JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 37, 167–184 (1972)

An Operational Calculus for a Class of Abstract Operator Equations

J. A. DONALDSON*†

Department of Mathematics and Statistics
University of New Mexico, Albuquerque, New Mexico 87106

and

Department of Mathematics, University of Illinois at Chicago Circle
Chicago, Illinois 60680

Submitted by Peter D. Lax

Received May 23, 1970

1. INTRODUCTION

In recent years the introduction of abstract analytic methods (see Hille [11, 12], Lions [14], Carroll [4], Feller [8], Schwartz [17]) into the study of differential equations has played a very important role in the advancement of the theory of these equations. This paper is concerned with the development of a rigorous operational calculus which may be used in the effective determination of solutions of the operator differential equation

$$\sum_{j=0}^{n} f_j(tA)\, A^j D_t^{n-j} Y_A(t) = 0 \tag{1.1}$$

satisfying the initial condition $Y_A(0) = I$. Here A is an operator defined in some complex Banach space X and $f_j(z)$ are polynomials in z, i.e.,

$$f_j(z) = \sum_{p=0}^{m_j} a_p z^p.$$

Here and in the sequel $D_t^k \equiv d^k/dt^k$. The solutions of (1.1) will be represented as contour or real integrals. Although the form of (1.1) looks quite special, it

* Work on this paper was carried out while the author was a postdoctoral research fellow at the University of New Mexico.

† Present address: Department of Mathematics, Howard University, Washington, D.C. 20001.

can be shown that a number of partial differential equations which arise in mathematical physics can be written in this form. For example, the wave equation

$$U_{yy} = AU \left(A = \frac{\partial^2}{\partial x^2}\right)$$

can be written in the form

$$4tU_{tt} + 2U_t = AU \tag{1.1'}$$

upon making the substitution $y^2 = t$. Now, if (1.1′) is multiplied by A, it becomes a special case of (1.1).

An incomplete sketch of the solution of another special case of (1.1), namely,

$$Y''(t) + (k/t)\, Y'(t) + AY(t) = 0, \qquad Y(0) = I, \tag{1.2}$$

was given by Rosenbloom [16, p. 84]. His method may be summarized as follows. He replaces the operator A by the parameter λ and notes that the resulting differential equation is a Bessel differential equation,

$$Y''(t) + (k/t)\, Y'(t) + \lambda Y(t) = 0.$$

The solution of this equation satisfying the initial condition $Y(0) = 1$ is expressed in terms of the power series

$$\begin{aligned} Y(t) &= \Gamma((k+1)/2) \sum_{m=0}^{\infty} \frac{(-\lambda t^2/4)^m}{m!\, \Gamma(m+1+(k-1)/2)} \\ &= \Gamma((k+1)/2)\, (t\lambda^{1/2}/2)^{(1-k)/2}\, J_{(k-1)/2}(t\lambda^{1/2}) \end{aligned}$$

for $k \neq -1, -3, -5, \ldots$. Then the parameter λ, in the series, is replaced by A and an operator series for the solution of (1.2) is obtained. He makes no attempt to attach a meaning to this operator series.

This method was extended by Bragg [1] to treat partial differential equations of the form (1.1) ($A \equiv P(x, D)$, a partial differential operator independent of t), where the ordinary differential equation which results after replacing A by the parameter λ has an ordinary or regular singular point at the origin. A series solution was obtained for this ordinary differential equation under assumption that the coefficients of the series satisfied a two-term recurrence relation. He replaced λ by A in this series and obtained an operator series as the solution of the operator equation in (1.1). The notion of related partial differential equations and properties of hypergeometric series were then used to attach a meaning to the resulting series.

For the special case of (1.1), where $X = \mathscr{C}$, the complex numbers, and where A is the operator defined by the relation $Ax = \lambda x$, (λ a fixed complex number) the operator equation becomes an ordinary differential equation

$$\sum_{j=0}^{n} f_j(t\lambda)\, \lambda^j D_t^{n-j}[Y_\lambda(t)] = 0. \tag{1.3}$$

Since the theory of this equation plays a considerable role in our development of an operational calculus for (1.1), an integral representation for its solution will be obtained in Section 2. In Sections 3 and 4 the exploitation of basic results from the theory of strongly continuous semigroups enables us to extend the result of Section 2, so that a solution of (1.1) can be obtained when A is a bounded or an unbounded operator in X. Several applications of our results are given in Section 5. The confluent hypergeometric and the Bessel operator differential equations are considered; the singular initial value problem is solved for the generalized Euler-Poisson-Darboux (EPD) equation

$$U_{tt} + (k/t)\, U_t = P(x, D)\, U,$$

where $P(x, D)$ is a differential operator which does not depend upon t. Another extension of the fundamental formula of Section 2 and an illustration of this extension by an example involving the generalized wave and heat equations are given in Section 6.

