On Approximate Inertial Manifolds to the Navier–Stokes Equations

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Submitted by C. Foias

Received February 27, 1989

Recently, the theory of Inertial Manifolds has shown that the long time behavior (the dynamics) of certain dissipative partial differential equations can be fully described by that of a finite ordinary differential system. Although we are still unable to prove existence of Inertial Manifolds to the Navier–Stokes equations, we present here a nonlinear finite dimensional analytic manifold that approximates closely the global attractor in the two-dimensional case, and certain bounded invariant sets in the three-dimensional case. This approximate manifold and others allow us to introduce modified Galerkin approximations.

INTRODUCTION

In numerical simulation of turbulent flows one is interested in numerical schemes that approximate the solutions of the Navier–Stokes equations (N.S.E.) for long intervals of time. This leads to the difficult question of relating the long time behavior of the exact solutions of the N.S.E. to that of the approximating finite dimensional system. The problem is difficult because of its unstable nature. Namely, if for example the exact solution of the N.S.E. is converging to an unstable stationary solution, it is most unlikely, in general, that the solution of the approximating scheme would converge to an approximation of this unstable stationary solution. The fact that the two-dimensional N.S.E. have finite dimensional universal (global) attractors (see [17, 3, 7] and the literature cited there) and a finite number of determining modes ([13, 11]) indicates that the dynamics of these equations are controlled by a finite number of parameters, which makes the above question more significant. Some attempts, restricted to simple configurations like stationary and time periodic solutions, are made to answer the above question when the approximating scheme under consideration is...
the usual Galerkin approximation associated to the eigenvectors of the Stokes operator (cf. [6, 27, 29]); for questions related to long time approximation by finite elements Galerkin approximation see [19].

In recent years the concept of Inertial Manifold (I.M.), which is a positively invariant finite dimensional Lipschitz manifold that attracts every trajectory exponentially, emerged as a significant feature for the study of long time behavior of dissipative partial differential equations (P.D.E.) (cf. [15, 4, 12, 21, 23, 16, 5]). Notice that the whole dynamics of the P.D.E. lies in the I.M., whenever it exists; moreover, the P.D.E. reduces to an ordinary differential equation (O.D.E.) on the I.M. As a result one can examine the dynamical properties (e.g., stability, bifurcation, etc.) of solutions of the P.D.E. by studying the inertial form, i.e., the reduced O.D.E. If in addition we know that the I.M. is asymptotically complete, as was established for instance in [4, 16], then the dynamics of the P.D.E. is completely described by the inertial form. Even in the cases where we know that I.M.'s exist, like the Kuramoto-Sivashinski equation and certain Reaction Diffusion equations, we are still unable to represent them in explicit forms. Hence alternatively we approximate them by simple manifolds which are well motivated by the dynamics of the equation (cf. [16]). Moreover, we use these approximate manifolds as substitutes in the real applications (see e.g., [8]). Inspired by the theory of I.M.'s Foias-Manley- Temam [9, 10] introduced an approximate I.M., say $\mathcal{M}_0$, to the two-dimensional N.S.E., even though the question of existence of I.M. is still open for the two-dimensional N.S.E.

This paper is organized as follows: In Section 1 we review the N.S.E. and we recall certain known estimates and facts that we use later. Section 2 is divided into two paragraphs. In paragraph 2.1 we recall the concept of I.M., inertial form, and approximate I.M. We state the results of [9, 10] in Theorem 2.1. Estimate (2.10) is of particular interest to us because we use it later in the proof of Theorem 2.4. To indicate the significance of the results of [9, 10] we present an illustrative example which we announced in [28] with the other results of this work. In paragraph 2.2 we recall the analytic manifold $\mathcal{M}^+$, which contains all the stationary solutions of the N.S.E. $\mathcal{M}^+$ was presented in two different ways in [18, 14]. We emphasize the fact that $\mathcal{M}^+$ exists in the two- as well as in the three-dimensional case. In this work we consider $\mathcal{M}^+$ as an approximate I.M. In Theorem 2.4 we consider the two-dimensional case and we show that orbits in the universal (global) attractor are closer to $\mathcal{M}^+$ than to $\mathcal{M}_0$. We remark that the proofs of Theorem 2.4 and the other results of this work can be easily extended to the tree-dimensional case when we replace orbits in the global attractor by orbits in invariant sets which are bounded in the $H^1$-Sobolev norm (cf. [7]). Therefore, we shall concentrate in this work on the two-dimensional case. Because $\mathcal{M}^+ = \text{graph } \Phi^+$, and $\Phi^+$ is given implicitly by (2.13), we
introduce in paragraph 2.3 an explicit approximating function \( \Phi_n^\varepsilon \) of \( \Phi^\varepsilon \) such that the graph(\( \Phi_n^\varepsilon \)) approximates the universal attractor as well as \( \mathcal{M}^\varepsilon \). We observe that when the N.S.E. are not supplemented with the periodic boundary condition (see (1.4a) and (1.4b)), we need to compute infinitely many Fourier coefficients in order to find \( \Phi_n^\varepsilon \). This fact prevents us from implementing these approximate inertial manifolds in real computations. Hence, we present in paragraph 2.3, in the case of boundary condition (1.4a), appropriate approximating functions \( \Phi_n^{+,k} \) to \( \Phi^\varepsilon \) and \( \Phi_n^\varepsilon \) that can be implemented in real computations. These approximating functions \( \Phi_n^{+,k} \) yield error estimates similar to those obtained for \( \Phi^\varepsilon \).

