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On Symmetric and Quasi-Symmetric Designs with the Symmetric Difference Property and Their Codes

DIETER JUNGNICKEL AND VLADIMIR D. TONCHEV*

Justus-Liebig-University, 6300 Giessen, Germany

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The four symmetric 2-(64, 28, 12) designs with the symmetric difference property are characterized as the only designs with the given parameters and minimal rank over GF(2). These designs give non-isomorphic quasi-symmetric 2-(36, 16, 12) and 2-(28, 12, 11) designs as residual and derived designs. The binary codes of the quasi-symmetric 2-(28, 12, 11) designs provide four inequivalent self-orthogonal doubly-even (28, 7, 12) codes. This gives a negative answer to the question for the uniqueness of the code of the Hermitian unital of order 3. © 1992 Academic Press, Inc.

1. SYMMETRIC DESIGNS WITH THE SYMMETRIC DIFFERENCE PROPERTY

We assume that the reader is familiar with the basic notions and facts from design and coding theory. Our notation follows that from [5, 8, 11, 16, 23].

A symmetric 2-design is said to have the symmetric difference property, or to be an SDP-design, if the symmetric difference of any three blocks is either a block or a complement of a block. SDP-designs were introduced by Kantor [14] (cf. also [7, 10, 15]), who studied a class of such designs invariant under a doubly-transitive group being an extension of the symplectic group Sp($2m$, 2) with the translation group of the affine space AG($2m$, 2). The parameters of these symplectic designs are

$$2\text{-}(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}). \tag{1}$$

* This paper was written while the author was at the University of Giessen as a Research Fellow of the Alexander von Humboldt Foundation, on leave from the University of Sofia, Bulgaria. Present address: Dept. of Mathematical Sciences, Michigan Technological University, 1400 Townsend Drive, Houghton, MI 49931-1295.

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As shown in [14], any SDP-design has parameters of this type. More recently, the following characterization of SDP-designs has been obtained by Dillon and Schatz:

THEOREM 1.1 [10]. *A design with parameters* (1) *has the symmetric difference property if and only if it is isomorphic to a design formed by the minimum weight codewords in a binary code spanned by a bent function on* $2m$ *variables* (*or equivalently, by the characteristic function of an elementary Abelian difference set in* AG$(2m, 2)$) *and the Reed–Muller code* RM$(1, 2m)$.

Since there are precisely four (up to weak affine equivalence) bent functions on six variables [19], the above theorem implies that there are four non-isomorphic SDP-designs with parameters 2-(64, 28, 12).

The rank of the incidence matrix of an SDP-design with parameters (1) over GF(2) is $2m+2$ [10]. In fact, as seen from the following lemma, this is the minimal possible value for the 2-rank of a design with the given parameters. This result is stated by Assmus and Key [3] without proof.

LEMMA 1.2. *The 2-rank of the incidence matrix of a symmetric design with parameters* (1) *is greater or equal to* $2m+2$.

Proof. Let C be the code spanned by the incidence vectors of the blocks of our design D. Define E to be the subcode generated by the hyperplanes $B+B'$ (which are codewords of weight 2^{2m-1}), where B and B' run over the pairs of distict blocks of D. We shall show that E contains at least $2^{2m}+2^m$ distinct hyperplanes which shows that $\dim(E) \geqslant 2m+1$. Since C also contains the 2^{2m} distinct blocks, we obtain the assertion 2-rank$(D) \geqslant 2m+2$. Thus let x denote the number of hyperplanes H. We count pairs (p, H) with $p \in H$ in two ways. To do so, note that each point p is on at least $v-k$ hyperplanes: we may select a fixed block B_0 through p and form the $v-k$ distinct hyperplanes B_0+B, where B runs over the $v-k$ blocks not containing p. Since each hyperplane has size 2^{2m-1}, we obtain $v(v-k) \leqslant x \cdot 2^{2m-1}$, i.e.,

$$2^{2m}(2^{2m}-2^{2m-1}+2^{m-1}) \leqslant x \cdot 2^{2m-1},$$

whence

$$x \geqslant 2^{2m}+2^m. \quad \blacksquare$$

Note. This shows, in fact, the slightly stronger assertion that $\dim(E) \geqslant 2m+1$. If one has $\dim(E)=2m+1$ and if E contains the all-one vector J, then D has, in fact, the SDP. This is seen as in the paper by Dillon and Schatz [10].

