ if $S_i$ can form an identical arrangement of symbols avoiding the arrangement identified by $X_i$. Suppose that for each cell of $\mathcal{T}$ containing $X_i$ for $1 \leq i \leq k - 1$, $S_i$ is suitable for $X_i$. Then we can form a $2 \times 2$ L.S. with $S_i$ avoiding the $2 \times 2$ (partial) L.S. identified by $X_i$. For $X_i$ in the last row and column, we can form an arrangement of symbols with $S_i$ avoiding the arrangement identified by $X_i$. Hence $\mathcal{T}$ becomes a partial $(2k - 1) \times (2k - 1)$ L.S. with empty cells on the leading diagonal. Without loss of generality, we can choose $S$ so that $2k - 1$ can be placed on the leading diagonal of $\mathcal{T}$ avoiding the leading diagonal of $\mathcal{R}$ since at least one of the leading diagonal cells of $\mathcal{R}$ is empty. Our goal then is to find an ordered partition of $S$ into $(k - 1)$ pairs of symbols such that for each $i$, $S_i$ is suitable for $X_i$.

Consider a $2 \times 2$ (partial) L.S. $X$ of $\mathcal{R}$ identified by $X_i$ for $1 \leq i \leq k - 1$. If a diagonal of $X$ contains two distinct symbols, say $a$ and $b$, then the pair of symbols $(a, b)$ is not suitable for $X_i$. We call a diagonal containing two distinct symbols a bad diagonal. Since $X$ has two diagonals, $X$ can have at most two unsuitable pairs of symbols. Consider also the arrangement identified by $X_i$ in the last row or column of $\mathcal{T}$. This too has at most two unsuitable pairs of symbols since we can permute the columns of $\mathcal{R}$ so that the back diagonal cells in $(j, j)$ of $\mathcal{R}$ do not contain the same symbol for $1 \leq i \leq k - 1$. These unsuitable pairs of symbols we will also call bad diagonals. Let $B$ be the set of all bad diagonals and let $E$ be the number of empty cells in $\mathcal{R}$. We will show that $|B| \leq 2k^2 - 5k + 2$ and to do this we consider two cases, $E \geq 4k - 2$ and $E \leq 4k - 3$.

**Case 1:** Suppose $E \geq 4k - 2$. Then $\mathcal{R}$ has at least $2k - 1$ empty cells excluding the last row. Define $l$ to be the fewest number of empty cells among the columns of $\mathcal{R}$, excluding the empty cells from the last row. We may assume that column 1 of $\mathcal{R}$ contains $l$ empty cells. If $l \geq 2$, then there are at least $2k - 2$ empty cells when we remove the first column and last row. By Lemma 2, we may arrange the rows and columns so that $2k - 2$ empty cells lie above the leading diagonal. Therefore, in $\mathcal{R}$, taking into account the appropriate permutation changes for the first column, these $2k - 2$ empty cells lie above the $n$th diagonal. Thus we are guaranteed at least $k$ good diagonals, taking into account the empty cells in column 1. Suppose $l = 1$. We repeat the argument as previously described and we are guaranteed $k$ good diagonals unless the empty cell in column 1 is cell $(1, 1)$ of $\mathcal{R}$ and there are exactly $4k - 2$ empty cells. In this case, interchange row 1 with row $j$ if $(1, j)$ is filled. Otherwise, row 1 is empty and so $\mathcal{R}$ can be completed by Theorem 1 and thus is avoidable. If $l = 0$, then we have enough empty cells above the $n$th diagonal to guarantee $k$ good diagonals. The arrangement of cells identified by $X_i$ in row $k$ of $\mathcal{T}$ contains no bad diagonals, and eliminating the leading diagonal since $X_k$ has not been identified with anything gives
\[
|B| \leq 2k^2 - 2k - 2(k - 1) = 2k^2 - 5k + 2.
\]