Let us also mention that recently R. Temam [26] introduced a different type of asymptotic approximation to the solutions of the two-dimensional N.S.E. The method introduced in [26] leads to error estimates similar to the ones we obtain in this work.

These approximate I.M.'s, that we introduce here, and others induce modified Galerkin approximations that will be studied in subsequent works.

1. PRELIMINARIES AND NOTATIONS

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \). The Navier–Stokes equations of two-dimensional viscous incompressible flows in \( \Omega \) are given as:

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = F \quad \text{in} \quad \Omega \times \mathbb{R}^+ \tag{1.1}
\]

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+ \tag{1.2}
\]

\[
u u(x, 0) = u_0(x) \quad \text{in} \quad \Omega, \tag{1.3}
\]

where \( u = u(x, t) \) is the velocity vector, \( p = p(x, t) \) is the pressure, \( \nu > 0 \) is the kinematic viscosity (which is given), and \( F = F(x) \) represents the external body forces (which are also given). We consider only the case when \( F \) is time independent.

We complete the N.S.E. with either the homogeneous Dirichlet boundary condition or the periodic boundary condition. Namely,

Homogeneous Dirichlet boundary condition:

\[
u u|_{\partial \Omega} = 0, \tag{1.4a}
\]

or

periodic boundary condition:

\[
\Omega = (0, L_1) \times (0, L_2) \quad \text{and} \quad u, p \quad \text{are periodic of period} \ L, \ \text{in} \ \text{the direction} \ x_i, \ i = 1, 2. \tag{1.4b}
\]
In the case (1.4a) we denote:
\[
\gamma = \{ v \in (C_0^\infty (\Omega))^2, \text{ div } v = 0 \},
\]
and in the case (1.4b) we denote:
\[
\gamma = \{ v = \text{trigonometric polynomial of period } L_i, \text{ the direction } x_i, i = 1, 2, \text{ with values in } \mathbb{R}^2, \text{ div } v = 0 \text{ and } \int_{\Omega} v(x) \, dx = 0 \}.
\]

In both cases we set
\[
H = \text{closure of } \gamma \text{ in } (L^2(\Omega))^2
\]
\[
V = \text{closure of } \gamma \text{ in } (H^1(\Omega))^2.
\]

where $H^s(\Omega)$ ($s = 1, 2, ..., $) denotes the Sobolev space of order $s$.

For $u \in H$ and $v \in V$ we denote by
\[
|u|^2 = \int_{\Omega} |u(x)|^2 \, dx \quad \text{and} \quad \|v\|^2 = \int_{\Omega} |\nabla v(x)|^2 \, dx
\]
the norms in $H$ and $V$ respectively. The corresponding scalar products will be denoted by $(\cdot, \cdot)$ and $((\cdot, \cdot))$, respectively.

From now on we shall assume that in the case (1.4a) the domain $\Omega$ has a sufficiently smooth boundary.

As usual let $P$ be the orthonormal projection of $(L^2(\Omega))^2$ onto $H$, set $D(A) = V \cap (H^2(\Omega))^2$, and define the Stokes operator
\[
Au = -PAu, \quad \forall u \in D(A),
\]
and the bilinear operator
\[
B(v, w) = P[(v \cdot \nabla) w], \quad \forall v, w \in D(A).
\]

It is well known that $A$ is a linear unbounded self-adjoint positive operator, with $A^{-1}$ compact (see e.g., [24, 25, 20, 2]). Since $D(A) \subset H$ is dense, then $H$ has an orthonormal basis \{ $w_j$ \}_{j=1}^\infty of eigenvectors of the operator $A$, $Aw_j = \lambda_j w_j$, $j = 1, 2, ..., $ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots $. Moreover, $\lambda_j$ satisfies
\[
c_0 \lambda_1 m \leq \lambda_m \leq c_1 \lambda_1 m, \quad m = 1, 2, ..., \tag{1.6}
\]
for some positive constants $c_0$ and $c_1$ (cf. [22]). (Here and elsewhere in this
work $c_0, c_1, c_2, \ldots$, denote positive absolute constants or nondimensional positive constants that depend on $\Omega$.

