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Low-distortion embeddings of graphs with large girth

Mikhail I. Ostrovskii [1]

Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, NY 11439, USA

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Abstract

The main purpose of the paper is to construct a sequence of graphs of constant degree with indefinitely growing girths admitting embeddings into ℓ_1 with uniformly bounded distortions. This result solves the problem posed by N. Linial, A. Magen, and A. Naor (2002).

Keywords: Distortion of embedding of a metric space into a Banach space; Girth of a graph; Graph lifts

Definition 1. Let $C < \infty$. A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

$$\forall u, v \in X \quad d_Y\big(f(u), f(v)\big) \leqslant C d_X(u, v).$$

A map f is called *Lipschitz* if it is C-Lipschitz for some $C < \infty$. For a Lipschitz map f we define its *Lipschitz constant* by

$$\operatorname{Lip} f := \sup_{d_X(u,v) \neq 0} \frac{d_Y(f(u), f(v))}{d_X(u, v)}.$$

E-mail address: ostrovsm@stjohns.edu.

[1] Participant, NSF supported Workshop in Analysis and Probability, Texas A & M University.

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A map $f : X \to Y$ is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in X \quad r d_X(u, v) \leqslant d_Y\bigl(f(u), f(v)\bigr) \leqslant rC d_X(u, v). \tag{1}$$

A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $C < \infty$. The smallest constant C for which there exists $r > 0$ such that (1) is satisfied is called the *distortion* of f. (It is easy to see that such smallest constant exists.)

The infimum of distortions of all embeddings of a finite metric space X into the Banach space ℓ_1 is called the *ℓ_1 distortion* of X and is denoted $c_1(X)$.

The ℓ_1 distortion of finite metric spaces plays an important role in the theory of approximation algorithms, see [10,11,16,17,19].

Our main purpose is to solve the following problem suggested in [12, p. 393] and repeated in [10, Open Problem 7] and [18, Problem 2.3]: does there exist a sequence of simple k-regular graphs, $k \geqslant 3$, with indefinitely growing girths and uniformly bounded ℓ_1 distortions? (All graphs mentioned in this paper are endowed with their shortest path distance.) We are going to show that such sequences exist.

The construction of this paper is inspired by the paper [2]. Recall that the *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. We start with a sequence of k-regular connected simple graphs $\{G_n\}$ with indefinitely increasing girths $g(G_n)$, such that

$$g(G_n) \geqslant \alpha \operatorname{diam}(G_n) \tag{2}$$

for some absolute constant $\alpha > 0$. Existence of such sequences of graphs is known for long time, see [6] and [3, Chapter III]. In the 1980s the constants involved in the construction were significantly improved, see [14,9,13,15]. For each graph G in the sequence $\{G_n\}$ we consider its lift $\widetilde{G}$ in the sense of the papers [1] and [5]. (We would like to warn the reader that somewhat different terminology (graph covers, voltage graphs) is used in other publications on the topic, such as in [2,7,8].) The particular version of the lift which we use is close to the lift used in [2], but it is applied to a different sequence of graphs. We changed the lift to make the argument simpler, also we need somewhat stronger estimates than those which were sufficient for [2]. Another difference of our presentation from the presentation in [2] is that we try to keep the presentation as elementary as possible, without assuming any topological and group-theoretical background of the reader. We use only some basic notions of graph theory and the definition of the space ℓ_1. We hope that our graph-theoretical terminology is standard, readers can find all unexplained terminology in [4].

Definition 2. Let L be a finite set. A *lift* $\widetilde{G}$ of a graph $G = (V(G), E(G))$ is a graph with vertex set $V(\widetilde{G}) = V(G) \times L$. The edge set $\widetilde{G}$ is the union of matchings corresponding to edges of $E(G)$. The matching corresponding to an edge uv matches all vertices of $\{u\} \times L$ with vertices of $\{v\} \times L$.