It follows from Theorem 1.1 and Lemma 1.2 (cf. also [1, 3]), that the SDP-designs can be characterized as the only designs with parameters (1) and minimum 2-rank provided that the all-one vector is always in the code of a design with such parameters. The three designs for $m=2$, i.e., the biplanes 2-(16, 6, 2), do contain the all-one vector. Now we prove that the same holds also for designs with $m=3$, i.e., for 2-(64, 28, 12) designs. This answers Question 1 of [3] for $m \leqslant 3$.

LEMMA 1.3. *The code of a design with parameters* (1) *contains the all-one vector provided that* $m \leqslant 3$.

Proof. Denote by C the code spanned by the incidence matrix M of a given design with parameters (1). The all-one vector is in the code C if and only if all codewords in the dual code C' are of even weight. Assume the contrary, that is, there is a codeword of an odd weight $2s+1$ in C'. This means that there are $2s+1$ columns in M such that each row meets an even number of them in ones. Let n_{2i} be the number of rows meeting exactly $2i$ of the $2s+1$ columns in an entry 1. Then

$$\sum 2in_{2i} = (2s+1)(2^{2m-1} - 2^{m-1})$$

$$\sum 2i(2i-1)\, n_{2i} = (2s+1)\, 2s(2^{2m-2} - 2^{m-1}),$$

whence

$$\sum 2i(2i-2)\, n_{2i} = 2^{m-1}(2s+1)(2s(2^{m-1}-1) - 2^m + 1).$$

The left side of the last equation is divisible by 8. The right side is divisible by 8 only if $2^{m-1} \geqslant 8$, i.e., $m \geqslant 4$. ∎

As a corollary of the last lemma and the Dillon–Schatz theorem we obtain the following characterization of the SDP-designs on 64 points:

THEOREM 1.4. *A* 2-(64, 28, 12) *design has minimum rank* (8) *over* GF(2) *if and only if it is an* SDP-*design.*

Consequently, up to isomorphism, there are precisely four designs with minimum rank defined by the four inequivalent bent functions on six variables. Representatives of these bent functions are (cf. [19])

$$f_1 = x_1x_2 + x_3x_4 + x_5x_6,$$

$$f_3 = x_1x_2x_3 + x_1x_4 + x_2x_5 + x_3x_6,$$

$$\begin{aligned} f_3 = {} & x_1x_2x_3 + x_2x_4x_5 + x_1x_2 \\ & + x_1x_4 + x_2x_6 + x_3x_5 + x_4x_5, \end{aligned}$$

$$\begin{aligned} f_4 = {} & x_1x_2x_3 + x_2x_4x_5 + x_3x_4x_6 \\ & + x_1x_4 + x_2x_6 + x_3x_4 + x_3x_5 \\ & + x_3x_6 + x_4x_5 + x_4x_6. \end{aligned}$$

The elementary Abelian difference sets corresponding to these bent functions are

$$\begin{aligned} D(f_1) &= \{3, 7, 11, 12, 13, 14, 19, 23, 27, 28, 29, 30, 35, 39, 43, \\ &\quad 44, 45, 46, 48, 49, 50, 52, 53, 54, 56, 57, 58, 63\}, \\ D(f_2) &= \{9, 11, 13, 15, 18, 19, 22, 23, 25, 26, 29, 30, 36, 37, 38, \\ &\quad 39, 41, 43, 44, 46, 50, 51, 52, 53, 56, 59, 61, 62\}, \\ D(f_3) &= \{6, 7, 10, 11, 17, 19, 21, 23, 25, 26, 29, 30, 36, 37, 42, \\ &\quad 43, 44, 45, 46, 47, 48, 50, 53, 55, 57, 58, 60, 63\}, \\ D(f_4) &= \{5, 6, 9, 10, 12, 14, 17, 19, 26, 27, 28, 29, 36, 39, 41, \\ &\quad 42, 45, 47, 49, 51, 52, 53, 54, 55, 56, 57, 60, 61\}. \end{aligned} \tag{2}$$

Here we assume that the 64 points are labeled by the 64 binary words of length 6 (ordered lexicographically), and any number in (2) corresponds to its binary expansion.