We recall the following inequalities which are satisfied by $B(u, v)$ (cf. [24, 25, 20, 21]):

\begin{align*}
|\langle B(u, v), w \rangle| & \leq c_2 \| u \|^{1/2} \| u \|^{1/2} \| v \| \| w \|^{1/2}, \quad \forall u, v, w \in V \quad (1.7) \\
|\langle B(u, v), w \rangle| & \leq c_3 \| u \|_{L^4(\Omega)} \| v \| \| w \|, \quad \forall u \in D(A), \forall v \in V, \forall w \in H. \quad (1.8)
\end{align*}

Also we recall from [1]

\[ \| u \|_{L^4(\Omega)} \leq c_4 \| u \| \left( 1 + \log \left( \frac{|Au|}{\lambda_1^{1/2} \| u \|} \right) \right)^{1/2}, \quad \forall u \in D(A). \quad (1.9) \]

From (1.8) and (1.9) one concludes:

\[ |\langle B(u, v), w \rangle| \leq c_5 \| v \| \| w \| \| u \| \left( 1 + 2 \log \left( \frac{|Au|}{\lambda_1^{1/2} \| u \|} \right) \right)^{1/2}, \quad \forall u \in D(A), \forall v \in V, \forall w \in H, \quad (1.10) \]

and

\[ |\langle B(u, v), w \rangle| \leq c_5 \| v \| \| u \| \| w \| \left( 1 + 2 \log \left( \frac{|Aw|}{\lambda_1^{1/2} \| w \|} \right) \right)^{1/2}, \quad \forall u \in H, \forall v \in V, \forall w \in D(A). \quad (1.10)' \]

(For other inequalities which are satisfied by the bilinear operator $B(u, v)$ the reader is referred to [24, 25, 20, 2, 11, 27]).

In addition, the operator $B$ enjoys the following fundamental property:

\[ (B(u, v), w) = -(B(u, w), v), \quad \forall u \in H, \forall v, w \in D(A). \quad (1.11) \]

Using the above notation, the system (1.1)–(1.3) completed by (1.4a) or (1.4b) is equivalent to the following functional differential equation (see, e.g., [24, 25, 20, 2]).

\[ \frac{du}{dt} + vAu + B(u, u) = f \quad (1.12) \]

\[ u(0) = u_0. \quad (1.13) \]

For results concerning existence, uniqueness, and regularity of solutions to (1.12), (1.13) see, for instance, [24, 25, 2].

It is also well known that there exist constants $M_0$ and $M_1$, which
depend only on \(v, |f|, \) and \(\lambda_1,\) such that for every solution \(u(t)\) of (1.12), (1.13) there is a time \(t_0\) depending on \(u_0, v, |f|, \) and \(\lambda_1\) such that
\[
|u(t)| \leq M_0 \quad \text{and} \quad \|u(t)\| \leq M_1 \quad \text{for all} \quad t \geq t_0. \tag{1.14}
\]
(see, e.g., [17, 24, 21]).

In particular if \(u_0\) belongs to the universal (global) attractor then (1.14) holds for all \(t \in \mathbb{R}.

We remark here that the existence of the constant \(M_1,\) such that (1.14) holds, is not known in the three-dimensional case. Therefore, in order to extend the results of this work to the three-dimensional case we need to consider the approximation of invariant sets which are bounded in \(V.\)

\section{2. Nonlinear Galerkin Approximation}

In this section we recall the concepts of Inertial Manifold, Inertial Form, and Approximate Inertial Manifold.

\subsection{2.1. Inertial Manifolds and Approximate Inertial Manifolds}

Denote by \(P_m\) the orthogonal projection of \(H\) onto \(H_m = \text{span}\{w_1, ..., w_m\},\) and \(Q_m = I - P_m,\) and set \(p = P_m u\) and \(q = Q_m u\). then Eq. (1.12) is equivalent to
\[
\begin{align*}
\frac{dp}{dt} + v A p + P_m B(p + q, p + q) &= P_m f \quad \text{(2.1)} \\
\frac{dq}{dt} + v A q + Q_m B(p + q, p + q) &= Q_m f \quad \text{(2.2)}
\end{align*}
\]

An Inertial Manifold (I.M.) for Eq. (1.12) is a subset \(\mathcal{M} \subset H\) which enjoys the following properties:

(i) \(\mathcal{M}\) is finite dimensional Lipschitz manifold,

(ii) \(\mathcal{M}\) is positively invariant under the flow (i.e., if \(u_0 \in \mathcal{M}\) then the solution of (1.12), (1.13) \(u(t) \in \mathcal{M}\) for all \(t > 0),\)

(iii) \(\mathcal{M}\) attracts every trajectory exponentially (i.e., for every solution \(u(t)\) of (1.12) \(\text{dist}(u(t), \mathcal{M}) \to 0\) exponentially.)

(See, e.g., [15, 4, 12, 23, 21, 16, 5] for existence and nonexistence results of I.M.'s for certain evolution equations.) Note that (2.3)(iii) implies that the universal (global) attractor is contained in \(\mathcal{M}.\)
If in addition we require \( \mathcal{M} \) to be a graph of a Lipschitz function \( \Phi: \mathcal{H}_m \to Q_m \mathcal{H}_m \), then the invariance condition (2.3)(ii) is equivalent to state that for every solution \( p(t) \) and \( q(t) \) of (2.1), (2.2) with \( q(0) = \Phi(p(0)) \) one has \( q(t) = \Phi(p(t)) \) for all \( t > 0 \). Hence, if such a function \( \Phi \) exists, then the reduction of the system (2.1), (2.2) to \( \mathcal{M} \) is equivalent to the ordinary differential system, which we call an inertial form:

\[
\frac{dp}{dt} + v A p + P_m B(p + \Phi(p), p + \Phi(p)) = P_m f, \quad p \in \mathcal{H}_m. \tag{2.4}
\]

Even for equations, like the Kuramoto–Sivashinski equation, that have an I.M. as a graph of a Lipschitz function, say \( \Phi \), we still cannot get an explicit analytic form for \( \Phi \) ([15, 4, 12, 21, 5]). Instead there were attempts, motivated by the dynamics of the evolution equation, to approximate \( \Phi \) by simple explicit functions, and to use these functions in new numerical schemes [16, 8].