Definition 2 immediately implies that there are well-defined projections $E(\widetilde{G}) \to E(G)$ and $V(\widetilde{G}) \to V(G)$: edges of the matching corresponding to uv are projected onto uv and vertices of $\{u\} \times L$ are projected onto u. We denote both of the projections by π. It is clear from the definition that the degrees of all vertices in $\widetilde{G}$ whose projection in G is u are the same as the degree of u. In particular, any lift of a k-regular graph is k-regular.

Remark 3. It is easy to see that for each walk $\{e_i\}_{i=1}^n$ in G and each vertex $\widetilde{u} \in V(\widetilde{G})$ of the form $\widetilde{u} = (u, \ell)$ with $\ell \in L$ and u being the initial vertex of the walk $\{e_i\}_{i=1}^n$; there is a uniquely determined *lifted walk* $\{\widetilde{e}_i\}_{i=1}^n$ in $\widetilde{G}$ for which $\pi(\widetilde{e}_i) = e_i$ and $\widetilde{u}$ is the initial vertex. It is important that if G is connected, this remark remains true if instead of $\widetilde{G}$ we consider a connected component of it.

Remark 4. It is clear that if a walk in G has an edge e which is backtracked (that is, the walk contains two consecutive edges e), then the corresponding edge in the lifted walk is also backtracked.

Remark 4 implies that the projection of a cycle in $\widetilde{G}$ to G cannot be such that its edges induce in G a subgraph having vertices of degree 1. In particular, the graph induced by edges of the projection of a cycle in $\widetilde{G}$ contains cycles in G. This immediately implies $g(\widetilde{G}) \geqslant g(G)$.

We apply the lift construction to the graphs $\{G_n\}$ mentioned above. The fact that we get k-regular graphs with indefinitely increasing girths follows immediately from the observations which we just made. It remains to specify lifts for which there are suitable estimates for ℓ_1 distortions of connected components of the obtained graphs. The bounds for the distortions which we get are in terms of the constant α in (2).

For each $G \in \{G_n\}_{n=1}^\infty$ we do the following. Let L be the set $\{0,1\}^{E(G)}$, so each element of L can be regarded as a $\{0,1\}$-valued function on $E(G)$. For each $uv \in E(G)$ we need to specify a perfect matching of $\{u\} \times L$ and $\{v\} \times L$. To specify the perfect matching it suffices, for each edge in E, to pick a bijection of the set L. We do this as follows. The bijection corresponding to $e \in E(G)$ maps each function f on $E(G)$ to the function h, which has the same values as f everywhere except the edge e, and on the edge e its value is the other one (recall that we consider $\{0,1\}$-valued functions).

We denote the graphs obtained from $\{G_n\}$ using such lifts by $\{\widehat{G}_n\}$. Observe that the graphs $\{\widehat{G}_n\}$ are disconnected. To see this consider two vertices: (v, f_1) and (v, f_2), where $v \in V(G_n)$, $f_1, f_2 \in \{0,1\}^{E(G_n)}$, $f_1 \neq f_2$. If (v, f_1) and (v, f_2) are connected in $\widehat{G}_n$ by a path P, it is clear that the projection $\pi(P)$ of this path is a closed walk in G which contains each of the edges corresponding to different values of f_1 and f_2 an odd number of times. Each of the other edges contained in $\pi(P)$ is contained there an even number of times. This implies that degrees of all vertices in the subgraph of G_n spanned by edges at which f_1 and f_2 have different values are even (we give a detailed proof of a more general statement in Lemma 8 below). Therefore (v, f_1) and (v, f_2) are not connected in $\widehat{G}_n$ if the condition of the last sentence does not hold.

For this reason (ℓ_1 distortions are well-defined for connected graphs only) we pick in each of $\widehat{G}_n$ a connected component which we denote $\widetilde{G}_n$. It is easy to see that a connected component of a k-regular graph with girth g is a k-regular graph with girth $\geqslant g$. The following theorem is the main result of this paper.

Theorem 5. $c_1(\widetilde{G}_n) = O(1)$.

