Optimal buffer size and dynamic rate control for a queueing system with impatient customers in heavy traffic

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Abstract

We address a rate control problem associated with a single server Markovian queueing system with customer abandonment in heavy traffic. The controller can choose a buffer size for the queueing system and also can dynamically control the service rate (equivalently the arrival rate) depending on the current state of the system. An infinite horizon cost minimization problem is considered here. The cost function includes a penalty for each rejected customer, a control cost related to the adjustment of the service rate and a penalty for each abandoning customer. We obtain an explicit optimal strategy for the limiting diffusion control problem (the Brownian control problem or BCP) which consists of a threshold-type optimal rejection process and a feedback-type optimal drift control. This solution is then used to construct an asymptotically optimal control policy, i.e. an optimal buffer size and an optimal service rate for the queueing system in heavy traffic. The properties of generalized regulator maps and weak convergence techniques are employed to prove the asymptotic optimality of this policy. In addition, we identify the parameter regimes where the infinite buffer size is optimal.
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1. Introduction

In this article, we address a stochastic control problem associated with a single server Markovian queueing system with impatient customers under heavy traffic conditions. Control features of the system allow the system manager to dynamically control the arrival and/or the service rates depending on the current state of the system. They also allow the manager to block incoming customers by choosing an appropriate 'buffer size' of the queue (or the size of the 'waiting room' for the waiting customers). The customers may abandon the queue if their service is not completed before an exponential 'impatience clock' rings. The system manager is faced with an infinite horizon discounted cost minimization problem where three types of costs are involved: A penalty for each blocked (rejected) customer, a control cost related to the adjustment of the arrival/service rates as well as a penalty for each abandoning customer. A linear holding cost can also be included in our setup without any significant change in the analysis (see Remark 2.5). We obtain a Brownian control problem (BCP) as the heavy traffic limit of the controlled queueing system and derive an explicit optimal strategy of the BCP, which consists of an optimal feedback-type drift control and a threshold-type rejection policy. This optimal policy enables us to propose a candidate policy for the queueing system in heavy traffic. We establish the asymptotic optimality of this candidate policy using generalized regulator maps (see [24,29] and the references therein) and weak convergence methods.

The idea of using Brownian system as a heavy traffic approximation of a queueing system has a long history and we refer the reader to [31] for a comprehensive list of references. In a series of recent articles [28,27,23,24], Ward and co-authors address heavy traffic analysis of queueing networks with impatient customers. These articles address the issue of *performance evaluation* of such queueing systems. For general queueing systems (with or without customer abandonment) there are numerous articles that address the issue of *system optimization* ([5,8,9,6,14,15,18,3,26, 29] is a partial list of such articles). Ours is also a system optimization problem for a queueing system with customer abandonment. The results of this paper are close to those of [13] (in terms of methodology used for solving the BCP) and [29] (in terms of the model and the cost structure), which we compare and contrast with the results of this article below.

In [13], the authors considered a stochastic processing system with variable arrival and service rates and general customer rejection policy (variable buffer size) for a long-term average cost minimization problem. A similar control problem for fixed buffer size was addressed in [2]. For such a model, an optimal policy which consists of a feedback-type drift control and an optimal buffer size was obtained in [13], but no asymptotically optimal policy for the corresponding queueing problem was derived there. Both of these articles [2,13] did not address the issue of customer impatience. In this work, we consider a Markovian queueing model with impatient customers with similar controls and address the infinite horizon discounted cost problem. In addition to solving the BCP, we interpret the solution of the BCP to construct an admissible control policy for the queueing model and prove its asymptotic optimality. Unlike the value function of [13], the value function of the BCP in our problem depends on the initial conditions and the corresponding Hamilton–Jacobi–Bellman (HJB) equation is a truly non-linear second-order equation. In [29], an optimal admission control problem was considered for a queueing system with general arrival and service processes and impatient customers. The model in [29] does not allow for dynamically controlled arrival and service rates; it is assumed that the arrival and service rates are constants (and not controlled) and satisfy a suitable heavy traffic condition. In the current work, we introduce state dependent arrival and service rates using a convenient time-change representation for jump-Markov processes (see [10]) and allow these rates to be

controlled by the system manager. Such rate control mechanisms are analogous to 'marginally state dependent' rates (see [21]) or 'thin controls' (in [1,4]). Under appropriate scaling, this leads to a controlled drift $u(\cdot)$ in the BCP. In fact, the BCP considered in this article reduces to the BCP of [29] when the control $u(\cdot)$ is identically zero. The only control in [29] is the admission control policy which is analogous to our rejection policy (or the choice of buffer size). The threshold-type optimal admission control derived in [29] indeed provides a finite optimal buffer size for their queue. The rejection process $U(\cdot)$ in our BCP represents the cumulative number of rejected customers, and is allowed to be any adapted, nondecreasing RCLL (right continuous with left limits) process, which includes all the threshold-type rejection processes.

