Note

A direct bijection between descending plane partitions with no special parts and permutation matrices

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ABSTRACT

We present a direct bijection between descending plane partitions with no special parts and permutation matrices. This bijection has the desirable property that the number of parts of the descending plane partition corresponds to the inversion number of the permutation. Additionally, the number of maximum parts in the descending plane partition corresponds to the position of the one in the last column of the permutation matrix. We also discuss the possible extension of this approach to finding a bijection between descending plane partitions and alternating sign matrices.

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1. Introduction

Alternating sign matrices have been objects of much inquiry over the past several decades with the most recent exciting development being the proof of the Razumov–Stroganov conjecture [4]. Descending plane partitions are objects equinumerous with alternating sign matrices, but there is no bijective proof known. In this paper, we find a direct bijection between these two sets of objects in the simplest special case.

Definition 1. A descending plane partition (DPP) is an array of positive integers \( \{a_{ij}\} \) with \( i \leq j \) (that is, with the \( i \)th row indented by \( i - 1 \) units) with weak decrease across rows, strict decrease down columns, and in which the number of parts in each row is strictly less than the largest part in that row and is greater than or equal to the largest part in the next row.

Definition 2. A descending plane partition is of order \( n \) if its largest part is at most \( n \).

Definition 3. A special part of a descending plane partition is a part \( a_{ij} \) such that \( a_{ij} \leq j - i \).

See Fig. 1 for the general form of a DPP and Fig. 2 for the seven DPPs of order 3. The only DPP in Fig. 2 with a special part is 31. The 1 is a special part since \( 1 = a_{1,2} \leq 2 - 1 \).

Though the definition of descending plane partitions seems a bit contrived, the history behind the counting formula and connections with other combinatorial objects make descending plane partitions interesting objects to study.

In 1982 Mills et al. proved that DPPs with largest part at most \( n \) are counted by the following expression [7].

\[
\prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n+j)!}.
\]
Lemma 5. There is a natural part-preserving bijection between descending plane partitions of order \( n \) with no special parts and partitions with largest part at most \( n \) and with at most \( i-1 \) parts equal to \( i \) for all \( i \leq n \).

The proof of Lemma 5 is slightly technical, but the bijection map is very simple, so we first state the map then prove that it is a well-defined bijection. To map from the DPPs to the partitions, take all the parts of the DPP and put them in one row in decreasing order. To map from the partitions to the DPPs, insert the parts of the partition into the shape of a DPP in decreasing order, putting as many parts in a row as possible and then moving on to the first position in the next row whenever the next part to be inserted would be forced to be special if added to the current row. So this bijection is simply a rearrangement of parts.
Proof of Lemma 5. We start with a DPP $\delta$ of order $n$ with no special parts. We then take all the parts of the DPP and put them in one row in decreasing order to form a partition. It is then left to show that $\delta$ has at most $i - 1$ parts equal to $i$ for all $i \leq n$. Suppose there exists an integer $i$ with $1 \leq i \leq n$ such that there are at least $i$ parts in $\delta$ equal to $i$. Since there is strict decrease down columns in $\delta$ it follows that there can be at most one $i$ in each column. Since $\delta$ has no special parts, $i$ must be greater than its column minus its row. So $i$ can appear no further right than entry $a_{i,j}$. So to have $i$ parts equal to $i$ in $\delta$, there must be an $i$ in each column from 1 to $i$. The tableau is shifted, so this means that there must be an $i$ in entry $a_{i,1}$. Thus, by the fact that the columns have to be strictly decreasing, all the $i$'s must appear in the first row. But the number of parts in each row must be strictly less than the largest part in that row, so we cannot have the first row of $\delta$ consisting of $i$ parts equal to $i$. Thus there are at most $i - 1$ parts equal to $i$.