The main steps in our proof are presented as lemmas, where G is one of the $\{G_n\}$.

Lemma 6. *For each edge $e \in E(G)$ the set of all edges $\widetilde{e} \in E(\widetilde{G})$ for which $\pi(\widetilde{e}) = e$ forms an edge cut in $\widetilde{G}$.*

Proof. Recall that G is assumed to be connected. Combining this with Remark 3 we get that for each $e \in E(G)$ and each $v \in V(G)$ the sets $\{\widetilde{e} \in E(\widetilde{G}): \pi(\widetilde{e}) = e\}$ and $\{\widetilde{v} \in V(\widetilde{G}): \pi(\widetilde{v}) = v\}$ are nonempty.

Now let $e \in E(G)$. We can just describe the sets separated by the set of edges $\{\widetilde{e} \in E(\widetilde{G}): \pi(\widetilde{e}) = e\}$: they are the sets $(V(G) \times A_{e,0}) \cap V(\widetilde{G})$ and $(V(G) \times A_{e,1}) \cap V(\widetilde{G})$ where $A_{e,0}$ and $A_{e,1}$ are the sets of functions in $\{0,1\}^{E(G)}$ whose values on e are equal to 0 and 1, respectively.

The only thing which is not obvious is that the sets $(V(G) \times A_{e,0}) \cap V(\widetilde{G})$ and $(V(G) \times A_{e,1}) \cap V(\widetilde{G})$ are nonempty. Let $e = uv$ and let $\widetilde{v} \in V(\widetilde{G})$ be such that $\pi(\widetilde{v}) = v$. Then $\widetilde{v}$ is in one of the sets $(V(G) \times A_{e,0}) \cap V(\widetilde{G})$ or $(V(G) \times A_{e,1}) \cap V(\widetilde{G})$. We lift the one-edge-walk e to $\widetilde{G}$ starting at $\widetilde{v}$. The other end of this walk is in the other set of the pair $(V(G) \times A_{e,0}) \cap V(\widetilde{G})$ and $(V(G) \times A_{e,1}) \cap V(\widetilde{G})$. $\square$

Let $\ell_1(E(G))$ be the space of real-valued functions on $E(G)$ with its ℓ_1-norm. By the ℓ_1-*norm* of a function $H : E(G) \to \mathbb{R}$ we mean the norm $\|H\|_1 = \sum_{e \in E(G)} |H(e)|$. It is easy to see that the space $\ell_1(E(G))$ is isometric to a subspace of ℓ_1 and therefore to prove Theorem 5 it suffices to find an embedding F of $V(\widetilde{G})$ into $\ell_1(E(G))$ with distortion bounded from above by a universal constant.

For each edge cut $R(e)$ defined by the set of edges $\widetilde{e}$ in $\widetilde{G}$ satisfying $\pi(\widetilde{e}) = e$, we call one of the sides of the cut $R(e)$ the 0-*side*, and the other side the 1-*side*, and define a function $F: V(\widetilde{G}) \to \ell_1(E(G))$ by

$$\big(F(x)\big)(e) = \begin{cases} 1 & \text{if } x \text{ is in the 1-side of } R(e), \\ 0 & \text{if } x \text{ is in the 0-side of } R(e). \end{cases}$$

The Lipschitz constant of this embedding is 1. In fact, the cuts $R(e)$ are disjoint and each edge of $\widetilde{G}$ is in exactly one of the cuts. Therefore $\|F(x) - F(y)\|_1 = 1$ if x and y are adjacent vertices of $\widetilde{G}$.

To estimate the Lipschitz constant of F^{-1} we consider $x, y \in V(\widetilde{G})$, denote by $d_{\widetilde{G}}(x, y)$ their distance in $\widetilde{G}$, and observe the following:

Observation 7. If P is an xy-walk in $\widetilde{G}$, then $d_{\widetilde{G}}(x, y) \leqslant \text{length}(P)$ and $\|F(x) - F(y)\|_1$ is the number of edges in the walk $\pi(P)$ which are repeated in the walk an odd number of times.