A novel feature of the analysis in this paper is that it addresses both issues of drift control as well as rejection control policy. In addition, we establish a necessary and sufficient condition for the finiteness of the optimal buffer size. More specifically, if $p > 0$ denotes the revenue lost per rejection, $\gamma > 0$ is the customer reneging rate, $\beta > 0$ represents the cost for each reneging customer (such as a refund given to these dissatisfied customers (as in [29])) and $\delta > 0$ is a discount factor (can be thought of as the continuously compounded bank interest rate), then let $p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$. For each $p > 0$, we derive an optimal feedback-type drift control u_p^* in Theorem 3.8. We show that when $0 < p < p_0$, there is an optimal rejection policy associated with a finite buffer size b_p^*. We also prove that when $p \geq p_0$, the optimal rejection process is identically zero (i.e. not rejecting any customer is optimal). Note that when $0 < p < p_0$, [29] also obtained a finite optimal buffer size and it conjectures that the condition $0 < p < p_0$ is necessary for having a finite buffer size. Here we establish this claim even in the presence of a drift control. Our analysis shows that the value of the threshold p_0 is independent of the control cost $C(\cdot)$ (see (3.14)). In the light of the results in [29], our work concludes that the introduction of a drift control does not affect the threshold value p_0. However, when $0 < p < p_0$, the value of the finite optimal buffer-length b_p^* is different from that of [29] due to the effect of optimal control $u^*(\cdot)$.

The paper is organized as follows: Section 2 has the problem description including the details of the queueing system, the cost structure for the control problem as well as the main result of the article. In Section 3 we discuss the approximating BCP and obtain its explicit solution. The BCP addressed here is a singular stochastic control problem and it can be read independent of the other sections. Section 4 begins with a short discussion of generalized regulator maps, which will be used later in the proofs of the theorems that follow. The rest of this section is devoted to proving the main theorem. Throughout this article, all the processes are assumed to be in the space $\mathcal{D}([0, \infty), \mathbb{R}^k)$ ($\equiv$ the space of right continuous functions with left limits) for some $k \geq 1$ and we use "$\Rightarrow$" to represent the weak convergence of the processes in the usual Skorokhod topology.

2. Problem description and the main result

2.1. Model formulation

We consider a sequence of queueing systems in heavy traffic indexed by $n \geq 1$. Each system is equipped with adjustable arrival and/or service rates and possibly a finite buffer size. The job of a "controller" is to choose these state dependent rates as well as the buffer size so that an infinite horizon discounted cost structure (see (2.14)) is minimized. In addition, customers waiting in the queue may abandon the system and this cannot be controlled. Thus, the control structure here is represented by $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \equiv (\underset{\sim}{\lambda} = \{\lambda_n(\cdot)\}, \underset{\sim}{\mu} = \{\mu_n(\cdot)\}, b_n)$, where λ_n, μ_n are functions of the current queue-length representing the state dependent arrival and service rates satisfying some
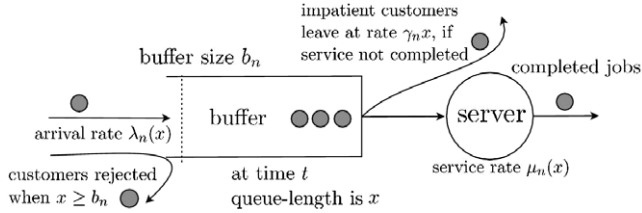
Fig. 1. Dynamics of the nth queueing system.

admissibility conditions (see Definition 2.2). The buffer size b_n of the nth system is chosen so that $b_n \equiv \sqrt{n}b$ ($b = \infty$ is allowed) for some $b > 0$. If b_n is not an integer, then $\lfloor b_n \rfloor =$ *the integer part of* b_n is the "effective" buffer size: the customers are allowed to join the queue as long as the current queue-length is less than (or equal to) b_n, and are rejected if the queue-length is greater than b_n.

We assume that all the processes defined for the queueing system are defined on some common probability space. For $n \geq 1$, the dynamics of the nth system under a control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ is described below. We assume that initially the queue is empty. The arrival time for the first customer is exponentially distributed with rate $\lambda_n(0)$ and the server immediately starts serving this customer. At this instant, the queue-length is 1 and the required service time to complete service to the first customer and the time until the second customer arrives is assumed to be independent and exponentially distributed with rates $\mu_n(1)$ and $\lambda_n(1)$, respectively. In addition, this customer can abandon the queue if the service is not completed within a random amount of time (patience time), which is assumed to be exponentially distributed with rate γ_n. We call a time instant an "event-time" if at that instant, either a new customer arrives or an existing customer leaves because of service completion or abandonment. At any "event-time", if the current queue-length is k, where $k \geq 0$, we assume the following memoryless structure: (remaining) inter-arrival time for the next customer, (remaining) service time for the current customer being served and (remaining) patience time for *each* of the existing customers in the queue are independent and distributed as exponential random variables with rates $\lambda_n(k)$, $\mu_n(k)I_{\{k>0\}}$ and γ_n, respectively. In addition, if the buffer size $b_n = \sqrt{n}b$ is finite, then every incoming customer is rejected if the buffer is full and no customer is rejected if $b_n = \infty$ is chosen. One can also think of the value b as an admission control threshold where the customers are allowed to join the queue only if the queue-length is less than b (see [29]). We assume that the server does not idle unless the buffer is empty (queue-length is zero). The sequence in which available jobs in the queue are served is irrelevant because of our Markovian structure. Fig. 1 describes the dynamics of the nth queueing system ($n \geq 1$) at any time point $t \geq 0$.