2. Solution of the Differential Equation in (1.3)

We start by considering the ordinary differential equation in (1.3). Upon making the change of scale $t = \lambda^{-1}z$, we may write (1.1) in the following form:

$$\sum_{j=0}^{n} f_j(z)\, D^{n-j} Y(z) = 0, \tag{2.1}$$

where

$$Y(z) = Y_\lambda(\lambda^{-1}z), \qquad \text{and} \qquad f_j(z) = \sum_{p=0}^{m_j} a_p z^p$$

are such that (2.1) has an ordinary or regular singular point at the origin. It is well known that the power series method is applicable if one wishes to find a solution of this equation. What is not commonly known (see Ince [13, Chaps. 8 and 18]) is that a solution of (2.1), in many cases, may be obtained in terms of a definite integral or a contour integral. In this paper we will be interested only in those Eqs. (2.1) whose solutions allow themselves to be represented in terms of generalized Laplace transforms of functions $F(x)$, which will be determined below. Thus, we seek a solution of (2.1) in the form

$$Y(z) = \int_{\alpha}^{\beta} F(x)\, e^{xz}\, dx, \tag{2.2}$$

where α and β are the endpoints (α may be equal to β) of a suitably chosen contour in the complex plane. We note that solutions of hypergeometric differential equations may be represented in this way.

No additional difficulty arises from studying the differential equation

$$L_z[Y(z)] = 0, \tag{2.3}$$

where L_z is a general linear ordinary differential operator. We define another differential operator M_x by the relation

$$L_z[e^{xz}] \equiv M_x[e^{xz}].$$

This can always be done when L_z has polynomial coefficients. Then the order of the operator M_x is equal to highest power of z appearing in the polynomial coefficients of L_z. We let $\bar{M}_x$ denote the formal adjoint of the operator M_x. The bilinear concomitant $\bar{P}(u, v)$ of two functions $u(x)$ and $v(x)$ with respect to the operator M_x is defined by the relation

$$vM_x[u] - u\bar{M}_x[v] = \frac{d}{dx}\{\bar{P}(u, v)\}.$$

The proof of the following theorem may be found in Ince's work [13, p. 186].

THEOREM 1. *If it is permissible to apply the operator* L_z *to the integral in* (2.2), *where* $F(x)$ *and* α *and* β *are such that*

$$\bar{M}_x[F] = 0, \qquad \bar{P}(F, e^{xt}) \Big|_{\alpha}^{\beta} = 0,$$

then (2.2) *provides a solution for* (2.3).

This method admits of considerable generalization. Here we shall generalize this method so that the operator equations in (1.1) may be treated. In order to simplify the exposition we shall assume that the contour of integration in (2.2) is a segment of the real axis. It will be indicated how the results are to be modified to cover cases where the contour of integration is a path in the complex plane.

It follows that a solution of (1.3) is provided by the expression

$$Y_\lambda(t) = C \int_\alpha^\beta F(x)\, e^{\lambda x t}\, dx, \tag{2.4}$$

where C is an arbitrary constant, and where $F(x)$ and α and β are chosen as indicated above.

3. A Brief Summary of Semigroup Theory

Here we shall describe the operational calculus arising in the theory of semigroups [10] because of its special importance in generalizing the integral in (2.4), so that the general operator equation can be solved.

Definition. (A) A one-parameter family of bounded linear operators $[S(t) : t \geqslant 0]$ on a Banach space $X \to X$ is called a strongly continuous semigroup if

(i) $S(s + t) = S(s)\, S(t)$ for all $s, t > 0$,

(ii) $S(0) = I$,

(iii) the map $t \to T(t)\phi$ is continuous from $[0, \infty)$ into X for all $\phi \in X$.

(B) The infinitesimal generator A of $S(t)$ is defined by

$$\mathscr{D}(A) = \{\phi \in X : \lim_{t \to 0^+} t^{-1}(S(t) - I)\phi \text{ exists in } X\}$$

and for $\phi \in \mathscr{D}(A)$,

$$A\phi = \lim_{t \to 0^+} t^{-1}(S(t) - I)\phi.$$

Then, for $t > 0$ and $\phi \in \mathscr{D}(A)$, we have

$$\frac{d}{dt}\{S(t)\phi\} = S(t)\, A\phi = AS(t)\phi.$$

(C) Let A generate a strongly continuous semigroup $S(t)$ and let $\phi \in \mathscr{D}(A^n)$. Then $S(t)\phi$ is n times continuously differentiable with respect to t and

$$D_t^k[S(t)\phi] = A^k[S(t)\phi] = S(t)\,[A^k\phi] \qquad \text{for } 0 \leqslant k \leqslant n. \tag{3.1}$$

For necessary and sufficient conditions for an operator A to generate a strongly continuous semigroup, see Ref. [7, p. 624-630]. For the definition and properties of strongly continuous groups, see Ref. [10]. It can be seen from (3.1) that $S(t)$ can be regarded in an appropriate sense as an exponential function of an unbounded operator. This observation will be utilized in the next section where Eq. (1.1) is considered when A is an unbounded closed operator.