Let us denote by $D(P)$ the number of edges repeated in $\pi(P)$ an odd number of times. Observation 7 shows that to complete the proof of Theorem 5 it suffices, for each $x, y \in V(\widetilde{G})$, to find an xy-walk P in $\widetilde{G}$ for which

$$\text{length}(P) \leqslant \beta D(P) \tag{3}$$

for some absolute constant β. This is our next goal.

Let $x = (u, f)$ and $y = (v, g)$ be two vertices of $\widetilde{G}$, where $u, v \in V(G)$ and $f, g \in \{0,1\}^{E(G)}$. Let $S \subset E(G)$ be the subset on which the functions f and g differ (recall that we consider f and g as $\{0,1\}$-valued functions on $E(G)$). Denote by H the subgraph of G induced by edges of S.

Lemma 8. *If $u \neq v$, degrees of all vertices of H, except u and v, are even. If $u = v$ degrees of all vertices of H are even.*

Proof. We use the assumption that $\widetilde{G}$ is connected, so there is an xy-walk Q in $\widetilde{G}$. We claim that $\pi(Q)$ has to contain each of the edges of S an odd number of times and each of the other edges an even number of times (possibly 0).

To see this we recall our construction of the lift of G and our definition of a lifted walk (see Remark 3). It is obvious that Q is a lifted walk of $\pi(Q)$. Our definitions are such that the change in the L-coordinate in each step (when we walk along the lifted walk) is made in exactly one value of the corresponding $\{0, 1\}$-valued function on $E(G)$, the choice of this coordinate depends only on the π-projection of the edge which we are passing, and not on the direction in which we pass it, or on the L-coordinate of the vertex we are at (this is a very important property of the graph lift which we consider). Also, we need an obvious observation that if we change some value of a $\{0, 1\}$-valued function twice, it returns to its original value.

We denote the subgraph of G induced by edges of $\pi(Q)$ by $I(Q)$ ($\pi(Q)$ and $I(Q)$ are slightly different objects: $\pi(Q)$ is a sequence of edges in which some edges can be repeated, $I(Q)$ is the subgraph of G induced by edges which are listed in $\pi(Q)$ at least once). Now we introduce a non-simple graph $N(Q)$ having $I(Q)$ as its underlying simple graph and having as many parallel edges for each edge of $I(Q)$, as many times the edge is repeated in $\pi(Q)$.

It is clear that the graph $N(Q)$ contains an Euler trail which is a uv-walk (we just follow the walk $\pi(Q)$, each time using a different parallel edge for edges repeated in $\pi(Q)$). Therefore, if $u \neq v$ degrees of all vertices of $N(Q)$, except u and v are even. If $u = v$ degrees of all vertices of $N(Q)$ are even. It is clear that if we delete from $N(Q)$ an even number of parallel edges, this property continues to hold. In particular, it continues to hold if we delete all edges parallel to edges repeated in $\pi(Q)$ an even number of times, and leave one copy of each edge repeated in $\pi(Q)$ an odd number of times. It is clear that what we get after this deletion is the graph H. □

Proof of Theorem 5. Our goal is to construct an xy-walk P satisfying (3). We use the graph H introduced above. Lemma 8 implies that all components of H, except possibly the one that contains u and v, contain cycles. Therefore each of them has at least $g(G)$ edges. It is also clear that the component containing u and v (if exists) has an Euler trail whose initial vertex is u and whose terminal vertex is v; and all other components have Euler tours (that is, closed Euler trails).

Let $H_1, \dots, H_t$ be the components of H. We assume that H_1 contains u and v. Observe that in the case where $u = v$ such component does not have to exist. In this case we introduce H_1 as a trivial component containing one vertex $u = v$. Observe that we may assume that this trivial component contains an Euler trail joining u and v, it is just a trail with no edges.