Although the existence problem of I.M. to the N.S.E. is still an open problem, the theory suggests to approximate the universal (global) attractor by smooth manifolds that will be called approximate inertial manifolds [9, 10, 26, 28].

Indeed, the usual Galerkin approximation method associated with the eigenvectors of the Stokes operator \( \mathcal{A} \), proposes the linear manifold \( \mathcal{H}_m \) as an approximate inertial manifold. Namely, we replace the mapping \( \Phi \) in (2.4) by zero to obtain the usual Galerkin approximation (see, e.g., [24, 25, 2])

\[
\frac{d u_m}{dt} + v A u_m + P_m B(u_m, u_m) = P_m f, \quad u_m \in \mathcal{H}_m.
\]

Heuristic and physical arguments led Foias–Manley–Temam in [9, 10] to introduce the finite dimensional analytic manifold \( \mathcal{M}_0 = \text{graph}(\Phi_0) \),

\[
\Phi_0(p) = (vA)^{-1} [Q_m f - Q_m B(p, p)], \quad p \in \mathcal{H}_m, \tag{2.5}
\]

as a better approximate manifold to the universal (global) attractor than \( \mathcal{H}_m \).

**Theorem 2.1** [9, 10]. Let \( m \) be large enough such that

\[
\lambda_{m+1} \geq \left( \frac{2c_2 M}{v} \right)^2. \tag{2.6}
\]

Then every solution \( u(t) = p(t) + q(t) \) of (2.1), (2.2) satisfies

\[
|q(t)| \leq K_0 \lambda_{m+1}^{1/2} L^{1/2} \tag{2.7}
\]
\[ \| q(t) \| \leq K_1 \lambda_{m+1}^{1/2} L^{1/2} \]  
(2.8)

\[ |Aq(t)| \leq K_2 L^{1/2} \]  
(2.9)

\[ \left| \frac{dq}{dt} (t) \right| \leq K_0 \lambda_{m+1}^{1/2} L^{1/2} \]  
(2.10)

\[ \| q(t) - \Phi_0(p(t)) \| \leq K_3 \lambda_{m+1} L \quad \text{for all } t \geq T_* \]  
(2.11)

where \( T_* > 0 \) depends on \( v, \lambda_1, |f|, \) and \( R_0, \) when \( |u(0)| \leq R_0, \)

\( L = (1 + \log(\lambda_m/\lambda_1)), \) and where \( K_0, K_1, K_2, K_3, K_4 \) denote positive constants that depend on \( v, \lambda_1, \) and \( |f|. \)

Remark 3.2.
(i) We remark that if \( u(t) \) is an orbit in the universal attractor then the estimates (2.7)–(2.11) hold for all \( t \in \mathbb{R}. \)

(ii) In the case of periodic boundary condition (i.e., (1.4b)) one can find in [10] an explicit analytic form for the above constants in terms of \( v, \lambda_1, \) and \( |f|. \)

(iii) To appreciate the significant improvement achieved by introducing \( \Phi_0, \) we recall the following example that we presented in [28].

**Example.** Let \( u' = \sum_{k=1}^{\infty} u_k w_k, \) where \( u_k = \sigma k^{3/2}(1 + \log k)^1 \) and \( \sigma > 0 \) to be chosen. Thanks to (1.6) \( u' \in D(A) \). \( u' \) is a stationary solution to (1.12) with \( f \) defined to be

\[ f = vAu' + B(u', u'). \]

By choosing \( \sigma \) large enough, for fixed \( v > 0, \) one can make the Grashof number \( G = |f|/v^{3/2}\lambda_1 \) sufficiently large so that the dynamics of (1.12) with the above \( f \) would not be trivial.

Because of (1.6) it is clear that

\[ \| Q_n u' \| \geq c_n \sigma^{1/2} \left( 1 + \log \left( \frac{\lambda_n}{\lambda_1} \right) \right) \quad \text{for all } n = 1, 2, \ldots \]  
(2.12)

Since \( u' \) belongs to the universal attractor, (2.12) makes the estimate (2.8) optimal up to the logarithmic terms. Accordingly, a comparison between (2.8) and (2.11) verifies the significant improvement achieved by introducing \( \Phi_0. \)

In the next paragraph we introduce another approximate inertial manifold, and show that it yields a smaller error in comparison to (2.11).
2.2. THE APPROXIMATE MANIFOLD $\mathcal{M}^i$

Denote by

$$\mathcal{B} = \{ p \in H_m: \| p \| \leq 2M_1 \},$$

and

$$\mathcal{B}^\perp = \{ q \in Q_m V: \| q \| \leq 2M_1 \},$$

where $M_1$ satisfies (1.14).