**Case 2:** Suppose $2k - 1 < E \leq 4k - 3$. This gives at most $2k - 2$ empty cells in $\mathcal{R}$ excluding the last row. Then there must be a symbol in $\mathcal{R}$ used $2k - 2$ times. Let $s$ be a symbol used $2k - 2$ times. There is a column not containing $s$ and we may assume that this is the last column. Then we can arrange the first $2k - 2$ columns and rows of $\mathcal{R}$ so that $s$ appears on the $j$th diagonal for $j > 2$. There also must be at least one empty cell in addition to the empty row, otherwise $\mathcal{R}$ is completeable. We may arrange the columns and rows of $\mathcal{R}$ so that this empty cell is not a cell on the leading diagonal of $\mathcal{R}$. Hence
\[
|B| \leq 2k^2 - 2k - 2(k - 1) - (k - 1) - 1 = 2k^2 - 5k + 2.
\]

Let $B_i$ be the set of pairs of symbols which are unsuitable for every $2 \times 2$ (partial) L.S. and every arrangement of cells identified by $X_i$. Then
\[
|B_1| + |B_2| + \cdots + |B_{k-1}| \leq |B|.
\]

We wish to exclude those partitions of $S$ for which $S_i \in B_i$ for $1 \leq i \leq k - 1$. There are
\[
\frac{(2k - 4)!}{2^{k-2}}
\]
partitions with \( S_i \in B_i \). We must, therefore, exclude at most
\[
\frac{(2k - 4)!}{2k - 1} \sum_{i} |B_i| \frac{k!}{2^{k-2}}
\]
partitions. Let \( N \) be the number of suitable ordered partitions of \( S \). Then
\[
N \geq \frac{(2k - 2)!}{2k - 1} - \sum_{i} |B_i| \frac{(2k - 4)!}{2^{k-2}}
\]
\[
\geq \frac{(2k - 2)!}{2k - 1} - |B| \frac{(2k - 4)!}{2^{k-2}}
\]
\[
= \frac{(2k - 4)!}{2^{k-2}} \left( \frac{(2k - 2)(2k - 3)}{2} - |B| \right)
\]
\[
= \frac{(2k - 4)!}{2^{k-2}} (2k^2 - 5k + 3 - |B|).
\]

However, we have established that \( |B| \leq 2k^2 - 5k + 2 \). Hence \( N > 0 \) for \( k \geq 4 \). Therefore, there is a partition of \( S \) for which \( S_i \) is suitable for each \( X_i \) for \( 1 \leq i \leq k - 1 \), and so \( \mathcal{R} \) avoids \( \mathcal{R} \).

3. Avoiding \( r \)-multi Latin squares

Let \( \mathcal{R} \) be an \( n \times n \) partial \( r \)-multi L.S. We begin by connecting the notion of \( \mathcal{R} \) being completable with the notion of \( \mathcal{R} \) being avoidable.

**Theorem 3.** An \( n \times n \) \( r \)-multi partial L.S. \( \mathcal{R} \) which is completable is avoidable.

**Proof.** We are given that \( \mathcal{R} \) is completable, therefore we extend \( \mathcal{R} \) to a complete \( n \times n \) \( r \)-multi L.S. Then the following permutation \( \sigma \) of rows yields an \( n \times n \) \( r \)-multi L.S. which avoids the completed \( \mathcal{R} \) and thus \( \mathcal{R} \); \( \sigma = (2, 3, \ldots , n, 1) \).

Thus we will only consider \( n \times n \) partial \( r \)-multi L.S. which are not completable.

Let \( G \) be a graph. For each \( e \in E(G) \) define \( L(e) \) to be a list of colors assigned to \( e \). Then \( G \) has an \( L \)-list coloring if the edges of \( G \) can be properly colored so that each edge \( e \) receives a color from \( L(e) \). We will use the following theorem of Borodin et al. in [4], and the solution to the Evans conjecture given independently by Smetanuik in [9] and Anderson and Hilton in [2].

**Theorem 4** (Borodin, Kostochka, and Woolall). Let \( G \) be a bipartite graph, and for each edge \( e \in E(G) \) let \( |L(e)| = \max\{d(x), d(y)\} \) where \( e = (x, y) \). Then \( G \) has an \( L \)-list coloring.