To map in the opposite direction, we begin with a partition $\pi$ with largest part at most $n$ and with at most $i - 1$ parts equal to $i$ for all $i \leq n$. We then arrange the parts of the partition into the shape of a DPP in order, putting as many parts in a row as possible before the part would be forced to be special, that is, before the part $i$ would be in position $(k, j)$ with $i \leq j - k$. Thus $i$ will be a non-special part in the first $i$ spots in any row, that is, in positions $(k, j)$ for $k \leq j < k + i$. Suppose we have filled the DPP with the parts $n, n - 1, \ldots, i + 1$ of $\pi$ by the above process and the result is a valid DPP with no special parts. We need to show that we can insert up to $i - 1$ parts equal to $i$ and still obtain a valid DPP with no special parts.

Suppose the last part greater than $i$ was inserted in position $(k, j)$. According to our algorithm, we must insert the first $i$ in position $(k, j + 1)\leq i + 1 - k$ and in position $(k + 1, k + 1)$ otherwise. In either case, there are at least $i - 2$ more columns into which we can insert $i$ as a non-special part while not violating the column inequality condition. So the columns are strictly decreasing.

If $i$ is the smallest part in row $k$, then there will be at most $i$ parts in row $k$. This is because the $ith$ part is in position $(k, k + i - 1)$ and so any additional part after the $ith$ part would have value less than or equal to $i$ and so would be a special part since its value would be less than or equal to $(k + i) - k$. Since the rows are weakly decreasing, $i$ is less than or equal to the largest part in row $k$. So we have that the number of parts of row $k$ is less than or equal to $i$ which is less than or equal to the largest part in row $k$. If the number of parts of row $k$ equals the largest part in row $k$, it must be that the largest part equals $i$. So $i$ would be both the largest and smallest in row $k$, meaning that there would be at most $i - 1$ parts in row $k$ since there are at most $i - 1$ parts equal to $i$ in $\pi$. Thus the number of parts in each row will always be strictly less than the largest part in that row.

If $i$ is the largest part in row $k + 1$ (where $k \geq 1$), then by our algorithm, $i$ would have been a special part if placed at the end of row $k$. That is, the first blank entry of row $k$ is in column $j$ where $i \leq j - k$. So row $k$ has nonempty entries in columns $k$ through $j - 1$ for a total of $(j - 1) - k + 1 = j - k \geq i$ entries. So the number of parts in row $k$ is greater than or equal to the largest part in row $k + 1$. Therefore the number of parts in each row is greater than or equal to the largest part in the next row. We have verified all the conditions of a DPP, so the result is a valid DPP.

We have shown that both maps result in the desired kind of objects. The first map is the only way to make a partition from the parts of a DPP. We now show that the DPP resulting from the second map is the unique DPP with no special parts made up of exactly the parts of the partition. For suppose there were another DPP with no special parts whose parts made up the same partition $\pi$ with largest part at most $n$ and with at most $i - 1$ parts equal to $i$ for all $i \leq n$. Let $\delta$ be the DPP produced by our algorithm and $\delta'$ be the other DPP. Look at the first row where they differ and the first position of difference in that row; call it position $(k, j)$. Suppose $i$ is in position $(k, j)$ in $\delta$ and $i'$ is in position $(k, j)$ in $\delta'$ ($i'$ may be the empty part). Then either $i' < i$ or $i'$ is the empty part. Then there is an additional $i$ in $\delta$ which must be put somewhere in a later row. Note that since $\delta$ and $\delta'$ agree up until position $(k, j)$ it must be the case that the largest part in row $k + 1$ of $\delta'$ is less than or equal to $i$ (since all parts larger than $i$ occupy a position in a previous row or earlier in row $k$). So the additional $i$ in $\delta'$ must be in row $k + 1$. This forces the number of elements in row $k$ of $\delta'$ to be at least $i$. So the position $(k, k + i - 1)$ must not be empty in $\delta'$. In order for this entry to be a non-special part, we need its value to be greater than $(k + i) - k = i - 1$. So the last entry in row $k$ must be greater than or equal to $i$ which forces entry $(k, j)$ to be greater than or equal to $i$. This is a contradiction. Therefore $\delta$ is the unique DPP with no special parts whose parts are those of $\pi$. Therefore the two maps are inverses and we have a bijection.