A more rigorous description of our model is as follows: Let $Q_n(t)$ denote the queue-length process at time t, $t \geq 0, n \geq 1$. We assume that $Q_n(0) = 0$ and $\{Q_n(t) : t \geq 0\}$ is a jump-Markov process with state space $\mathbb{Z}^+$ ($=$ *set of all non-negative integers*) and jump intensities are given by

$$q_{k,k+1}^n = \lambda_n(k)I_{\{k<\sqrt{n}b\}}, \qquad q_{k,k-1}^n = \mu_n(k)I_{\{k>0\}} + k\gamma_n, \quad k \in \mathbb{Z}^+,$$

and $q_{k',\ell'}^n = 0$ for all other values of $k', \ell' \in \mathbb{Z}^+$. It is well known (see Chapter 6 of [20]) that such a process can be represented as a linear combination of time-changed independent Poisson processes:

$$Q_n(t) = Y_n^A \left(\int_0^t \bar{\lambda}_n(Q_n(s)) \mathrm{d}s \right) - Y_n^S \left(\int_0^t \bar{\mu}_n(Q_n(s)) \mathrm{d}s \right)$$

$$- Y_n^R \left(\int_0^t \gamma_n Q_n(s) \mathrm{d}s \right), \quad t \geq 1, \tag{2.1}$$

where $\bar{\lambda}_n(k) \doteq \lambda_n(k) I_{\{k < \sqrt{n}b\}}$, $\bar{\mu}_n(k) \doteq \mu_n(k) I_{\{k>0\}}$ are the "effective" rates, and Y_n^A, Y_n^S, Y_n^R are independent Poisson processes with intensities 1. We will use (2.1) as the definition of the queue-length process in our model (see [32] for similar queueing models with state dependent rates).

2.2. Heavy traffic and admissible controls

First we state our assumption on the reneging rates. A similar assumption was used in [29].

Assumption 2.1. There exists $\gamma > 0$ such that

$$n\gamma_n \to \gamma > 0 \quad \text{as } n \to \infty.$$

We assume that the system operates under heavy traffic (i.e. the long-run average arrival and service rates are equal), under any *admissible* control policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ that the controller chooses.

Definition 2.2 (*Admissible Controls*). A control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \equiv (\{\lambda_n(\cdot)\}, \{\mu_n(\cdot)\}, b)$ is called admissible for the queueing system if $\lambda_n(\cdot)$, $\mu_n(\cdot)$ are non-negative, continuous functions defined on $[0, \infty)$ and $b \in (0, \infty]$ such that for some $\lambda > 0$ and $\mu > 0$ the following holds:

(i)

$$\sup_{x \geq 0} |\lambda_n(x) - \lambda| \to 0, \qquad \sup_{x \geq 0} |\mu_n(x) - \mu| \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

(ii) For $n \geq 1$, define

$$u_n(x) \doteq \sqrt{n}(\mu_n(\sqrt{n}x) - \lambda_n(\sqrt{n}x)), \quad \text{for each } x \geq 0, \tag{2.3}$$

then $\{u_n(\cdot)\}$ is a sequence of uniformly Lipschitz continuous functions (with a Lipschitz constant κ_u) and for some bounded function $u(\cdot)$,

$$\sup_{x \geq 0} |u_n(x) - u(x)| \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Note that the assumption in (2.2) clearly implies that for some positive constants c and c', $\sup_{x \geq 0}[\lambda_n(x) \vee \mu_n(x)] \leq c$ and $\inf_{x \geq 0}[\lambda_n(x) \wedge \mu_n(x)] > c'$, for $n \geq n_0$ (for a suitable $n_0 \geq 1$). However, the lower bound c' on the rates given above and their continuity guarantee that the representation of queue-length in (2.1) is possible (see [20]), and hence we will simply take $n_0 = 1$ without loss of generality and thus the following holds:
For some positive constants c and c'

$$\sup_{n \geq 1} \sup_{x \geq 0}[\lambda_n(x) \vee \mu_n(x)] \leq c, \qquad \inf_{n \geq 1} \inf_{x \geq 0}[\lambda_n(x) \wedge \mu_n(x)] > c' > 0. \tag{2.5}$$

From Assumption 2.1 it follows that $\gamma_n \to 0$ as $n \to \infty$ and hence the customer abandonment rates do not influence the long-run average departure rate. Parts (ii) and (iii) of Definition 2.2

imply that the system is in "heavy traffic", i.e.

$$\lambda = \mu. \tag{2.6}$$

As is often the case in heavy traffic analysis of queueing systems, (because of an underlying functional central limit theorem) the diffusion scaled queue-length

$$\hat{Q}_n(\cdot) = \frac{Q_n(n\cdot)}{\sqrt{n}}, \quad n \geq 1, \ t \geq 0, \tag{2.7}$$

stabilizes. This is the reason for studying the asymptotic behavior of the system and the associated cost criterion (see (2.14) below) under the diffusion scaling.