Remark 1.5. The elementary Abelian difference sets (2) in the group of translations of AG(6, 2) also provide 2-(64, 28, 12) designs. However, only two of these difference set designs, namely those of $D(f_1)$ and $D(f_2)$, have 2-rank 8, while the designs of $D(f_3)$ and $D(f_4)$ have 2-rank 12 and 14, respectively.

2. Quasi-Symmetric Designs Derived from SDP-Designs

A 2-design is quasi-symmetric with intersection numbers x, y $(x<y)$ if any two blocks intersect in either x or y points. Obvious examples of quasi-symmetric designs are: the multiples of symmetric designs; the Steiner 2-designs which are not projective planes; the strongly resolvable designs; the residuals of biplanes. A quasi-symmetric design which does not belong to any of these four classes is called exceptional [18]. The block graph of a quasi-symmetric design, where two blocks are adjacent if they intersect in x points, is strongly regular.

In 1982 Neumaier [18] gave a list of possible parameters of small exceptional quasi-symmetric designs surviving various existence criteria for strongly regular graphs. Since then many of the entries in Neumaier's table have been updated (cf. the survey [22]). In particular, the existence or non-existence and enumeration of all non-isomorphic designs up to 31 points has been settled with only one exception; it was not known whether the quasi-symmetric 2-(28, 12, 11) design is unique or not. This design and the 2-(36, 16, 12) design are members of an infinite class of quasi-symmetric designs constructed by P. Cameron as derived and residual designs, respectively, of the symplectic designs. The essential feature ensuring the quasi-symmetry is the symmetric difference property of the symplectic designs. Namely, one has

LEMMA 2.1. *The derived* $2\text{-}(2^{2m-1}-2^{m-1}, 2^{2m-2}-2^{m-1}, 2^{2m-2}-2^{m-1}-1)$ *and residual* $2\text{-}(2^{2m-1}+2^{m-1}, 2^{2m-2}, 2^{2m-2}-2^{m-1})$ *designs of an SPD-design with parameters* (1) *are quasi-symmetric.*

Proof. Since the symmetric difference of any three blocks in the initial symmetric design is either a block or a complement of a block, the same holds for symmetric differences of pairs of blocks in a derived or residual design, whence the quasi-symmetry follows. ∎

Any of the four SDP-designs with parameters 2-(64, 28, 12) gives quasi-symmetric 2-(28, 12, 11) and 2-(36, 16, 12) designs as derived and residual, respectively. It turns out that quasi-symmetric designs obtained from non-isomorphic symmetric designs are also non-isomorphic. Some designs can be distinguished by the number of maximal cliques in their block graphs. The graphs of the 2-(28, 12, 11) designs where two blocks are adjacent if they intersect in four points, derived from the SDP-designs of the bent functions f_1, f_2, f_3, f_4, contain 135, 135, 55, and 47 cliques of size 7, respectively. Curiously enough, the graphs of the designs related to f_1 and f_2 are isomorphic. The parameters of these strongly regular graphs are $n=63$, $a=30$, $c=13$, $d=15$ (in the notation of [8 or 22]). Adding the identity matrix to the adjacency matrix of such a graph, one obtains the incidence matrix of a symmetric 2-(63, 31, 15) design. The 2-ranks of the 2-(63, 31, 15) designs thus obtained are

$$f_1 : 7; f_2 : 7; f_3 : 11; f_4 : 13.$$

It follows now by a famous result of Hamada and Ohmori [12] that the first two designs are isomorphic to the design formed by the hyperplanes in PG(5, 2). However, as we shall see in the next section, the quasi-symmetric designs related to bent functions f_1 and f_2 can be distinguished by some properties of their codes. Therefore, the following is true.

THEOREM 2.2. *There are four non-isomorphic quasi-symmetric* 2-(28, 12, 11) (*resp.* 2-(36, 16, 12)) *designs obtained as derived* (*resp. residual*) *from the four SDP-designs on* 64 *points.*

Remark 2.3. Kantor [15] has shown that the number of non-isomorphic SDP-designs grows exponentially. We conjecture that it is possible to obtain an exponentially increasing number of non-isomorphic quasi-symmetric designs as derived and residual of Kantor's SDP-designs.