The above bijection yields an interesting generating function for DPPs with no special parts.

**Corollary 6.** The generating function for DPPs of order $n$ and no special parts (with weight equal to the sum of the parts) is

$$
\prod_{i=1}^{n} [i]_q
$$

where $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$.

**Proof.** From basic partition theory, (3) is the generating function of partitions with largest part at most $n$ and at most $i - 1$ parts equal to $i$ for all $i \leq n$. Since the bijection of Lemma 5 preserves the value of the parts, this is also the generating function for DPPs of order $n$ with no special parts. \(\square\)
There is a simple bijection between descending plane partitions of order n with a total of p parts, k parts equal to n, and no special parts and n × n permutation matrices with inversion number p whose 1 in the last column is in row n − k.}

**Proof.** We first describe the bijection map. An example of this bijection is shown in Fig. 4.

Begin with a DPP δ of order n with no special parts. From Lemma 5 we know that the parts of δ form a partition with largest part at most n and at most i − 1 parts equal to i for all i ≤ n. Use these parts to make a monotone triangle of order n in the following way. The bottom row of a monotone triangle is always 1 2 3 · · · n. Let ci denote the number of parts of δ equal to i. Beginning with i = n, make a continuous path (border strip) of i's in the triangle starting at the i in the bottom row and at each step going northeast if possible or else northwest. The path continues until there are a total of ci northwest steps in the path. In this way, the path stays as far to the east as possible and has exactly ci entries equal to their southeast diagonal neighbor. Decrement i by one and repeat until reaching i = 1. Since there are at most i − 1 parts equal to i, this process is well-defined. The resulting array is a monotone triangle of order n such that there are no entries satisfying ai,j < ai−1,j < ai,j+1 (i.e., either aij = ai−1,j or ai−1,j = ai,j+1). Thus the monotone triangle corresponds to an n × n permutation matrix A, since permutation matrices are alternating sign matrices with no −1 entries.

The inverse map first takes a permutation matrix A to its monotone triangle. We claim that the parts of the corresponding DPP δ are exactly the entries of the monotone triangle which are equal to their southeast diagonal neighbor, that is, entries aij such that aij = ai+1,j+1. Because of the shape of the monotone triangle, there are at most i − 1 such entries equal to i. Thus these entries form a partition with largest entry at most n and at most i − 1 parts equal to i for all i ≤ n. Using Lemma 5 the parts of this partition can always be formed into a unique DPP.

This is a bijection because the monotone triangle entries aij such that aij = ai+1,j+1 are exactly the entries coming from the northwest steps in the border strips which are exactly the entries of δ.

We now show that this map takes a DPP with p parts to a permutation matrix with p inversions. The inversion number of any ASM A (with the matrix entry in row i and column j denoted Aij) is defined as I(A) = ∑ AijAkℓ where the sum is over all i, j, k, ℓ such that i > k and j < ℓ. This definition extends the usual notion of inversion in a permutation matrix. In [9] we
noted (using slightly different notation) that \( I(A) \) satisfies \( I(A) = E(A) + N(A) \), where \( N(A) \) is the number of \(-1\)'s in \( A \) and \( E(A) \) is the number of entries in the monotone triangle equal to their southeast diagonal neighbor (i.e. entries \( a_{ij} \) satisfying \( a_{ij} = a_{i+1,j+1} \)). Since in our case, \( N(A) = 0 \) and \( E(A) \) equals the number of parts of the corresponding DPP, we have that \( I(A) \) equals the number of parts of \( \delta \).