Remark 2.3. (a) For a concrete example of rates satisfying all our admissibility conditions in Definition 2.2, consider the following class of rates:

$$\lambda_n(x) = \lambda + \frac{1}{\sqrt{n}} u_1 \left(\frac{x}{\sqrt{n}} \right) + \frac{1}{n} v_1^n \left(\frac{x}{\sqrt{n}} \right),$$

$$\mu_n(x) = \lambda + \frac{1}{\sqrt{n}} u_2 \left(\frac{x}{\sqrt{n}} \right) + \frac{1}{n} v_2^n \left(\frac{x}{\sqrt{n}} \right), \quad x \geq 0, \ n \geq 1,$$

where $\lambda > 0$, $u_1(\cdot)$, $u_2(\cdot)$ are any two Lipschitz continuous functions with Lipschitz constants $\kappa_1 > 0$, and $\kappa_2 > 0$, respectively. Furthermore, $\sup_{x \geq 0} v_i^n(x) = o(\sqrt{n})$ for $i = 1, 2$.

(b) Note that in our setup, any admissible policy will affect the system behavior (in diffusion scale) marginally, via $u_n(\cdot)$. We call this the "marginal drift function" and its limiting version $u(\cdot)$ as the "asymptotic marginal drift function" for a given (λ, μ, b). From the properties of the marginal drift functions in (2.3)–(2.4) in Definition 2.2, we conclude that $u(\cdot)$ is also a Lipschitz continuous function with the same Lipschitz constant κ_u.

2.3. Scaled processes

First we define the lower- and upper-"reflection" processes: For $n \geq 1$

$$L_n(t) \dot{=} \mu_n(0) \int_0^t I_{\{Q_n(s)=0\}} ds, \qquad U_n(t) \dot{=} \lambda_n(\sqrt{n}b) \int_0^t I_{\{Q_n(s) \geq \lfloor \sqrt{n}b \rfloor\}} ds, \quad t \geq 0. \tag{2.8}$$

This combined with (2.1), yields that

$$
\begin{aligned}
Q_n(t) = & \left[Y_n^A \left(\int_0^t \bar{\lambda}_n(Q_n(s)) ds \right) - \int_0^t \bar{\lambda}_n(Q_n(s)) ds \right] \\
& - \left[Y_n^S \left(\int_0^t \bar{\mu}_n(Q_n(s)) ds \right) - \int_0^t \bar{\mu}_n(Q_n(s)) ds \right] \\
& - \left[Y_n^R \left(\int_0^t \gamma_n Q_n(s) ds \right) - \int_0^t \gamma_n Q_n(s) ds \right] \\
& - \int_0^t [\mu_n(Q_n(s)) - \lambda_n(Q_n(s)) + \gamma_n Q_n(s)] ds + L_n(t) - U_n(t),
\end{aligned} \tag{2.9}
$$

for all $n \geq 1$ and $t \geq 0$. Next, we define the following diffusion scaled Poisson processes:

$$\hat{Y}_n^A(t) = \frac{1}{\sqrt{n}} (Y_n^A(nt) - nt), \qquad \hat{Y}_n^S(t) = \frac{1}{\sqrt{n}} (Y_n^S(nt) - nt),$$

$$\hat{Y}_n^R(t) = \frac{1}{\sqrt{n}}(Y_n^R(nt) - nt), \quad t \geq 0. \tag{2.10}$$

and the diffusion scaled versions of the reflection processes in (2.8) are given by:

$$\hat{L}_n(t) \doteq \frac{1}{\sqrt{n}} L_n(nt) = \sqrt{n}\mu_n(0) \int_0^t I_{\{\hat{Q}_n(s)=0\}} \mathrm{d}s, \quad t \geq 0,$$

$$\hat{U}_n(t) \doteq \frac{1}{\sqrt{n}} U_n(nt) = \sqrt{n}\lambda_n(\sqrt{n}b) \int_0^t I_{\{\hat{Q}_n(s)\geq b_n\}} \mathrm{d}s, \quad t \geq 0, \tag{2.11}$$

where $b_n = \frac{\lfloor \sqrt{n}b \rfloor}{\sqrt{n}}$. Using (2.7) and the definitions in (2.3) and (2.9)–(2.11), one can easily verify that the following identity holds: For each $t \geq 0$,

$$\hat{Q}_n(t) = \hat{W}_n(t) - \int_0^t [u_n(\hat{Q}_n(s)) + n\gamma_n \hat{Q}_n(s)] \mathrm{d}s + \hat{L}_n(t) - \hat{U}_n(t), \tag{2.12}$$

where,

$$\hat{W}_n(t) \doteq \hat{Y}_n^A \left(\int_0^t \bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s)) \mathrm{d}s \right) - \hat{Y}_n^S \left(\int_0^t \bar{\mu}_n(\sqrt{n}\hat{Q}_n(s)) \mathrm{d}s \right)$$

$$- \hat{Y}_n^R \left(\int_0^t \gamma_n \sqrt{n}\hat{Q}_n(s) \mathrm{d}s \right). \tag{2.13}$$

2.4. The cost structure and the main result

Note that in diffusion scaling, when the diffusion scaled queue-length is $\hat{Q}_n(t)$ at any time $t \geq 0$, the customers abandon the queue at the collective rate of $n\gamma_n\hat{Q}_n(t)$. As mentioned in Remark 2.3 (and the discussion before that), our objective here is to study the asymptotic performance of the system under diffusion scaling. We assume that the cost of each abandoning customer is a constant $\beta > 0$, cost of controlling marginal rates is given by "a control cost function" $C(\cdot)$, and the income lost due to each rejected customer is a constant amount $p > 0$. We impose the following assumption on the control cost function $C(\cdot)$:

Assumption 2.4 (*Control Cost*). $C(\cdot)$ is a non-negative, twice continuously differentiable function defined on $(-\infty, \infty)$ which satisfies $C(x) = 0$ for $x \leq 0$, $C'(0) = 0$ and $C''(x) > 0$ for all $x \geq 0$.