4. Solution of (1.1) when A is an Unbounded Operator

Throughout this section, X will denote a complex Banach space; $S(t)$ will denote a strongly continuous semigroup (group) of operators; and A will denote the infinitesimal generator of $S(t)$. In general, A will be closed and may be unbounded. It will be shown that if ϕ is an element of the domain of A^n, where n is sufficiently large, then a solution of (1.1) can be expressed in terms of $S(t)\phi$. Let

$$\bar{P}(F, S(xt)\phi) = \sum_{j=0}^{n} \sum_{p=1}^{m_j} \sum_{q=0}^{p-1} (-1)^q a_p \{D_x^{p-q-1}[S(xt)\phi]\} D_x{}^q[x^{n-j}F(x)]. \tag{4.1}$$

Furthermore, let $F(x)$ be a solution of the mth-order ordinary differential equation

$$B[F] = \sum_{j=0}^{n} \sum_{p=0}^{m_j} (-1)^p a_p D_x{}^p[x^{n-j}F(x)] = 0, \tag{4.2}$$

where $m = \max\{m_j : 0 \leqslant j \leqslant n\}$. We are now able to prove the following result which provides an operational calculus for equation (1.1) when A is an unbounded closed operator.

THEOREM 2. (i) *Let $F(x)$ and α and β be selected so that $F(x)$ is a solution of* (4.2) *and*

$$\{A^n \bar{P}(F, S(xt)\phi)\} \Big|_{x=\alpha}^{x=\beta} = 0.$$

(ii) *Let A generate a strongly continuous semigroup (group if $\alpha < 0$ and $\beta > 0$) on X.*

(iii) *Let $\phi \in \mathscr{D}(A^\gamma)$, where $\gamma = n + m$.*

Then

$$Y_A(t) = \int_\alpha^\beta F(x)\, S(xt)\phi\, dx \tag{4.3}$$

is a solution of (1.1).

Proof. We compute:

$$D_t^k[Y_A(t)] = \int_\alpha^\beta F(x)\, D_t^k[S(xt)\,\phi]\, dx$$

$$= \int_\alpha^\beta F(x)\, x^k(A^k[S(xt)\,\phi])\, dx \qquad \text{[by (3.1)]}$$

for $0 \leqslant k \leqslant n$. Thus

$$\sum_{j=0}^{n} f_j(tA)\, A^j D_t^{n-j}[Y_A(t)]$$

$$= \sum_{j=0}^{n} \sum_{p=0}^{m_j} a_p(tA)^p\, A^j \int_\alpha^\beta F(x)\, x^{n-j} A^{n-j}[S(xt)\,\phi]\, dx$$

$$= A^n \sum_{j=0}^{n} \sum_{p=0}^{m_j} a_p \int_\alpha^\beta x^{n-j} F(x)\, (tA)^p\, [S(xt)\,\phi]\, dx$$

$$= A^n \sum_{j=0}^{n} \sum_{p=0}^{m_j} a_p \int_\alpha^\beta x^{n-j} F(x)\, D_x^p[S(xt)\,\phi]\, dx \qquad \text{[by (3.1)]}$$

$$= A^n \int_\alpha^\beta \left\{ \sum_{j=0}^{n} \sum_{p=0}^{m_j} a_p x^{n-j} F(x)\, D_x^p[S(xt)\,\phi] \right\} dx$$

$$= A^n \bar{P}(F, S(xt)\,\phi) \Big|_\alpha^\beta - \int_\alpha^\beta B[F]\, A^n[S(xt)\,\phi]\, dx \quad \text{(by integration by parts)}$$

$$= 0 \qquad \text{by (i)},$$

whence it follows that (4.3) is a solution of (1.1). In the above considerations we have assumed tacitly that it was permissible to differentiate under the integral sign in (4.3). This assumption is clearly valid when α and β are both finite as can be seen from property (3.1) of strongly continuous semigroups. For the case when α or β is infinite, a sufficient condition for the validity of the operation of differentiating under the integral sign in (4.3) is that $\| A^k S(t)\| \, | F(t)|$ be integrable on $[\alpha, \beta]$ for $0 \leqslant k \leqslant n$.