Observe that any two components of H can be joined by a path in G of length $\leqslant \operatorname{diam}(G)$. Let $M_1, \dots, M_{t-1}$ be paths of length $\leqslant \operatorname{diam}(G)$ each, such that

1. M_i joins H_i and H_{i+1}.
2. The terminal vertex of M_i coincides with the initial vertex of M_{i+1}.

Now we form the following uv-walk M in G:

- It starts at u and follows an Euler trail of H_1 to the initial vertex of M_1.
- It follows M_1 to H_2.
- It follows the Euler tour of H_2.
- It follows M_2 to H_3.
- It continues in an obvious way to H_t.

- It follows the Euler tour of H_t.
- It follows M_{t-1} back to H_{t-1}.
- It follows M_{t-2} back to H_{t-2}.
- It continues in an obvious way till it reaches H_1.
- It follows the final part of the Euler trail of H_1 (the initial part of that Euler trail was followed in the first step) and completes it at v.

We lift the walk M taking $x = (u, f)$ as the initial vertex of the lifted walk. Denote the obtained walk by P. It is clear that all edges of S are used in M an odd number of times and that all other edges are used in M an even number of times. Therefore the lifted walk P has $y = (v, g)$ as its terminal vertex (to see this we use observations made in the proof of Lemma 8). So P is an xy-walk. Also it is clear that $D(P) = |S|$.

Counting the number of edges in P, we get that its length does not exceed

$$2(t-1)\operatorname{diam}(G) + \bigl|E(H_1)\bigr| + \bigl|E(H_2)\bigr| + \cdots + \bigl|E(H_t)\bigr| = 2(t-1)\operatorname{diam}(G) + |S|. \tag{4}$$

In the case where $t = 1$, we use the right-hand side of (4) and get

$$\operatorname{length}(P) \leqslant D(P),$$

so (3) holds with $\beta = 1$. In the case where $t > 1$ we use the left-hand side of (4) to get

$$\operatorname{length}(P) \leqslant \bigl|E(H_1)\bigr| + \sum_{i=2}^{t}\bigl(\bigl|E(H_i)\bigr| + 2\operatorname{diam}(G)\bigr).$$

After that we combine the fact that $|E(H_i)| \geqslant g(G)$ for $i \geqslant 2$ with the assumption that $g(G) \geqslant \alpha \operatorname{diam}(G)$, and get

$$\begin{aligned}
\operatorname{length}(P) &\leqslant \bigl|E(H_1)\bigr| + \sum_{i=2}^{t}\bigl|E(H_i)\bigr|\left(1 + \frac{2}{\alpha}\right) \\
&\leqslant \left(1 + \frac{2}{\alpha}\right)|S| \\
&= \left(1 + \frac{2}{\alpha}\right)D(P).
\end{aligned}$$

Thus (3) holds with $\beta = (\frac{2}{\alpha} + 1)$. This completes the proof of Theorem 5. □

Remark 9 *(Remark on coarse embeddings).* The main purpose of [2] is to construct metric spaces with bounded geometry which are coarsely embeddable into a Hilbert space, but do not have property A introduced by Yu in [23].

Definition 10. A discrete metric space X has *property A* if for every $\varepsilon > 0$ and every $R > 0$ there is a family $\{A_x\}_{x\in X}$ of finite subsets of $X \times \mathbb{N}$ and a number $S > 0$ such that

- $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \varepsilon$ whenever $d(x, y) \leqslant R$,
- $A_x \subseteq B(x, S) \times \mathbb{N}$ for every $x \in X$, where $B(x, S)$ is the ball of radius S centered at x.

Definition 11. A metric space X is said to have a *bounded geometry* if for each $r > 0$ there exists a positive integer $M(r)$ such that each ball in X of radius r contains at most $M(r)$ elements.