We recall (cf. [18, 14]) that for $m$ large enough there exist a mapping $\Phi^i: \mathcal{B} \to Q_m V$, which satisfies

$$\Phi^i(p) = (vA)^{-1} [Q_m f - Q_mB(p + \Phi^i(p), p + \Phi^i(p))], \quad \forall p \in \mathcal{B}. \quad (2.13)$$

Moreover, the graph of $\Phi^i$, which we denote $\mathcal{M}^i$, is a C-analytic manifold. Notice that $\mathcal{M}^i$ contains all the stationary solutions of (1.12).

In the next theorem we recall the existence of $\Phi^i$ from [18] and give a lower bound for $m$.

**THEOREM 2.3.** Let $m$ be large enough such that

$$\lambda_{m+1} \geq \max \left\{ 4r_2^2, \frac{r_1^2}{4M_1^2} \right\}. \quad (2.14)$$

Then there exists a unique mapping $\Phi^i: \mathcal{B} \to Q_m V$ that satisfies (2.13). Moreover

$$\| \Phi^i(p) \| \leq \lambda_{m+1}^{-1/2} r_1, \quad (2.15)$$

where $r_1 = v^{-1}c_5 8M_1^2 L^{1/2} + v^{-1}c_2 8M_1^2 + v^{-1}\lambda_{m+1}^{-1/2} | f |$,

$$r_2 = [v^{-1}c_5 2M_1 L^{1/2} + v^{-1}c_2 6M_1], \text{ and, as before,}$$

$$L = \left(1 + \log \left(\frac{\lambda_m}{\lambda_1}\right)\right).$$

**Proof.** Let $p \in \mathcal{B}$ be fixed; we define $T_p: \mathcal{B}^\perp \to Q_m V$ such that

$$T_p(q) = (vA)^{-1} [Q_m f - Q_mB(p + q, p + q)].$$

It is sufficient to show that $T_p$ has a unique fixed point. First we show that $T_p: \mathcal{B}^\perp \to \mathcal{B}^\perp$. Let $q \in \mathcal{B}^\perp$, and let $w \in H$, with $|w| = 1$; then
\[ |(A^{1/2} T_p(q), w)| \leq v^{-1} \left[ \left| (B(p + q, p + q), A^{-1/2} Q_m w) \right| + \left| A^{-1} Q_m f \right| \left| w \right| \right] \]
\[ \leq v^{-1} \left[ \left| (B(p, p + q), A^{-1/2} Q_m w) \right| + \left| (B(q, p + q), A^{-1/2} Q_m w) \right| + \left| (v \lambda_{m+1})^{-1} \right| \right]. \]

Using (1.10) and (1.7) we get
\[ \leq v \left| c_5 \right| p + q \left| A^{-1/2} Q_m w \right| \left| p \right| \left( 1 + \log \left( \frac{|A p|}{\| p \| \lambda_1^{1/2}} \right) \right)^{1/2} \]
\[ + v \left| c_2 \right| q^{1/2} \left| q \right|^{1/2} \left| p + q \right| \]
\[ \times \left| A^{-1/2} Q_m w \right|^{1/2} |w|^{1/2} + \left| (v \lambda_{m+1})^{-1} \right| \left| f \right| \]
\[ \leq v^{-1} c_5 8 M_1^2 \lambda_{m+1}^{-1/2} \left( 1 + \log \left( \frac{\lambda_m}{\lambda_1} \right) \right)^{1/2} \]
\[ + v \left| c_2 \right| \lambda_{m+1}^{-1/2} 8 M_1^2 + \left| (v \lambda_{m+1})^{-1} \right| \left| f \right|. \]

Therefore,
\[ \| T_p(q) \| \leq \lambda_{m+1}^{1/2} r_1, \quad (2.16) \]

by (2.14)
\[ \| T_p(q) \| \leq 2 M_1. \]

Now we show that $T_p$ is a contraction. Observe that
\[ \frac{\partial}{\partial q} T_p(q) \eta = (vA)^{-1} Q_m [B(p + q, \eta) + B(\eta, p + q)], \quad \forall \eta \in Q_m V. \]

Let $w \in H$, with $|w| = 1$, then
\[ \left| \left( A^{1/2} \frac{\partial}{\partial q} T_p(q) \eta, w \right) \right| \]
\[ \leq v^{-1} \left| (B(p, \eta), A^{-1/2} Q_m w) \right| + v^{-1} \left| (B(q, \eta), A^{-1/2} Q_m w) \right| \]
\[ + v^{-1} \left| (B(\eta, p + q), A^{-1/2} Q_m w) \right|, \]

using (1.10) and (1.7),
Thus (2.17) thanks to (2.14) we conclude from (2.17) that

$$|q(t) - \Phi^t(p(t))| \leq K_0' v^{-3/2} L^{1/2} t^{-3/2} \lambda_{m+1}^{-1/2}$$

for all $t \geq T_\ast$, (2.18)

where $T_\ast$ and $K_0'$ are as in Theorem 2.1.