The assumption that the contour of integration be an interval of the real axis may be dropped when $S(t)$ admits of an analytic extension to a region of the complex plane. For this case it is only necessary that the path of integration lie in the region of analyticity of $S(t)$.

Finally, we remark that if a solution is sought for the operator equation

$$L_t[Y_A(t)] = 0, \tag{4.4}$$

where

$$A^k\{L_t[Y_A(t)]\} = \sum_{j=0}^{n} f_j(tA)\, A^j D_t^{n-j}[Y_A(t)],$$

then it is permissible to replace the integer n in Theorem 2 by $(n - k)$, and (4.3) provides a solution for (4.4). Let us note that (1.1′) is of the form (4.4).

5. Applications

Examples illustrating how the operational calculus developed in Section 4 may be used in solving certain partial differential equations with unbounded coefficients will be given in this section. Our first example involves the confluent hypergeometric operator equation

$$tY_A''(t) + [b - tA]\, Y_A'(t) - aA[Y_A(t)] = 0, \tag{5.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on a Banach space X. This equation is related to the ordinary differential equation

$$zy''(z) + (b - z)\, y'(z) - ay(z) = 0. \tag{5.2}$$

A solution of this equation satisfying the initial condition $y(0) = 1$ is provided by the expression

$${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \sigma^{a-1}(1-\sigma)^{b-a-1}\, e^{z\sigma}\, d\sigma, \qquad b > a > 0.$$

It is easily verified that the hypotheses of Theorem 2 are all satisfied. Thus, a solution of (5.1) satisfying the initial condition $Y_A(0) = \phi$, where $\phi \in \mathscr{D}(A^2)$ is provided by

$$Y_A(t) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \sigma^{a-1}(1-\sigma)^{b-a-1}\, [S(\sigma t)\,\phi]\, d\sigma, \tag{5.3}$$

where $V(t) = S(t)\,\phi$ is the solution of the "abstract" Cauchy problem

$$\frac{dV}{dt} = AV, \qquad V(0) = \phi. \tag{5.4}$$

Here we need to verify only that the initial condition is satisfied. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\| Y_A(t) - \phi \| \leqslant \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \sigma^{a-1}(1-\sigma)^{b-a-1} \| S(\sigma t)\phi - \phi \| \, d\sigma$$
$$< \epsilon, \qquad \text{whenever} \qquad 0 < t < \delta,$$

since $S(t)$ is a strongly continuous semigroup.

As a concrete example we set $A = f(x)\,\partial/\partial x$, where $0 < \delta < f(x) < M$, $x \in R$, and $f(x) \in C(R)$. For this case, $S(t)\,\phi(x) = V(x, t)$ is the solution of the Cauchy problem

$$\frac{\partial V}{\partial t} = f(x)\frac{\partial V}{\partial x},$$
$$V(x, 0) = \phi(x), \qquad \text{namely}$$
$$V(x, t) = S(t)\,\phi(x) = \phi(K^{-1}(t + K(x))),$$

where

$$K(x) = \int_0^x \frac{dz}{f(z)}.$$

Thus

$$u(x, t) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \sigma^{a-1}(1-\sigma)^{b-a-1}\,\phi(K^{-1}(\sigma t + K(x)))\, d\sigma$$

is a solution of

$$t\left[\frac{\partial^2 u}{\partial t^2} - f(x)\frac{\partial^2 u}{\partial x\,\partial t}\right] + \left[b\frac{\partial u}{\partial t} - af(x)\frac{\partial u}{\partial x}\right] = 0$$

satisfying the initial condition

$$u(x, 0) = \phi(x).$$

Our next example involves the Bessel differential operator equation

$$D_\rho[Y_A(t)] = A^2[Y_A(t)], \tag{5.5}$$

where $D_\rho[u] = u_{tt} + (\rho/t)\,u_t$ is Bessel's differential operator. This operator has been studied for over two hundred years by such mathematicians as Euler, Poisson, Bessel, etc. In recent times Weinstein [18], Bureau [3], Lions [14], and others have carried out investigations of this operator. Now in (5.5) A could be a differential operator, an integral operator, a difference operator, etc. If A^2 is a constant operator, then (5.5) is an ordinary differential equation (Bessel's differential equation).