**Theorem 5** (Evans conjecture). Let \( \mathcal{R} \) be an \( n \times n \) partial L.S. with at most \( n - 1 \) cells filled. Then \( \mathcal{R} \) is completable.

**Theorem 6.** An \( n \times n \) partial \( r \)-multi L.S. \( \mathcal{R} \) with at most \( (n - 1) \) cells filled is avoidable.

**Proof.** We begin by forming \( r \) \( n \times n \) partial L.S. \( \mathcal{R}_1, \ldots , \mathcal{R}_r \) from \( \mathcal{R} \) such that for \( 1 \leq i \leq r \), \( \mathcal{R}_i \) contains symbols \( \{(i - 1)n + 1, \ldots , in\} \). In doing this, first consider \( r \) \( n \times n \) grids \( A_1, \ldots , A_r \). For each \( i \), fill \( A_i \) with the symbols \( \{(i - 1)n + 1, \ldots , in\} \) in such a way that these symbols appear in \( A_i \) as they do in \( \mathcal{R} \). Note that the now partially filled \( A_i \) is another L.S. since we may have more than one symbol appearing in a cell. Define \( S \) to be the set of symbols appearing in \( A_i \). Without loss of generality we may assume that \( S \neq \{(i - 1)n + 1, \ldots , in\} \), since \( \mathcal{R} \) uses at most all \( (n - 1) \) symbols.

To transform \( A_i \) into a partial L.S., consider the bipartite graph \( G = (A, B) \) where \( A = \{p_1, \ldots , p_p\} \) is the set of rows of \( A_i \) containing filled cells and \( B = \{c_1, \ldots , c_q\} \) is the set of columns of \( A_i \) containing filled cells. Include the edge \( (p_s, c_t) \) in \( E(G) \) if \( (s, t) \) is a filled cell in \( A_i \). Define \( L(s, t) \) to be a list of colors for the edge \( (p_s, c_t) \) such
Theorem 7. Let $\mathcal{P}$ be a $n \times n$ partial $r$-multi L.S. and consider an $n \times n$ L.S. $\mathcal{L}$. For each symbol $i$ of $\mathcal{L}$, define $A_i$ to be the set of symbols which appear in the corresponding cells of $\mathcal{P}$. If $i$ appears in cell $(j, k)$ of $\mathcal{L}$, then $A_j$ will contain the symbols that appear in cell $(j, k)$ of $\mathcal{P}$. Of course, there is more than one $n \times n$ L.S. Therefore, let $\mathcal{L}_1, \ldots, \mathcal{L}_m$ be different L.S.s up to permutations of symbols, rows, and columns and set $\mathcal{P} = \{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$. Thus each element of $\mathcal{P}$ may yield a different set of symbols for $A_i$.

Let $\mathcal{P}$ be a $n \times n$ partial $r$-multi L.S. Then $\mathcal{P}$ is avoidable if for some $\mathcal{L}_k \in \mathcal{P}$, there is a partition of $[nr]$ into $r$ $n$-sets $S_1, \ldots, S_r$ such that for each set $S_i$ and each $U \subseteq S_i$, $|U| \leq |I|$, where $I = \{i : s_j \notin A_i \text{ for some } s_j \in U\}$.

Proof. For each $i \in [n]$, form a bipartite graph $G$ with $V(G) = (S_i, R)$ where, $S_i = \{s_{i-1}, \ldots, s_{in}\}$ and $R = \{A_1, \ldots, A_n\}$. Recall that $A_j$ is the set of symbols according to $\mathcal{L}_k$ for some $k$. Include the edge $(s_l, A_j)$ in $E(G)$ if $s_l \notin A_j$. By defining $G$ this way, $I = \{i : s_j \notin A_i \text{ for some } s_j \in U\} = N(U)$. Since $|U| \leq |I|$ for every $U \subseteq S_i$, by Hall’s theorem there is a matching $M_i$, which saturates $S_i$. We may assume without loss of generality that $M_i = \{(s_{i-1}, A_i), (s_{i-1}+2, A_2), \ldots, (s_{in}, A_n)\}$.