We consider an infinite horizon, discounted cost criterion, i.e. for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$, we define the associated asymptotic cost by

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \doteq \liminf_{n \to \infty} E \int_0^\infty e^{-\delta t} \left\{ \left[\beta(n\gamma_n)\hat{Q}_n(t) + C(u_n(\hat{Q}_n(t))) \right] \mathrm{d}t + p\mathrm{d}\hat{U}_n(t) \right\}, \tag{2.14}$$

where $\delta > 0$ is a constant discount factor. The control problem here is to find an asymptotically optimal policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ which minimizes the cost defined in (2.14) among all the admissible policies. In other words, the problem is to find $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ such that

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \inf J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b),$$

where the infimum is taken over all admissible policies $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ as in Definition 2.2.

Remark 2.5. (a) Notice that for the *n*th system, if we control only the service rate, then the control cost is a nondecreasing function of the service rate.

(b) [Linear holding costs] We can include a linear holding cost in our analysis and obtain the corresponding optimal strategy. In particular, if $\kappa \geq 0$ represents the rate of holding cost per customer in the system, then using the structure of the cost functional in (2.14) (also see (3.4)), we can simply change the parameter β to $\left(\beta + \frac{\kappa}{\gamma}\right)$ and the value of the threshold p_0 to $\frac{\beta\gamma+\kappa}{\delta+\gamma}$ and our analysis and the conclusions will remain valid.

Theorem 2.6. *There exists a real number b_p^* (b_p^* is considered as $+\infty$ in the case of $p \geq p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$) and a C^2-function $\mathcal{V}_p$ which satisfies*

$$\frac{\sigma^2}{2}\mathcal{V}_p''(x) - \Phi(\mathcal{V}_p'(x)) - \gamma x \mathcal{V}_p'(x) - \delta \mathcal{V}_p(x) + \beta\gamma x = 0 \quad \text{for } 0 \leq x \leq b_p^*, \tag{2.15}$$

$$\mathcal{V}_p'(0) = 0 \quad \text{and} \quad \mathcal{V}_p'(x) = p, \quad \text{for } x \geq b_p^*, \tag{2.16}$$

with $\sigma^2 = 2\lambda > 0$. Moreover, the pair $(b_p^*, \mathcal{V}_p(\cdot))$ is unique.

We provide the proof of the above theorem in Section 3.

Definition 2.7 (*A Candidate for Optimality*)**.** Let $p > 0$, $\mathcal{V}_p$ and b_p^* be as in (2.15) and (2.16) in Theorem 2.6. Define $u_p^*(\cdot) = (C')^{-1}(\mathcal{V}_p^*(\cdot))$. Choose any two functions $\theta_1^*(\cdot)$, and $\theta_2^*(\cdot)$ defined on $[0, \infty)$, such that

$$0 \leq \theta_2^*(x) - \theta_1^*(x) = u_p^*(x), \quad \text{for all } x \geq 0.$$

Define

$$\lambda_n^*(x) \doteq \lambda + \frac{1}{\sqrt{n}}\theta_1^*\left(\frac{1}{\sqrt{n}}x\right), \quad \text{and} \quad \mu_n^*(x) = \mu + \frac{1}{\sqrt{n}}\theta_2^*\left(\frac{1}{\sqrt{n}}x\right). \tag{2.17}$$

Then, $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*) \equiv (\{\lambda_n^*(\cdot)\}_{n\geq 1}, \{\mu_n^*(\cdot)\}_{n\geq 1}, b_p^*)$ is a candidate for an optimal policy. The admissibility and asymptotic optimality of this policy will be shown in the proof of Theorem 2.8.

Theorem 2.8 (*Main Result*)**.** *Our proposed policy $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*)$ in Definition 2.7 is asymptotically optimal, i.e.*

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*) \leq J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$$

for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$.

The proof of this theorem will be given in Section 4.2.

We have used lim inf in our definition of the asymptotic cost function in (2.14). Alternatively, one could define the asymptotic cost using lim sup as follows:

$$I_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \doteq \limsup_{n\to\infty} E\int_0^\infty e^{-\delta t}\left\{\left[\beta(n\gamma_n)\hat{Q}_n(t) + C(u_n(\hat{Q}_n(t)))\right]dt + pd\hat{U}_n(t)\right\}. \tag{2.18}$$

In the proof of Theorem 2.8, it turns out that for the proposed optimal policy in Definition 2.7, the limit (as $n \to \infty$) is actually achieved. Hence, using the simple fact that $\liminf a_n \leq \limsup a_n$, we also obtain the following corollary and its proof is given at the end of Section 4.

Corollary 2.9. *The proposed policy* $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*)$, *given in* Definition 2.7 *is asymptotically optimal also for the cost criterion defined in* (2.18), *i.e.*

$$I_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b_p^*) \leq I_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$$

for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$.