Equation (5.5) suggests that we begin our considerations with the differential equation

$$zy''(z) + \rho y'(z) - zy = 0, \qquad \rho > 0. \tag{5.6}$$

If we assume a solution of the form

$$y(z) = \int_\alpha^\beta F(\sigma)\, e^{\sigma z}\, d\sigma,$$

we find after an easy calculation that $F(\sigma) = (1 - \sigma^2)^{(\rho/2)-1}$ and $\alpha = -1$, $\beta = 1$. Thus,

$$y(z) = C_\rho \int_{-1}^{1} (1 - \sigma^2)^{(\rho/2)-1}\, e^{\sigma z}\, d\sigma, \tag{5.7}$$

where

$$C_\rho = \frac{\Gamma(\rho)\, 2^{1-\rho}}{\Gamma\left(\frac{\rho}{2}\right) \Gamma\left(\frac{\rho}{2}\right)}$$

is the solution of (5.6) satisfying the initial conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

One can now verify that if $\phi \in \mathscr{D}(A^2)$, then

$$Y_A(t) = C_\rho \int_{-1}^{1} (1 - \sigma^2)^{(\rho/2)-1}\, [S(\sigma t)\, \phi]\, d\sigma, \tag{5.8}$$

where $S(t)$ is the strongly continuous group with infinitesimal generator A, is the solution of (5.5) satisfying the initial conditions

$$\lim_{t\to 0} \| Y_A(t) - \phi \| = 0 \qquad \text{and} \qquad \lim_{t\to 0} \| Y_A'(t) - 0 \| = 0.$$

For a more detailed account of this singular initial value problem for (5.5), see Ref. [6]. For other treatments of this problem see Carroll [4] and Lions [14].

As our first concrete application of (5.8), we consider the singular hyperbolic differential equation

$$u_{tt} + \frac{\rho}{t}\, u_t = u_{xx}, \qquad \rho > 0, \tag{5.9}$$

which is the one-dimensional EPD equation. For this case $A = \partial/\partial x$; $S(t)\phi = \phi(x + t)$. Thus,

$$u(x, t) = C_\rho \int_{-1}^{1} (1 - \sigma^2)^{(\rho/2)-1} \phi(x + \sigma t)\, d\sigma$$

is the solution of (5.9) satisfying the initial conditions

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = 0,$$

where

$$\phi(x) \in \mathscr{D}\left(\frac{\partial^2}{\partial x^2}\right).$$

In order to apply (5.8) one must be able to determine the strongly continuous group $S(t)$ generated by the operator A. For the above example it was not difficult to show that the group generated by the operator $A = \partial/\partial x$ is the family of translation operators. For more general operators A the following procedure may be used to obtain the group $S(t)$. The resolvent $R(\lambda, A)$ of the operator A is the Laplace transform of $S(t)$, that is

$$R(\lambda, A)\phi = \int_0^\infty S(t)\phi e^{-\lambda t}\, dt, \tag{5.10}$$

whence it follows that $S(t)$ is obtained from the resolvent by the inverse Laplace transform formula. Thus,

$$S(t)\phi = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} R(\lambda, A)\phi e^{\lambda t}\, d\lambda. \tag{5.11}$$

The key step in applying (5.8) is that of determining the resolvent $R(\lambda, A)$ of A.

Now we consider the singular initial-value problem for the generalized EPD equation. Let $x = (x_1, x_2, x_3, ..., x_n)$ and $D = (D_1, ..., D_n)$, where $D_i\phi(x) = \partial\phi/\partial x_i$. Let $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \cdots D_n^{\beta_n}$ and let

$$P(x, D) = \sum_{\beta : 0 \leqslant |\beta| \leqslant m} f_\beta(x) D^\beta,$$

where $|\beta| = \beta_1 + \cdots + \beta_n$ and the $f_\beta(x)$ are given functions of x. We consider the two initial-value problems:

(I)
$$U_{tt} = P(x, D)\, U, \qquad t > 0, \tag{5.12}$$
$$U(x, 0) = \phi(x), \qquad U_t(x, 0) = 0; \tag{5.13}$$

(II)
$$V_{tt} + (\rho/t)\, V_t = P(x, D)\, V, \qquad t > 0, \tag{5.14}$$
$$V(x, 0) = \phi(x), \qquad V_t(x, 0) = 0. \tag{5.15}$$

Equation (5.14) depends on a parameter ρ, and for $\rho \neq 0$, a coefficient of this equation becomes infinite for $t = 0$. In this case we meet a singular Cauchy problem when the initial data are given on the hyperplane $t = 0$. For $\rho = 0$, problem (II) reduces to problem (I). We note that if $P(x, D)$ is a second-order elliptic operator having a positive definite form, then the equations in (5.12) and (5.14) are hyperbolic.