Proof. Let $\Delta(t) = q(t) - \Phi^t(p(t))$. From (2.2) and (2.13) we have

$$vA\Delta + Q_m[B(\Delta, p + \Phi^t(p)) + B(p + q, \Delta)] + \frac{dq}{dt} = 0.$$
and applying (1.7) we get
\[
\nu \|A\|^2 \leq c_2 |A| \|A\| p + \Phi'(p) \| + \frac{d\bar{q}}{dt}|A|.
\]
(2.19)

For \( t \geq T_* \) we have \( \|p(t)\| \leq M_1 \); by virtue of Theorem 2.3 we have \( \|\Phi'(p(t))\| \leq 2M_1 \). Substituting (2.10) in (2.19) implies
\[
\nu \|A\|^2 \leq c_2 \hat{\lambda}_{m+1}^{1/2} \|A\|^2 (M_1 + 2M_1) \| K_0 \hat{\lambda}_{m+1}^{1/2} L^{1.2} \| A \|.
\]

Using (2.14) we conclude
\[
\|A\| \leq \frac{2K_0}{\nu} \hat{\lambda}_{m+1}^{1/2} L^{1.2}.
\]

which concludes our proof.

### 2.3. Explicit Approximations to \( \Phi' \)

Notice that a straight comparison between (2.11) and (2.18) shows that \( \Phi' \) gives a better approximation than \( \Phi_0 \). However, \( \Phi_0 \) has the advantage of having an explicit analytic form by (2.5), while \( \Phi' \) is given implicitly by (2.13). Nevertheless, \( \Phi' \) was constructed in Theorem 2.3 by the contraction principle, therefore it can be approximated by simple explicit functions, as we shall see in Theorem 2.5, using the successive approximations procedure. Later in this paragraph we introduce appropriate “truncations” to these simple functions that can be implemented in real computations.

**Theorem 2.5.** Let \( m \) be large enough such that (2.14) holds. For every \( p \in \mathcal{B} \) we define \( T_p : \mathcal{B}^+ \to \mathcal{B}^+ \) as in Theorem 2.3:

\[
T_p(q) = (vA)^{-1}[Q_m f - Q_m B(p + q, p + q)]. \quad \forall q \in \mathcal{B}^+.
\]

Denote by

\[
\Phi_0^*(p) = T_p(0), \quad \forall p \in \mathcal{B},
\]
\[
\Phi_{n+1}^*(p) = T_p(\Phi_n^*(p)), \quad \forall p \in \mathcal{B} \text{ and for } n = 0, 1, 2, \ldots.
\]

(2.20)

Then
\[
\|\Phi'(p) - \Phi_n^*(p)\| \leq 2(r_2 \hat{\lambda}_{m+1}^{1.2})^{n+1} \hat{\lambda}_{m+1}^{-1.2} \nu^{-1} [f] + 4c_s M_1^2 L^{1.2}],
\]

(2.21)

where \( r_2 \) is as specified in Theorem 2.3.
Proof. First notice that $\Phi_s^p(p) = \Phi_0(p)$, for every $p \in \mathcal{B}$. By virtue of Theorem 2.3 and because of (2.17) and (2.14) one can easily show that
\[
\| \Phi_s^p(p) - \Phi_n^s(p) \| \leq 2(r_2 \lambda_m^{-1/2})^n + 1 \| \Phi_0^s(p) \|. \tag{2.22}
\]
Hence, it is enough to estimate $\| \Phi_0^s(p) \|$. From (2.20) we have
\[
\Phi_0^s(p) = \Phi_0(p) = (vA)^{-1} [Q_m f - Q_m B(p, p)], \tag{2.23}
\]
then
\[
| A\Phi_0^s(p) | \leq v^{-1} | f | + v^{-1} | B(p, p) |;
\]
applying (1.10)
\[
| A\Phi_0^s(p) | \leq v^{-1} | f | + v^{-1} c_5 \| p \|^2 \left( 1 + \log \left( \frac{| A p |}{\| p \| \lambda_m^{1/2}} \right) \right)^{1/2}
\]
\[
| A\Phi_0^s(p) | \leq v^{-1} | f | + v^{-1} c_5 4M_1^2 L^{1/2},
\]

hence,
\[
\| \Phi_0^s(p) \| \leq \lambda_m^{-1/2} v^{-1} [ | f | + 4c_5 M_1^2 L^{1/2} ] . \tag{2.24}
\]
Combine (2.24) and (2.22) to conclude (2.21).

**Corollary 2.6.** Let $m$ be large enough such that (2.14) holds. Then for every solution $u(t) = p(t) + q(t)$ of (2.1), (2.2) we have
\[
\| q(t) - \Phi_n^s(p(t)) \| \leq \lambda_m^{-1/2} \frac{2K_0}{v} L^{1/2} + 2(r_2 \lambda_m^{-1/2})^n + 1 \lambda_m^{-1/2} v^{-1} [ | f | + 4c_5 M_1^2 L^{1/2} ] \tag{2.25}
\]
for all $t \geq T_*$ and all $n = 0, 1, 2, \ldots$, where $\Phi_n^s$ is given by (2.20), $T_*$, $L$, and $K_0$ are as in Theorem 2.1 and $r_2$ as in Theorem 2.3.

Proof. The proof is a direct consequence of Theorem 2.4 and Theorem 2.5.