3. The Codes of the Quasi-Symmetric Designs

The 2-rank of the incidence matrix of a derived design of an SDP-design with parameters (1) is $2m+1$. Moreover, one has

Lemma 3.1. *The 2-rank of a* $2\text{-}(2^{2m-1}-2^{m-1},\ 2^{2m-2}-2^{m-1},\ 2^{2m-2}-2^{m-1}-1$ *design is greater than or equal to* $2m+1$. *If the equality holds then the design is quasi-symmetric.*

Proof. Since the number of blocks through a given point $r=2^{2m-1}-2^{m-1}-1$ is odd, the all-one vector belongs to the code spanned by the incidence vectors of the blocks. Therefore, in addition to the $2^{2m}-1$ words of weight $2^{2m-2}-2^{m-1}$ corresponding to the blocks, there are $2^{2m}-1$ words of weight 2^{2m-2} corresponding to the complements of the blocks. Hence the dimension of the code is at least $2m+1$. If there are no more codewords, the symmetric difference of any pair of blocks must be either a block or a complement of a block. Hence the design is quasi-symmetric. ∎

A quasi-symmetric 2-(28, 12, 11) design which is a derived design of an SDP-design 2-(64, 28, 12) generates in this way a code of length $n=28$, dimension $k=7$, and weight distribution

$$A_0=A_{28}=1, \qquad A_{12}=A_{16}=63. \tag{3}$$

As we have seen in the previous section, the designs related to the bent functions f_3 and f_4 are pairwise non-isomorphic and also non-isomorphic to a design related to f_1 or f_2. Consequently, the four designs define at least three inequivalent codes. We will see later on that the codes of designs related to f_1 and f_2 are also inequivalent, whence the corresponding designs are non-isomorphic.

The dual of a (28, 7) code with weight distribution (3) is a (28, 21) code containing 315 codewords of weight 4. In the case of the code of the 2-(28, 12, 11) design related to f_1 these 315 words form a 2-design (in fact, a 2-(28, 4, 5)), since the initial design and consequently the code is invariant under the doubly-transitive symplectic group Sp(6, 2). This (28, 21) code can be considered as generated by the incidence matrix of the Hermitian unital of order 3 [6, 2]. Recall that the Hermitian unital of order q (q a prime power) is the $2\text{-}(q^3+1, q+1, 1)$ design of the absolute points and non-absolute lines of a unitary polarity of the projective plane $PG(2, q^2)$, and more generally, any design with such parameters is called a unital of order q (cf., e.g., [5]). Therefore, the 2-(28, 4, 5) design in the code of the Hermitian unital of order 3 is a union of a unital 2-(28, 4, 1) and a 2-(28, 4, 4) design. In fact, this 2-(28, 4, 5) design is a member of an infinite class of 2-(q^3+1, $q+1$, $q+2$) designs constructed by Hölz [13] as the union of a 2-(q^3+1, $q+1$, $q+1$) design and the Hermitian unital of order q, with the property that any two blocks meet in at most two points.

The words of weight 4 in the dual (28, 21) codes of the remaining three quasi-symmetric 2-(28, 12, 11) designs related to f_2, f_3, and f_4 also form 2-(28, 4, 5) designs, clearly again with the property that any two blocks meet in at most two points. This follows by the fact that the words of any weight in the (28, 7) codes form 2-designs; hence by Theorem 13 of [16, Chap. 6] or Theorem 1 of [20] the codewords of a given weight in the dual code also form 2-designs. We are thankful to Ed Assmus for pointing this argument to us. However, only the 2-(28, 4, 5) design in the code of the Hermitian unital contains a unital 2-(28, 4, 1) as a subdesign (as mentioned in [2], one can extract both the Hermitian and Ree unitals as subdesigns in this case).