First we would like to mention that Theorem 5 also provides examples of metric spaces with bounded geometry, which are coarsely embeddable into a Hilbert space, but do not have property A. In fact, since ℓ_1 admits a coarse embedding into a Hilbert space (see [19, Corollary 3.1]), Theorem 5 implies that the graphs $\widetilde{G}_n$ admit uniformly coarse embeddings into a Hilbert space. Therefore, combining our Theorem 5 with a recent result of Willett [22], we get more examples of metric spaces with bounded geometry but without property A, admitting coarse embeddings into a Hilbert space. (It is worth mentioning that without the bounded geometry condition such examples were known earlier [20].)

Also, it is worth mentioning that in [21] it was proved that locally finite metric spaces which do not admit coarse embeddings into a Hilbert space contain substructures which are "locally expanding" (see [21] for details). Our example, as well as the example in [2], show that the converse is false, since families of simple graphs with constant degree $\geqslant 3$ and indefinitely growing girth are "locally expanding" in the sense of [21].

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References

[1] A. Amit, N. Linial, Random lifts of graphs: edge expansion, Combin. Probab. Comput. 15 (2006) 317–332.
[2] G. Arzhantseva, E. Guentner, J. Špakula, Coarse non-amenability and coarse embeddings, Geom. Funct. Anal., in press; arXiv:1101.1993.
[3] B. Bollobás, Extremal Graph Theory, Academic Press, London–New York, 1978.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Grad. Texts in Math., vol. 244, Springer, New York, 2008.
[5] Y. Drier, N. Linial, Minors in lifts of graphs, Random Structures Algorithms 29 (2006) 208–225.
[6] P. Erdős, H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, Wiss. Z. Martin-Luther-Univ. Halle-Wittenb. Math.-Natur. Reihe 12 (1963) 251–257.
[7] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.
[8] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, Inc., New York, 1987.
[9] W. Imrich, Explicit construction of regular graphs without small cycles, Combinatorica 4 (1) (1984) 53–59.
[10] N. Linial, Finite metric spaces–combinatorics, geometry and algorithms, in: Proceedings of the International Congress of Mathematicians, vol. III, Beijing, 2002, Higher Ed. Press, Beijing, 2002, pp. 573–586.
[11] N. Linial, E. London, Y. Rabinovich, The geometry of graphs and some of its algorithmic applications, Combinatorica 15 (2) (1995) 215–245.
[12] N. Linial, A. Magen, A. Naor, Girth and Euclidean distortion, Geom. Funct. Anal. 12 (2002) 380–394.
[13] A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan graphs, Combinatorica 8 (1988) 261–277.

[14] G.A. Margulis, Explicit constructions of graphs without short cycles and low density codes, Combinatorica 2 (1) (1982) 71–78.

[15] G.A. Margulis, Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators, Problemy Peredachi Informatsii 24 (1) (1988) 51–60 (in Russian); English translation: Probl. Inf. Transm. 24 (1) (1988) 39–46.

[16] J. Matoušek, Lectures on Discrete Geometry, Springer-Verlag, New York, 2002.

[17] J. Matoušek, Embedding finite metric spaces into normed spaces, an updated version of Chapter 15 of the book [16], prepared for a Japanese translation published in 2005. English version is available at http://kam.mff.cuni.cz/~matousek/.

[18] J. Matoušek (Ed.), Open problems on embeddings of finite metric spaces, last revision: July 2010, jointly with Assaf Naor; can be downloaded from http://kam.mff.cuni.cz/~matousek/.

[19] A. Naor, L_1 embeddings of the Heisenberg group and fast estimation of graph isoperimetry, in: Proceedings of the International Congress of Mathematicians, vol. III, 2010, Hyderabad India, 2011, pp. 1549–1575.

[20] P.W. Nowak, Coarsely embeddable metric spaces without property A, J. Funct. Anal. 252 (2007) 126–136.

[21] M.I. Ostrovskii, Expansion properties of metric spaces not admitting a coarse embedding into a Hilbert space, C. R. Acad. Bulgare Sci. 62 (2009) 415–420.

[22] R. Willett, Property A and graphs with large girth, J. Topol. Anal. 3 (3) (2011) 377–384.

[23] G. Yu, The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. 139 (2000) 201–240.