Remark 2.10. (a) We suppress the parameter $p > 0$ in $\underset{\sim}{\lambda}^*$ and $\underset{\sim}{\mu}^*$ for simplicity of the notation.

(b) Also notice that, in the above proposed optimal policy, our arrival and service rates in (2.17) are not unique, even if the optimal asymptotic drift function $u^*(\cdot)$ is unique. This general setup covers more realistic special cases. For example, if the $\lambda > 0$ is a given constant and if $\lambda_n(x) \equiv \lambda$ for all x and the control problem is to choose an optimal state dependent service rate $\mu_n(\cdot)$, then $\theta_1^* \equiv 0$ and $\theta_2^* \equiv u^*$, $\mu_n^*(\cdot)$ will be an optimal solution. Similarly, if $\mu > 0$ is given and $\mu_n(x) \equiv \mu$, then choosing $\theta_1^* = -u^*$, one can obtain an optimal state dependent arrival rate λ_n^* for this problem.

3. Brownian control problem

In this section, we describe a diffusion model that approximates the behavior of the queueing model under diffusion scaling. The associated diffusion control problem is usually referred to as the Brownian control problem (BCP). From the functional central limit theorem for the Poisson processes (with unit intensity), it follows that

$$(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R) \Rightarrow (W^A, W^S, W^R) \quad \text{as } n \to \infty,$$

where W^A, W^S, W^R are independent standard Brownian motions defined on some filtered probability space (see (4.35)). Intuitively, this suggests that from (2.13) and the definition of an admissible control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ (Definition 2.2) that

$$\hat{W}_n \Rightarrow \sigma W$$

where W is a standard Brownian motion with zero drift and infinitesimal variance 1 and the constant $\sigma > 0$ is given by $\sigma^2 = 2\lambda$. In Proposition 4.4, we will verify this assertion. Also, from the definition of $\hat{L}_n, \hat{U}_n$ in (2.11), it is clear that these processes start from the origin, they are nondecreasing and increase only when $\hat{Q}_n = 0$ or b, respectively. Thus, if $u(\cdot)$ is the associated asymptotic marginal drift function of $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$, one expects that the limit of diffusion scaled queues for each admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ will satisfy:

$$X(t) = \sigma W(t) - \int_0^t [u(X(s)) + \gamma X(s)] \mathrm{d}s + L(t) - U(t), \quad t \geq 0,$$

where (X, L, U) is a weak limit of $(\hat{Q}_n, \hat{L}_n, \hat{U}_n)$. As is the case in many queueing system control problems, studying the diffusion control problem with a cost structure similar to that in the queueing control problem often provides insights for the search of an asymptotically optimal control policy for the queueing control problem. Throughout this section, the positive constants δ, β, γ, p and the function $C(\cdot)$ are as in Section 2.

We consider a state process $X_x(\cdot)$ which is a weak solution to

$$X_x(t) = x - \int_0^t u(s) \mathrm{d}s - \gamma \int_0^t X_x(s) \mathrm{d}s + \sigma W(t) + L(t) - U(t), \quad t \geq 0, \quad (3.1)$$

where $x \geq 0$, $\{W(t) : t \geq 0\}$ is a one-dimensional Brownian motion, with no drift and variance 1 (and $\sigma^2 = 2\lambda$), adapted to a right continuous Brownian filtration $\{\mathcal{F}_t : t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, P)$. The σ-algebra $\mathcal{F}_0$ is assumed to contain all the null sets in $\mathcal{F}$. The processes $u(\cdot)$ and $U(\cdot)$ are the control processes and they satisfy the following conditions.

The drift control process $\{u(t) : t \geq 0\}$ is real-valued progressively measurable with respect to $\{\mathcal{F}_t\}$. To ensure that Eq. (3.1) makes sense, we will also assume that

$$E \int_0^T |u(s)| \mathrm{d}s < +\infty, \quad \text{for all } T > 0. \tag{3.2}$$

The singular control process $U(\cdot)$ is adapted to $\{\mathcal{F}_t\}$, nondecreasing, right continuous with left limits and $U(0) = 0$. These processes also satisfy the property that the associated state process $X_x(\cdot)$ in (3.1) always remain non-negative.

The other nondecreasing process $L(\cdot)$ represents the local-time process of $X_x(\cdot)$ at the origin. Therefore

$$\int_0^T I_{\{X_x(s)>0\}} \mathrm{d}L(s) = 0, \quad \text{for all } T > 0. \tag{3.3}$$

Definition 3.1 (*Brownian Control Problem (BCP)*). For any given $x \geq 0$, any non-negative solution $X_x(\cdot)$ to (3.1) together with the associated controls $u(\cdot)$ and $U(\cdot)$, which satisfy the above assumptions yield an admissible control system. More precisely, $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}, X_x(\cdot), u(\cdot), U(\cdot))$ is called an admissible control system. With a slight abuse of notation, we simply write (X_x, u, U) for an admissible control policy. For such an admissible control policy (X_x, u, U), we define the cost criterion

$$\widetilde{J}_p(x, u, U) \doteq E \int_0^\infty \mathrm{e}^{-\delta t} [(\beta\gamma X_x(t) + C(u(t)))\mathrm{d}t + p\mathrm{d}U(t)]. \tag{3.4}$$