Notice that the function $\Phi_0$, which was introduced in [9, 10], appears here as a first-order (iterate) approximation of $\Phi^s$. However, we need to take here the second or maybe even higher iterates in order to approximate the universal attractor as well as $\Phi^{s_1}$ does (see (2.25) and (2.18)). It is clear from (2.25) that for any improvement in the estimate (2.18) we can always choose a sufficiently large $n$ such that $\Phi_n^s$ approximates the universal attractor as well as $\Phi^s$ does.

Next we introduce appropriate finite approximations to $\Phi^s$ and $\Phi_n^s$.
which will allow us to implement this idea of approximate inertial manifolds in real computations. Notice that this step is not necessary in the case of periodic boundary condition \((1.4b)\).

**Lemma 2.7.** Let \(m\) be large enough such that \((2.14)\) holds. Then for every integer \(k \geq m + 1\) we have

\[
\|Q_k \Phi^\gamma(p)\| \leq K_1 \lambda_k^{1/2}, \quad \forall p \in \mathcal{S}, \tag{2.26}
\]

where

\[
K_1 = \frac{16c_5 M_{\lambda}}{v} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2} + |f|.
\]

**Proof.** From \((2.13)\) we have

\[
vAQ_k \Phi^\gamma(p) + Q_k B(p + \Phi^\gamma(p), p + \Phi^\gamma(p)) = Q_k f.
\]

Taking the scalar product in \(H\) with \(\Phi^\gamma(p)\) we get

\[
v \|Q_k \Phi^\gamma(p)\|^2 \leq |(B(p + \Phi^\gamma(p), p + \Phi^\gamma(p)), Q_k \Phi^\gamma(p))| + |f| \|Q_k \Phi^\gamma(p)\|
\]

\[
v \|Q_k \Phi^\gamma(p)\|^2 \leq |(B(p + P_k \Phi^\gamma(p), p + \Phi^\gamma(p)), Q_k \Phi^\gamma(p))| + |(B(Q_k \Phi^\gamma(p), p + \Phi^\gamma(p)), Q_k \Phi^\gamma(p))| + |f| \|Q_k \Phi^\gamma(p)\|;
\]

using \((1.7)\) and \((1.10)\) we obtain

\[
v \|Q_k \Phi^\gamma(p)\|^2 \leq c_5 \|p + \Phi^\gamma(p)\| \|Q_k \Phi^\gamma(p)\| \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}
\]

\[
+ c_2 \|Q_k \Phi^\gamma(p)\| \|Q_k \Phi^\gamma(p)\| \|p + \Phi^\gamma(p)\| + |f| \|Q_k \Phi^\gamma(p)\|;
\]

and from \((2.15)\) and the definition of \(\mathcal{S}\) we get

\[
v \|Q_k \Phi^\gamma(p)\| \leq c_5 8M_{\lambda}^{1/2} \lambda_k^{-1/2} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}
\]

\[
+ c_2 \sqrt{8} \lambda_k^{1/2} \|Q_k \Phi^\gamma(p)\| + \lambda_k^{1/2} |f|.
\]

Because of \((2.14)\) we conclude from the above \((2.26)\).

By virtue of the example in Remark 2.2 (iii) one can easily verify that for large \(k\)'s \((2.26)\) is sharp up to the logarithmic terms.

Let \(k \geq m + 1\), where \(m\) large enough satisfies \((2.14)\). We consider the standard Galerkin approximation of order \(k\):

\[
\frac{d}{dt} u_k + vAu_k + P_k B(u_k, u_k) = P_k f, \quad \text{where} \quad u_k \in H_k. \tag{2.27}
\]
One can easily verify, by applying the proof of Theorem 2.3 to Eq. (2.27), that there exists a unique function $\Phi^{s,k}: \mathcal{B} \to P_k Q_m V$ which satisfies

$$vA\Phi^{s,k}(p) + P_k Q_m B(p + \Phi^{s,k}(p), p + \Phi^{s,k}(p)) = P_k Q_m f, \quad \forall p \in \mathcal{B}. \tag{2.28}$$

Notice that the graph of $\Phi^{s,k}$ passes through all the stationary solutions of (2.27).

**Lemma 2.8.** Let $m$ be large enough such that (2.14) holds. Then for every integer $k \geq m + 1$, we have

$$\| \Phi^s(p) - \Phi^{s,k}(p) \| \leq K_2 \lambda_k^{-1/2}, \quad \forall p \in \mathcal{B}, \tag{2.29}$$

where

$$K_2 = \left[ 1 + \frac{(2c_5 + c_2)}{\nu} \sqrt{8 M_1 \lambda_k^{-1/2}} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2} \right] K_1$$

and $K_1$ is as in Lemma 2.7.

**Proof.** For $p \in \mathcal{B}$ we denote $u = p + \Phi^s(p)$, $v = p + \Phi^{s,k}(p)$, $\Delta = P_k(u - v)$, and $\eta = Q_k(u - v) = Q_k \Phi^s(p)$. We are interested in estimating $u - v = \Delta + \eta$.