Now fill an $n \times n$ array as a L.S. with the symbols $X_1, \ldots, X_n$ such that $X_i$ appears exactly where $i$ appears in $\mathcal{L}_k$. From this new L.S. we can form an $n \times n$ r-multi L.S. which avoids $\mathcal{P}$ as follows: set $X_i = \{s_{i}, s_{i+1}, s_{2i+1}, \ldots, s_{(r-1)n+i}\}$ for every $i \in [n]$. □

Theorem 7 gives sufficient conditions for avoiding a partial $r$-multi L.S. However, with these conditions, what does such an r-multi L.S. look like? The following corollaries paint a picture of what these could look like.

Corollary 1. Let $\mathcal{P}$ be an $n \times n$ partial $r$-multi L.S. with at most $nr/2$ of the symbols used. Then $\mathcal{P}$ is avoidable if each symbol used in $\mathcal{P}$ appears at most $n/r$ times.

Proof. Consider $\mathcal{L}_k \in \mathcal{P}$ for any $k$. Then partition $[nr]$ into $r$ $n$-sets $S_1, \ldots, S_r$ such that for each $i$, $S_i$ contains at most $n/2$ symbols which appear in $\mathcal{P}$ and at least $nr/2$ symbols which do not appear in $\mathcal{P}$. Form a bipartite graph $G$ with $V(G) = (S_i, R)$ as in Theorem 7. Then at least half of the vertices in $S_i$ have degree $n$ with the remaining vertices having degree at least $nr/2$. Therefore, $|N(U)| \geq |U|$ for each $U \subseteq S_i$. Hence by Theorem 7, $\mathcal{P}$ is avoidable. □

Corollary 2. Let $\mathcal{P}$ be an $n \times n$ partial $r$-multi L.S. Then $\mathcal{P}$ is avoidable if there is an $\mathcal{L}_k \in \mathcal{P}$ such that

1. Each symbol in $[nr]$ is used no more than $(nr/2 - 1)$ times, and
2. There is a partition of $[nr]$ into pairs of symbols such that for the pair $j, l \in [nr]$ if $j \in \bigcap_{i \in [n]} A_i$ then $k \notin \bigcup_{i \in [n]} A_i$.

Proof. Consider $\mathcal{L}_k \in \mathcal{P}$. Partition $[nr]$ into $r$ $n$-sets $S_1, \ldots, S_r$ such that $S_i$ contains $nr/2$ pairs of symbols for which the pairing is defined as above. For $U \subseteq S_i$, if $|U| \leq (nr/2 + 1)$, then clearly $|N(U)| \geq (nr/2 + 1)$ and if $|U| > (nr/2 + 1)$, then $|N(U)| = n$. Therefore by Theorem 7, $\mathcal{P}$ is avoidable. □
In continuing our analysis on avoiding partial $r$-multi L.S.s, we use the following lemma, an application of Lovasz’s local lemma, found in [1], Corollary 1.4(i), found in [3], p. 11.

Lemma 3. Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent of a set of all the other events $A_j$ but at most $d$, and that $P(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d + 1) \leq 1$, then $P(\bigwedge_{i=1}^{n} \bar{A_i}) > 0$.

Define $S_{n,p}$ to be the random variable with a binomial distribution with parameters $n$ and $p$. Thus, $P(S_{n,p} = k)$ is the probability that we get $k$ heads when tossing a biased coin $n$ times with the probability of getting a head $p$, namely $\binom{n}{k} p^k q^{n-k}$ where $q = 1 - p$.

Lemma 4. If $1 \leq h < \min\{pqn/10, (pn)^{2/3}/2\}$ with $h = x(pqn)^{1/2}$, then

$$P(|S_{n,p} - pn| \geq h) < \frac{1}{x} e^{-x^2/2}.$$ 

Lemma 5. Let $\{A_1, \ldots, A_m\}$ be a set of subsets of $[mr]$, some of which may be empty. Then, provided $r$ is large enough, there is a partition of $[mr]$, $X_1, \ldots, X_m$, so that for $i \in [m]$ and $j \in [m],$