Although problem (I) has been subjected to an enormous amount of research [5, 9, 15], only special cases of (II) have received treatment in the literature. Of these investigations, the work of A. Weinstein [18] on the EPD equation

$$U_{tt} + (\rho/t)\, U_t = \Delta_n U, \tag{5.16}$$

and the generalized axially symmetric potential equation

$$U_{tt} + (\rho/t)\, U_t = -\,\Delta_n U \tag{5.17}$$

is foremost. Here, Δ_n is the n-dimensional Laplacian operator. It is noted that (5.16) and (5.17) are both special cases of (5.14). We will outline a procedure for obtaining a solution of (II). Although the general formula in (5.8) provides a solution of (II), when the square root of the differential operator $P(x, D)$ generates a strongly continuous group $S(t)$ on some function space X, we wish to extend this result to a wider class of differential operators. The extension given below is suggested by the form of the integral representation in (5.8) and a result of Hille concerning the "abstract" Cauchy problem

$$\begin{gathered} Y''(t) = A^2[Y(t)], \\ \lim_{t\to 0} \| Y(t) - \phi \| = 0, \qquad \lim_{t\to 0} \| Y'(t) - 0 \| = 0, \end{gathered} \tag{5.18}$$

where A generates a strongly continuous group $S(t)$ on X. The integral in (5.8) may be written in the from

$$Y_A(t) = 2C_\rho \int_0^1 (1 - \sigma^2)^{(\rho/2)-1}\, Y(\sigma t)\, d\sigma, \tag{5.19}$$

where

$$Y(t) = \tfrac{1}{2}\,[S(t)\phi + S(-\,t)\phi]. \tag{5.20}$$

The right member of (5.20) is the unique solution of (5.18) obtained by E. Hille [12]. Thus, (5.19) may be viewed as a transformation which maps solutions of the regular initial-value problem (5.18) into solutions of the singular initial-value problem for (5.5). This observation suggests the following result which can be established directly, although the operational calculus developed in Section 4 is insufficient for this purpose.

THEOREM 3. *If $U(x, t)$ is a strict solution of* (I), *then*

$$V(x, t) = 2C_\rho \int_0^1 (1 - \sigma^2)^{(\rho/2)-1} U(x, \sigma t)\, d\sigma \qquad (5.21)$$

is a solution of (II), *valid for $\rho > 0$.*

By a procedure of Weinstein (see [18]), we may employ the two recurrence formulas

$$V_t^\rho(x, t) = t V^{\rho+2}(x, t) \qquad (5.22)$$

and

$$V^\rho(x, t) = t^{1-\rho} V^{2-\rho}(x, t), \qquad (5.23)$$

which are satisfied by solutions of (5.14), along with the expression in (5.21) to obtain a solution of problem (II), valid for $\rho \leqslant 0$, but $\rho \neq -1, -3, \ldots$. We remark that the continuity requirements on U may be considerably relaxed in many cases by the introduction of generalized functions.

Formula (5.21) is a transformation which maps solutions of (I) into solutions of (II). Consequently, it is useful in solving (II) only when it is possible to solve the Cauchy problem (I). For the case when $P(x, D)$ is a second-order elliptic operator with positive definite form, one can indeed solve the Cauchy problem for the hyperbolic equation

$$U_{tt} = P(x, D)\, U.$$

When $P(x, D)$ is a second-order elliptic operator with negative definite form, the equation

$$U_{tt} = P(x, D)\, U$$

is elliptic. In general, the Cauchy problem for this equation is not well-posed. However, if the initial data in (I) are sufficiently smooth, it is possible in certain cases to find a solution of (I) when the equation is elliptic as can be seen by the following example involving the generalized axially symmetric potential equation. Let $F(x, t)$ be a harmonic function in the plane satisfying the condition that $F(x, t) = F(x, -t)$ for all $t \in R$. Then $U(x, t) = F(x, t)$ is the solution of

$$U_{tt} = -U_{xx}, \qquad U(x, 0) = f(x), \qquad U_t(x, 0) = 0,$$

where $f(x) = F(x, 0)$. Thus, by (5.21),

$$V(x, t) = 2C_\rho \int_0^1 (1 - \sigma^2)^{(\rho/2)-1} F(x, \sigma t)\, d\sigma, \qquad \rho > 0,$$

is a solution of the generalized axially symmetric potential problem

$$V_{tt} + \frac{\rho}{t} V_t = - V_{xx} ,$$
$$V(x, 0) = f(x), \qquad V_t(x, 0) = 0,$$

as can be shown by a direct computation. As our final example we consider a singular initial-value problem where $P(x, D)$ is a first-order partial differential operator. The function $U(x, t) = e^x \cosh t$ is a solution of the initial-value problem