We can see that the parts of \( \delta \) correspond to permutation inversions directly by noting that to convert from the monotone triangle representation of a permutation to a usual permutation \( \sigma \) such that \( i \rightarrow \sigma(i) \), one simply sets \( \sigma(i) \) equal to the unique new entry in row \( i \) of the monotone triangle. Thus for each entry of the monotone triangle \( a_{ij} \) such that \( a_{ij} = a_{i+1,j+1} \), there will be an inversion in the permutation between \( a_{ij} \) and \( \sigma(i+1) \). This is because \( a_{ij} = \sigma(k) \) for some \( k \leq i \) and \( \sigma(k) = a_{ij} > \sigma(i) \). These entries \( a_{ij} \) such that \( a_{ij} = a_{i+1,j+1} \) are exactly the parts of the DPP. Thus if a DPP has \( p \) parts, its corresponding permutation will have \( p \) inversions.

Also, observe that if the number of parts equal to \( n \) in \( \delta \) is \( k \), then \( k \) determines the position of the 1 in the last column of the permutation matrix. This is because the path of \( n \)'s in the monotone triangle must consist of exactly \( k \) northwest steps (no northeast steps). So by the bijection between monotone triangles and ASMs, the 1 in the last column of \( A \) is in row \( n - k \). So finally, we have a bijection between descending plane partitions of order \( n \) with a total of \( p \) parts, \( k \) parts equal to \( n \), and no special parts and \( n \times n \) permutation matrices with inversion number \( p \) whose 1 in the last column is in row \( n - k \).

See Fig. 4 for an example of this bijection. Note that there is a direct correspondence between parts of the DPP and entries of the monotone triangle equal to their southeast neighbor.

3. Toward a bijection between all DPPs and ASMs

This perspective has some nice characteristics which may make the problem of finding a full bijection between ASMs and DPPs easier to approach. One such characteristic is the fact that this bijection uses monotone triangles rather than one-line permutations, which translate directly to ASMs. Another is that there is a one-to-one correspondence between parts of the DPP and certain entries in the monotone triangle (or inversions of the permutation). This indicates that a full bijection should proceed by finding a one-to-one correspondence between parts of the DPP and inversions of the ASM. As discussed in the proof of Theorem 8, the inversion number of an ASM satisfies \( I(A) = E(A) + N(A) \) where \( N(A) \) is the number of \(-1\)'s in \( A \) and \( E(A) \) is the number of entries in the monotone triangle equal to their southeast diagonal neighbor [9]. Thus there should be a one-to-one correspondence between these diagonal equalities of the monotone triangle and the non-special parts of the DPP. Difficulties arise quickly, though, since even for \( n = 4 \) there are examples of DPPs, such as \( 4 \quad 4 \quad 3 \quad 3 \), whose non-special parts cannot correspond exactly to diagonal equalities of the same number in the monotone triangle. In the above example, 1 is a special part and 4 4 3 3 are non-special parts. If the parts 4 4 3 3 are each entries in the monotone triangle equal to their southeast neighbors, there is no way to fill in the rest of the entries of the monotone triangle so that there is a \(-1\) in the ASM (to correspond to the special part of 1 in the DPP). Evidently, the addition of special parts to the DPP makes the relationship between non-special parts and monotone triangle diagonal equalities more complex.

Another complicating factor is that, though there is at most one way to write any collection of numbers as a DPP with no special parts (as shown in Lemma 5), the same is not true of DPPs with special parts. For example, the parts 5 5 5 3 1 can form either the DPP \( 5 \quad 5 \quad 5 \quad 3 \quad 1 \) or \( 5 \quad 5 \quad 3 \quad 5 \quad 1 \). Therefore, the position of the parts in the DPP matters when special parts are present (such as the 1 in this example).

Despite these difficulties, the simplicity of the bijection presented in this paper in the case of no special parts gives hope that a nice bijection exists between all DPPs and ASMs. The discovery of such a bijection would, in particular, allow us to find a weight on ASMs corresponding to the \( q \)-generating function (2) and illuminate not only the relationship between DPPs and ASMs, but also their relationships to other combinatorial objects.

References