Let

$$\mathcal{A} = \{(X_x, u, U) : (X_x, u, U) \text{ is an admissible control policy }\}.$$

The value function of the control problem is defined by

$$\mathrm{V}_p(x) = \inf_{\mathcal{A}} \widetilde{J}_p(x, u, U). \tag{3.5}$$

Note that the value function also depends on the other parameters of the system such as δ, β, γ, etc., but we suppress this dependence in our notation for the clarity of the presentation. Our analysis shows that the value function V_p is the unique solution to (2.15)–(2.16) of Theorem 2.6 and our optimal feedback control satisfies

$$u_p^*(\cdot) = (C')^{-1}(\mathrm{V}_p(\cdot)).$$

We next introduce the $\mathcal{A}^+$ defined as

$$\mathcal{A}^+ = \{(X_x, u, U) : (X_x, u, U) \text{ is an admissible control policy,}$$
$$\text{and } u(t) \geq 0 \text{ for all } t \geq 0\}. \tag{3.6}$$

Our next proposition shows that it is enough to consider only non-negative drift control processes $u(\cdot)$ for the BCP.

Proposition 3.2. *Let* V_p *be the value function given in* (3.5). *Then for all* $x \geq 0$,

$$V_p(x) = \inf_{\mathcal{A}^+} \tilde{J}_p(x, u, U).$$

Proof. Let (X_x, u, U) be any admissible policy as defined in Definition 3.1. Introduce $u^+(t) = \max\{0, u(t)\}$ for all $t \geq 0$. Our aim here is to construct an admissible policy $(\tilde{X}_x, u^+(\cdot), U)$ so that $\tilde{J}_p(x, u, U) \geq \tilde{J}_p(x, u^+, U)$, which proves the claim.

We begin with introducing the conventional one-sided regulator map (ϕ, ψ) defined on $\mathcal{C}[0, \infty)$ (see [25,17]): For each continuous function f defined on $[0, \infty)$, let $\psi(f)(t) \doteq -\inf_{0 \leq s \leq t}(f(s) \wedge 0)$ and $\phi(f)(t) \doteq f(t) + \psi(f)(t)$ for all $t \geq 0$. We also use the following well-known property of the regulator maps (see [17]): if f, g are continuous functions with $f(0) \geq g(0) \geq 0$ then

$$(f - g) \text{ is a nondecreasing function} \implies \phi(f)(t) \geq \phi(g)(t) \quad \text{for all } t \geq 0. \tag{3.7}$$

Note that $X_x(\cdot)$ in (X_x, u, U) satisfies (3.1). Hence, using Itô's Lemma and properties of the regulator map, we obtain

$$e^{\gamma t} X_x(t) = \phi\left(x - \int_0^{\cdot} e^{\gamma s} u(s) \mathrm{d}s + \sigma \int_0^{\cdot} e^{\gamma s} \mathrm{d}W(s) - \int_0^{\cdot} e^{\gamma s} \mathrm{d}U(s)\right)(t), \quad t \geq 0. \tag{3.8}$$

Next, we define

$$\tilde{X}_x(t) = e^{-\gamma t} Z(t), \quad \text{where } Z(t) = \phi\left(x - \int_0^{\cdot} e^{\gamma s} u^+(s) \mathrm{d}s\right.$$

$$\left. + \sigma \int_0^{\cdot} e^{\gamma s} \mathrm{d}W(s) - \int_0^{\cdot} e^{\gamma s} \mathrm{d}U(s)\right)(t), \ t \geq 0. \tag{3.9}$$

Then for all $t \geq 0$, $\tilde{X}_x(t) \geq 0$ and

$$\tilde{X}_x(t) e^{\gamma t} = x - \int_0^t e^{\gamma s} u^+(s) \mathrm{d}s + \sigma \int_0^t e^{\gamma s} \mathrm{d}W(s) - \int_0^t e^{\gamma s} \mathrm{d}U(s) + L_1(t), \tag{3.10}$$

where $L_1(0) = 0$ and L_1 is a nondecreasing process which satisfies

$$\int_0^t I_{\{\tilde{X}_x(s)>0\}} \mathrm{d}L_1(s) = 0 \quad \text{a.s.} \tag{3.11}$$

Introduce $\tilde{L}(t) = \int_0^t e^{-\gamma s} \mathrm{d}L_1(s)$ for all $t \geq 0$. Using Itô's Lemma again, we get for all $t \geq 0$,

$$\tilde{X}_x(t) = x - \int_0^t u^+(s) \mathrm{d}s - \int_0^t \gamma \tilde{X}_x(s) \mathrm{d}s + \sigma W(t) - U(t) + \tilde{L}(t), \tag{3.12}$$

and using (3.11),

$$\int_0^t I_{\{\tilde{X}_x(s)>0\}} \mathrm{d}\tilde{L}(s) = 0 \quad \text{a.s.} \tag{3.13}$$