$$|X_i| - r < \sqrt{2r \ln(6m)}$$

and

$$|X_j \cap A_j| - \frac{|A_j|}{m} < \sqrt{2r \ln(6m)}.$$ 

Proof. Randomly color $[mr]$ with $m$ colors, thereby assigning sets of symbols to $X_1, \ldots, X_m$. For each $i \in [m]$, let $C_i$ be the event that

$$|X_i| - r \geq \sqrt{2r \ln(6m)}$$

and for each $j \in [m]$, let $D_j$ be the event that

$$|X_j \cap A_j| - \frac{|A_j|}{m} \geq \sqrt{2r \ln(6m)}.$$ 

Then we can form the dependency graph $G$ where $V(G) = \{C_1, \ldots, C_m, D_1, \ldots, D_m\}$ and since $|V(G)| = 2m, d(C_i)$ and $d(D_j)$ can be at most $2m - 1$. Therefore, according to the local lemma, if $P(C_i) < 1/e2m$ and $P(D_j) < 1/e2m$, then

$$P\left( \bigwedge_{i=1}^{k} \bar{C}_i \land \left( \bigwedge_{j=1}^{k} \bar{D}_j \right) \right) > 0$$

and so the theorem must hold. For $r$ being large enough, $1 \leq h < \min\{pqn/10, (pq)^{2/3}/2\}$, and so by Theorem 4 and setting $x = \sqrt{2\ln(6m)}$,

$$P(|X_i| - r \geq \sqrt{2r \ln(6m)}) = P(|S_{mr,1/m} - r| \geq \sqrt{2r \ln(6m)})$$

$$\leq P\left( |S_{mr,1/m} - r| \geq \sqrt{\frac{2r(m-1)\ln(6m)}{m}} \right)$$

$$\leq \frac{1}{\sqrt{2\ln(6m)}} e^{-((\sqrt{2\ln(6m)})^2)/2},$$

$$P(|X_i| - r \geq \sqrt{2r \ln(6m)}) \leq e^{-\left((\sqrt{2\ln(6m)})^2\right)/2}.$$
Theorem 8. Let
\[
\frac{|A_j|}{m} \geq \sqrt{2r \ln(6m)}
\]
which is true for
\[\delta n \sqrt{2er \ln(6n)} < (1 - \varepsilon)nr,
\]
can be used the unused symbols in \([nr]\) to give \(X_i\) exactly \(r\) symbols for \(1 \leq i \leq \delta n\). Our objective is to fill an \(n \times n\) array with \(X_1, \ldots, X_n\) as a L.S. such that \(X_j \cap A_j = \emptyset\), thereby avoiding \(R\). As it stands, for \(1 \leq j \leq \delta n\),
\[
|A_j \cap X_j| < \frac{|A_j|}{n} + \sqrt{2er \ln(6n)}.
\]
If there are enough unused symbols, then we can replace \(|A_j \cap X_j|\) with unused symbols. This is possible if
\[
\delta n \left( \frac{|A_j|}{n} + \sqrt{2er \ln(6n)} \right) + \delta n \sqrt{2er \ln(6n)} < (1 - \varepsilon)nr.
\]
Since \(|A_j| \leq enr\), we need only that
\[
\delta nr + 2\delta n \sqrt{2er \ln(6n)} < (1 - \varepsilon)nr,
\]
which is true for \(r\) large enough and \(\varepsilon < 1/(1 + \delta)\). The remaining unused symbols and the symbols removed from \(X_i\) for \(1 \leq i \leq \delta n\) can be used for the remaining \(X_i\) since for \(\delta n + 1 \leq i \leq n\), \(A_i = \emptyset\). □

Corollary 3. Let \(0 < \delta < 1\) and let \(R\) be a partial \(n \times n\) \(r\)-multi L.S. using at most \(nr/2\) symbols. Suppose there is an \(n \times n\) L.S. \(L\) such that \(\delta n\) of the symbols in \(L\) cover the filled cells in \(R\). Then \(R\) is avoidable provided \(r\) is large enough.

Proof. In Theorem 8, let \(\varepsilon = \frac{1}{2}\). Then clearly \(\varepsilon < 1/(1 + \delta)\). □

References