We have checked this by computer in the following way. Suppose that a 2-(28, 4, 5) design D is given. A spread in D through a given point P is a collection of nine blocks, through P meeting pairwise only in P. Evidently the design D contains a 2-(28, 4, 1) design as a subdesign if and only if there is a collection of 28 spreads, one through each point, such that any two spreads have precisely one block in common. For the design in the code of the Hermitian unital, the last code being the dual of the code of the quasi-symmetric 2-(28, 12, 11) design related to f_1, there are 200 spreads through each point. The design related to f_2 has 40 spreads through each point; whence the quasi-symmetric 2-(28, 12, 11) designs related to f_1 and f_2 are non-isomorphic and their codes inequivalent. The points of the design related to f_3 are divided into two classes: there are points with 20 spreads through them, and there are points without any spreads through them. Hence the automorphism group of the related design and code is not transitive, and the absence of a 2-(28, 4, 1) subdesign is obvious. Finally, the design related to f_4 has 12 spreads through each point. A computer search showed that no collection of spreads can produce a 2-(28, 4, 1) design in the designs related to f_2 or f_4. Hence one has

THEOREM 3.2. *The four quasi-symmetric* 2-(28, 12, 11) *designs derived from the SDP-designs on* 64 *points generate four inequivalent* (28, 7) *codes with weight distribution* (3).

The last theorem gives a negative answer to the question for the uniqueness of a code with the given parameters. This question was asked by Brouwer [6] and Assmuss and Key [2] in connection with a possible characterization of the Hermitian unital of order 3. In our turn, we would like to ask the following

QUESTION 3.3. *Are there* (28, 7) *codes with weight distribution* (3) *other than those described in Theorem* 3.2?

Another interesting property of the Hölz 2-$(q^3+1, q+1, q+2)$ design pointed out by Thas [21] is that its derived 1-$(q^3, q, q+2)$ design is a generalized quadrangle, i.e., an 1-design such that any two blocks meet in at most one point and for each pair of non-incident point p and block B there is a unique block C through p meeting B. We have checked by computer that among the four 2-(28, 4, 5) designs only the Hölz design is an extension of a generalized quadrangle.

Assmus and Key [2] have constructed designs analogous to the Hölz designs and containing the Ree unital of order $q = 3^{2n+1}$. In the case $q = 3$ the Assmus–Key and Hölz designs coincide. An interesting question is whether the Assmus–Key designs are extensions of generalized quadrangles.

Similarly one can consider the codes generated by the quasi-symmetric designs obtained as residual of SDP-designs. The codes of the four 2-(36, 16, 12) designs related to the SDP-designs on 64 points are (36, 7) codes with weight distribution $A_0 = A_{36} = 1$, $A_{16} = A_{20} = 63$. The minimum weight codewords in the dual (36, 29) codes form 2-(36, 4, 9) designs. This can be explained again as in the case of the designs on 28 points. The symplectic 2-(36, 4, 9) design can be viewed as the design of the totally singular affine planes of a hyperbolic quadric in AG(6, 2), on which the symplectic group Sp(6, 2) is acting doubly transitively on points (cf. [9]). The inequivalence of the four codes and consequently the non-isomorphism of the 2-(36, 16, 12) designs related to f_1 and f_2, which have isomorphic block graphs, can be seen by comparing the corresponding 2-(36, 4, 9) designs. Since any two blocks in such a 2-(36, 4, 9) design meet in at most two points, given a block there are precisely 48 blocks meeting that block in two points. Consider a graph with vertices such a set of 48 blocks, two bocks being adjacent when meeting in precisely one point. Since the automorphism group Sp(6, 2) of the design related to f_1 is block transitive, the graphs with respect to any block are isomorphic; each contains exactly 2816 cliques of size 3. The graph with respect to a randomly chosen block of the design related to f_2 turned out to contain 2752 cliques of size 3. Therefore, the designs are non-isomorphic and the related codes are inequivalent.

4. TRANSFORMATIONS OF SDP-DESIGNS

In [14] Kantor gives a construction of SDP-designs by a transformation of the symplectic design with respect to a maximal totally isotropic subspace. In fact, this is a special case of a general transformation which applies to any design with parameters (1) containing subdesigns known as maximal arcs. A maximal n-arc in a 2-(v, k, μ) design is a subset S of the point set such that each block intersects S in either n or no points [17]. The size of a maximal n-arc is $|S| = r(n-1)/\mu + 1$, where r denotes the number of blocks through a point. The points of the arc together with the non-empty intersections of the arc with the blocks form a 2-$(|S|, n, \mu)$ design. Moreover, the blocks disjoint from the arc form a maximal n'-arc in the dual design where $n' = (r - \mu)/n$.