$$U_{tt} = U_x , \qquad U(x, 0) = e^x, \qquad U_t(x, 0) = 0.$$

Hence,

$$V(x, t) = 2C_\rho \int_0^1 (1 - \sigma^2)^{(\rho/2)-1} e^x \cosh(\sigma t) \, d\sigma, \qquad \rho > 0$$

is a solution of the singular initial-value problem

$$V_{tt} + \frac{\rho}{t} V_t = V_x ,$$
$$V(x, 0) = e^x, \qquad V_t(x, 0) = 0.$$

We conclude this section by noting that Theorem 3 can be extended so that a wide class of mixed boundary-value problems for (5.14) may be treated. For another treatment of (II), when $P(x, D)$ is a second-order elliptic operator with positive definite form, see Ref. [3].

6 Another Extension of Formula (2.4)

In this section, we consider another extension of the formula in (2.4). The integral in (2.4) may be written in the form

$$Y_\lambda(t) = C\lambda^{-1} \int_{\alpha'}^{\beta'} F(\lambda^{-1}u) \, e^{tu} \, du, \tag{6.1}$$

where we have employed the transformation $u = \lambda x$. The object now is to assign a meaning to

$$Y_A(t) = \int_{\alpha'}^{\beta'} [A^{-1}F(uA^{-1}) \, \phi] \, e^{tu} \, du, \tag{6.2}$$

where A is now an operator. By again employing semigroup theory, a meaning can always be assigned to (6.2) when $F(u)$ can be expressed in terms of the exponential function. It can be shown for this case that (6.2) provides another solution of (1.1). When $\alpha' = \alpha - i\infty$ and $\beta' = \alpha + i\infty$ and the contour of integration is the line $x = \alpha$, we have formally that $A^{-1}F(uA^{-1})\phi$ is a constant multiple of the Laplace transform of $Y_A(t)$, namely

$$A^{-1}F(uA^{-1})\phi = 2\pi i \int_0^\infty Y_A(t)\, e^{-ut}\, dt. \tag{6.3}$$

This extension will be illustrated by the following example where the solution of a Cauchy problem for a generalized wave equation is expressed in terms of the solution of a generalized heat equation.

We begin here with ordinary differential equation

$$4zy'' + \rho y'(z) - y(z) = 0, \qquad \rho < 4, \tag{6.4}$$

and determine a solution of it in terms of a generalized Laplace transform. One can show that

$$Y(z) = \frac{\Gamma\left(1 - \frac{\rho}{4}\right)}{2\pi i} \int_{\alpha' - i\infty}^{\alpha' + i\infty} \sigma^{(\rho/4)-2} e^{(1/4)\sigma} e^{\sigma z}\, d\sigma \tag{6.5}$$

is the solution of (6.4) satisfying the initial conditions

$$y(0) = 0, \qquad \lim_{z\to 0} z^{\rho/4} y'(z) = 0. \tag{6.6}$$

Upon making the transformation $z \to \lambda t$ in (6.5), followed by the transformation $\sigma \to \xi/\lambda$, it can be seen that (6.5) may be written in the form

$$y_\lambda(t) = \frac{\Gamma\left(1 - \frac{\rho}{4}\right)}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \xi^{\rho/4-2} \lambda^{1-(\rho/4)} e^{\lambda/4\xi} e^{t\xi}\, d\xi. \tag{6.7}$$

Now set

$$W_\lambda(t) = \lambda^{(\rho/4)-1} y_\lambda(t).$$

Then

$$W_A(t) = \frac{\Gamma\left(1 - \frac{\rho}{4}\right)}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \xi^{(\rho/4)-2} \left[S\left(\frac{1}{4}\,\xi\right)\phi\right] e^{t\xi}\, d\xi, \tag{6.8}$$

where $S(t)$ is the semigroup generated by A, is a solution of the operator equation

$$4tW_A''(t) + \rho W_A'(t) = A[W_A(t)], \tag{6.9}$$

satisfying the initial conditions

$$W_A(0) = 0, \qquad \lim_{t \to 0} t^{\rho/4} W_A'(t) = \phi. \tag{6.10}$$

A more straightforward computation shows that (6.8) satisfies (6.9). A slightly more involved calculation shows that the initial conditions are satisfied.