From (2.13) and (2.28) we have

$$vA\Delta + P_k Q_m [B(u - v, u) + B(v, u - v)] = 0$$

$$vA\Delta + P_k Q_m [B(\Delta + \eta, u) + B(v, \Delta + \eta)] = 0.$$

Take the scalar product in $H$ with $\Delta$ and use (1.11) to get

$$v \| \Delta \|^2 \leq |(B(\Delta + \eta, u), \Delta)| + |(B(v, \eta), \Delta)|,$$

use (1.11)

$$\leq |(B(\Delta, u), \Delta)| + |(B(\eta, u), \Delta)| + |(B(P_k v, \Delta), \eta)| + |(B(Q_k v, \Delta), \eta)|,$$

and apply (1.7), (1.10)', (1.10), and (1.7), respectively, to obtain

$$v \| \Delta \|^2 \leq c_2 |\Delta| |\Delta| |u| + c_5 |\eta| |u| |\Delta| \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}$$

$$+ c_5 \|v\| |\Delta| |\eta| \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}$$

$$+ c_2 |Q_k v|^{1/2} |Q_k v|^{1/2} |\Delta| |\eta|^{1/2} \|\eta\|^{1/2}, \tag{2.30}$$
Notice that (2.15) implies that \( \| u \| \leq \sqrt{8} M_1 \), and similarly we can obtain \( \| v \| \leq \sqrt{8} M_1 \). Therefore, (2.30) implies

\[
v \| A \| \leq c_2 \lambda_{m+1}^{1/2} \| A \| \sqrt{8} M_1 + c_5 2 \sqrt{8} M_1 \| \eta \| \lambda_{k+1}^{1/2} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2} + c_2 \sqrt{8} M_1 \| \eta \| \lambda_{k+1}^{1/2} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}.
\]

and because of (2.14) we get

\[
\| A \| \leq \frac{2(c_5 + c_2)}{v} \sqrt{8} M_1 \| \eta \| \lambda_{k+1}^{1/2} \left( 1 + \log \frac{\lambda_k}{\lambda_1} \right)^{1/2}. \tag{2.31}
\]

Use (2.26) and (2.31) to conclude (2.29).

**Theorem 2.9.** Let \( m \) be large enough such that (2.14) holds, and let \( k \geq m + 1 \) be given. For every \( p \in B \) we define \( T_p : B \to B \) as in Theorem 2.3,

\[
T_p(q) = (v A)^{-1} \left[ Q_m f - Q_m R(p + q, p + q) \right], \quad \forall q \in B.
\]

Denote by

\[
\Phi_{n}^{0,k}(p) = P_k T_n(0), \quad \forall p \in B, \quad \text{and}
\]

\[
\Phi_{n}^{n,k}(p) = P_k T_n(\Phi_{n}^{n,k}(p)), \quad \forall p \in B \quad \text{and for } n = 0, 1, 2, \ldots . \tag{2.32}
\]

Then

\[
\| \Phi_n^{0,k}(p) - \Phi_n^{n,k}(p) \| \leq 2(r_2 \lambda_{m+1}^{1/2})^{n+1} \lambda_{m+1}^{-1/2} v^{-1} \left[ \| f \| + 4c_5 M_1^2 L^{1/2} \right]. \tag{2.33}
\]

Moreover, for every solution \( u(t) = p(t) + q(t) \) of (2.1), (2.2) we have

\[
\| q(t) - \Phi_n^{n,k}(p(t)) \| \leq \frac{2K_0'}{v} \lambda_{m+1}^{-3/2} L^{1/2} + 2(r_2 \lambda_{m+1}^{1/2})^{n+1} \lambda_{m+1}^{1/2} v^{-1}
\times \left[ \| f \| + 4c_5 M_1^2 L^{1/2} \right] + K_2 \lambda_{k+1}^{1/2}. \tag{2.34}
\]

for all \( t \geq T_* \) and every \( n = 0, 1, 2, \ldots \), where \( T_* \), \( L \) and \( K_0' \) are as in Theorem 2.1, \( r_2 \) as in Theorem 2.3, and \( K_2 \) is as in Lemma 2.8.

**Proof.** To deduce (2.33) we repeat the proof of Theorem 2.5, while replacing \( \Phi^1 \) by \( \Phi^{n,k} \). The estimate (2.34) is a direct consequence of (2.18), (2.29), and (2.33).

Here again estimate (2.18) is the leading term in all our error analysis. Namely, for any improvement in the estimate (2.18) we can choose
appropriate $n$ and $k$ such that $\Phi_n^{\times k}$ approximates the universal attractor as well as $\Phi^t$ does. However, we believe that any improvement in our approximation algorithm should involve a better approximation to the dynamics (i.e., to $dq/dt$). To match the estimate that we currently have in (2.18) it is enough to choose $n \sim \log m$ and to choose $k$, in the case of (1.4a), large enough such that $\lambda_{k+1}^{3} \sim \lambda_{m+1}^{3}$, i.e., $k \sim m^{3}$ (see (1.6)).

ACKNOWLEDGMENTS

It is my pleasure to thank Peter Constantin and Cipriean Foias for the interesting and stimulating discussions. Part of this work was done when the author enjoyed the hospitality of the Technion-I.I.T. and Tel Aviv University. This work was supported in part by DOE Grant DE-FG02-86ER25020, NSF Grant DMS 881-0684, and the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.

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