A maximal 2^{m-1}-arc in a symmetric design with parameters (1) contains 2^m points, and the blocks disjoint from such an arc form a maximal 2^{m-1}-arc in the dual design.

LEMMA 4.1. *Let D be a symmetric design with parameters* (1) *and S be a maximal 2^{m-1}-arc in D. Then interchanging incidences with non-incidences for all points from S and all blocks not disjoint from S transforms D into a design D' with the same parameters.*

Proof. The transformation is equivalent to replacement of the 2-$(2^m, 2^{m-1}, 2^{2m-2} - 2^{m-1})$ design of the arc S with its complementary design which has the same parameters. Consider a point P from the arc and a point Q not lying on the arc. Since the blocks disjoint from S form a 2^{m-1}-arc in the dual design, Q is contained in 2^{m-1} of the 2^m blocks disjoint from S. Furthermore, Q occurs together with P in $2^{2m-2} - 2^{m-1}$ blocks not disjoint from S. Any of the remaining $2^{2m-2} - 2^{m-1}$ blocks through Q does not contain P. Therefore, Q occurs together with P in precisely $2^{2m-2} - 2^{m-1}$ blocks of the new design D'.

Remark 4.2. The transformation described in the lemma is equivalent to adding (mod 2) the incidence vector of the set of blocks not disjoint from the arc S to all columns of the incidence matrix indexed by S. If the binary code spanned by the columns of the incidence matrix does not contain words of weight $2^{2m} - 2^m$ the new design is clearly non-isomorphic to the initial one. For instance, the code of an SDP-design does not contain a codeword of such weight, therefore transforming an SDP-design with respect to a maximal arc as in Lemma 4.1 produces a design which is not an SDP-design.

However, a slightly modified transformation described in the next lemma does preserve the symmetric difference property.

LEMMA 4.2. *With the assumption of Lemma* 4.1, *transform the design both with respect to the arc S and the set of blocks disjoint from S. The resulting design is an SDP-design if and only if the initial design is such a design.*

Proof. This transformation is equivalent to taking as new blocks the symmetric differences of the old blocks with S, and replacing the symmetric differences obtained from the blocks disjoint from S by their complements. ▌

A transformation described in the last lemma does not always produce non-isomorphic designs. For example, only one of the three biplanes 2-(16, 6, 2) is an SDP-design, and although this design contains maximal 2-arcs (which are in this case also ovals in the sense of [4]), any transformation obviously gives a design that is isomorphic to the initial one. However, it is possible to transform this SDP-design into any of the remaining two 2-(16, 6, 2) designs by transformations of the type of Lemma 4.1.

We have checked by computer that all four 2-(64, 28, 12) SDP-designs can be obtained starting from any of them by transformations with respect to maximal 4-arcs as in Lemma 4.2. For example, taking as $(-1, 1)$-incidence matrix of the symplectic design the tensor cube of $J-2I$, where J is the all-one and I the identity matrix of order 4, and transforming consequently with respect to rows 27, 28, 31, 32, 43, 44, 47, 48 and columns 1, 2, 5, 6, 49, 50, 53, 54 and then with respect to rows 11, 12, 19, 20, 43, 44, 51, 52 and columns 1, 2, 9, 10, 37, 38, 45, 46, one obtains a design isomorphic to that of the bent function f_3.

Note added in proof. In a sequel to the present paper (D. Jungnickel and V. D. Tonchev, Exponential number of quasi-symmetric designs with the symmetric difference property, *Designs, Codes Cryptography* **1** (1991), to appear), we show that quasi-symmetric designs which are derived or residual designs of non-isomorphic symmetric designs with the symmetric difference property are also non-isomorphic. In particular, this settles the conjecture made in Remark 2.3, since it implies that the number of quasi-symmetric designs with the parameters of Lemma 2.1 grows exponentially with m. We also give a transformation of quasi-symmetric designs by means of maximal arcs which, in particular, transforms the residual of a symmetric SDP-design into a quasi-symmetric design with the same block graph but higher rank over GF(2). As an application, we obtain at least seven non-isomorphic quasi-symmetric 2-(36, 16, 12) designs.

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