Formally, we have upon taking Laplace transforms of both sides of (6.8)

$$S\left(\frac{1}{4\xi}\right)\phi = \frac{\xi^{2-(\rho/4)}}{\Gamma(1-(\rho/4))} \int_0^\infty W_A(t)\, e^{-t\xi}\, dt. \tag{6.11}$$

We consider now the following Cauchy problem for the abstract wave equation. Let X be a Banach space and let A be an operator defined in X. We are to find a function $U(t)$ satisfying

$$U_{tt} = AU, \qquad U(0) = 0, \qquad U'(0) = \phi. \tag{6.12}$$

Upon making the substitution $t^2 = z$, (6.12) may be written in the form

$$\begin{gathered} 4z\tilde{U}_{zz} + 2\tilde{U}_z = A\tilde{U}, \qquad \tilde{U}(0) = 0, \\ \lim_{z \to 0} 2z^{1/2}\tilde{U}'(z) = \phi, \end{gathered} \tag{6.13}$$

where $\tilde{U}(z^2) = U(z)$. This problem is the special case $\rho = 2$ of (6.9) and (6.10). Thus,

$$\tilde{U}(z) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \sigma^{-3/2}\, V\left(\frac{1}{4\sigma}\right) e^{\sigma z}\, d\sigma, \tag{6.14}$$

where $V(t) = S(t)\phi$ is the solution of the abstract Cauchy problem

$$\frac{dV}{dt} = AV, \qquad V(0) = \phi. \tag{6.15}$$

Hence,

$$U(t) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \sigma^{-3/2}\, V\left(\frac{1}{4\sigma}\right) e^{\sigma t^2}\, d\sigma \tag{6.16}$$

is the solution of (6.12). Upon employing (6.11) along with (6.14) and then after simplifying, we obtain

$$V(t) = \frac{t^{-3/2}}{2\Gamma(\frac{1}{2})} \int_0^\infty \xi U(\xi)\, e^{-\xi^2/4t}\, d\xi. \tag{6.17}$$

We remark that the expressions in (6.16) and (6.17) were obtained by Bragg and Dettman [2]. Their method may be outlined as follows: They showed that the Cauchy problems in (6.12) and (6.15) could both be transformed into the same Cauchy problem by making suitable transformations of the independent and dependent variables and then by employing the Laplace transformation.

Acknowledgments

The author is grateful to Professor R. Hersh and Professor E. Hille for many stimulating discussions during the preparation of this paper.

References

1. L. R. Bragg, Hypergeometric operator series and related partial differential equations, *Trans. Amer. Math. Soc.* **143** (1969), 319–337.
2. L. R. Bragg and J. W. Dettman, Related problems in partial differential equations, *Bull. Amer. Math. Soc.* **74** (1968), 375–378.
3. F. Bureau, Divergent integrals and partial differential equations, *Commun. Pure Appl. Math.* **8** (1955), 143–202.
4. R. Carroll, "Abstract Methods in Partial Differential Equations," Harper and Row, New York, 1969.
5. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. II, Wiley (Interscience), New York, 1962.
6. J. A. Donaldson, A singular abstract Cauchy problem, *Proc. Nat. Acad. Sci. U.S.A.* **66** (1970), 269–274.
7. N. Dunford and J. Schwartz, "Linear Operators," Part I, Wiley (Interscience), New York, 1958.
8. W. Feller, The parabolic differential equations and the associated semigroups of transformations, *Ann. of Math.* **55** (1952), 468–519.
9. G. Hellwig, "Partial Differential Equations," Blaisdell Publishing Company, New York, 1964.
10. E. Hille, "Functional Analysis and Semigroups," Amer. Math. Soc. Colloq. Publ. **31** (1948).
11. E. Hille, The abstract Cauchy problem and Cauchy's problem for parabolic differential equations, *J. Analyse Math.* **3** (1953–54), 81–196.
12. E. Hille, Une généralization du problème de Cauchy, *Ann. Inst. Fourier (Grenoble)* **IV** (1953), 31–48.
13. E. L. Ince, "Ordinary Differential Equations," Longmans, Green, London, 1927.

14. J. L. Lions, Opérateurs de Delsarte et problèmes mixtes, *Bull. Soc. Math. France* **84** (1956), 9–95.
15. M. Riesz, L'intégrale de Riemann–Liouville et le problème de Cauchy, *Acta Math.* **81** (1949), 1–223.
16. P. C. Rosenbloom, Numerical analysis and partial differential equations, *in* "Surveys in Applied Mathematics," V. Wiley, New York, 1958.
17. L. Schwartz, "Théorie des distributions," Vols. 1 and 2, Hermann, Paris, 1950–51.
18. A. Weinstein, On the Wave Equation and the Equation of Euler–Poisson, *in* American Mathematics Society Proceedings', "Fifth Symposium of Applied Mathematics, 1952," pp. 137–147, McGraw-Hill, New York, 1954.