Hence, $\tilde{X}_x(\cdot)$ satisfies (3.1) with controls (u^+, U) and $(\tilde{X}_x, u^+, U)$ is an admissible control policy in $\mathcal{A}^+$. Since $\int_0^{\cdot} e^{\gamma s}(u^+(s) - u(s)) \mathrm{d}s$ is a non-negative nondecreasing function, using (3.7) and (3.8)–(3.9), we obtain $e^{\gamma t} X_x(t) \geq e^{\gamma t} \tilde{X}_x(t)$, and hence $X_x(t) \geq \tilde{X}_x(t)$ for all $t \geq 0$. Also, it is evident that $C(u(t)) \geq C(u^+(t))$ for all $t \geq 0$. Hence, by the definition of the cost function in (3.4), we have $\tilde{J}_p(x, u, U) \geq \tilde{J}_p(x, u^+, U)$. This completes the proof. $\quad \square$

The above proposition implies that it suffices to minimize $\widetilde{J}_p(x, u, U)$ over the control policies involving only non-negative $u(\cdot)$ (i.e. the control policies in $\mathcal{A}^+$). Hence, for the rest of the paper, we will assume that $u(\cdot)$ is a non-negative function. In the next assumption, we introduce a critical value p_0 associated with the cost parameter p. We also describe a "control space" $\mathcal{D}$, where the control processes $u(\cdot)$ take values. This control space $\mathcal{D}$ is a priori given to the controller and is typically assumed to be a compact subset of $[0, \infty)$ (See Chapter III.6 and Chapter IV of [11], in particular, Assumption 4.1 and (6.1) of Chapter IV).

Assumption 3.3 (*Control Space*). Let

$$\mathcal{D} = \{z \geq 0 : \exists (X_x, u, U) \in \mathcal{A}^+ \text{ such that } u(t) = z \text{ for some } t \geq 0\}$$

and

$$p_0 = \frac{\beta\gamma}{(\delta + \gamma)}. \tag{3.14}$$

We assume that there exists a positive real number θ_0 such that $[0, \theta_0] \subseteq \mathcal{D}$ and it satisfies

$$C'(\theta_0) = p_0. \tag{3.15}$$

Remark 3.4. Since $C''(x) > 0$ for all $x \geq 0$, the above θ_0 which satisfies (3.15) is unique. In particular, from Assumption 2.4 it follows that for each $0 < p < p_0$, there exists a unique $\theta_p \in \mathcal{D}$ such that $C'(\theta_p) = p$.

Now we state the formal connections of the processes in the BCP above to the processes introduced in Section 2: The process $X_x(t)$ represents the diffusion limit of the queue-length process at time t, such that at time $t = 0$, the (diffusion scaled) queue-length is equal to $x \geq 0$. The controller can choose the state dependent drift rate function $u(\cdot)$ from the control space $\mathcal{D}$. The drift rate is analogous to the scaled difference between the service and the arrival rates in the queueing system (see (2.3) and (2.4) in Definition 2.2). We do not restrict to feedback-type drift control in the BCP, and $u(\cdot)$ is any progressively measurable process which satisfies (3.2). However, the optimal drift turns out to be of the feedback type. The other control $U(t)$ is analogous to the cumulative number of customers rejected from the queueing system during the time-interval $[0, t]$, for all $t \geq 0$. A trivial choice of such U is the identically zero function which is associated with the infinite buffer-length situation. In such a situation, the controller makes no effort to reduce the queue-length process by rejecting customers and this can be a good control policy if the penalty for rejecting the customers is prohibitively high. Later in this section, we will show the optimality of the no rejection policy under such circumstances. A more interesting choice for U corresponds to a finite buffer situation, which rejects customers if the queue-length exceeds a predetermined threshold $b > 0$ (the buffer-length). This case corresponds to $U(\cdot)$ being the local-time process of $X_x(\cdot)$ at the buffer-length $b > 0$. In general, this "rejection process" U can be chosen from any criteria (with jumps allowed) to reduce the queue-length (and need not be a local-time process), as far as it satisfies the constraints in Definition 3.1 above.

Before we discuss the solution of the BCP in the next two subsections, we introduce the following two functions Φ and Ψ which are essential in finding an optimal control policy. Introduce the function Φ on $[0, \infty)$ by

$$\Phi(y) \doteq \sup_{a \in \mathcal{D}}[ay - C(a)] \quad \text{for } y \geq 0, \tag{3.16}$$

where $\mathcal{D}$ is as in Assumption 3.3. Clearly $\Phi(y)$ is finite for each $y \geq 0$. For each $y \in [0, p_0]$, the supremum in (3.16) is achieved at a unique point $\Psi(y) \in \mathcal{D}$, where

$$\Psi(y) = (C')^{-1}(y), \quad \text{for } 0 \leq y \leq p_0. \tag{3.17}$$

Note that, with Assumption 2.4, the function $\Psi(\cdot)$ is continuously differentiable. For a detailed discussion on the properties of Φ and Ψ and their use in a discrete-time optimal control problem, we refer to [12]. In [2] and in [13], these functions were used in the construction of the optimal drift control processes and we follow the same approach here. In all these articles, these functions are denoted by ϕ and ψ (instead of Φ and Ψ, respectively), but to distinguish these from the conventional Skorokhod maps (which will be described in Section 4.1), we intend to use this different notation in this article. By Assumption 2.4, Ψ is strictly increasing on $[0, p_0]$. Furthermore, for each $0 < p \leq p_0$,

$$0 \leq \Psi(y) \leq \theta_p, \quad \text{when } 0 \leq y \leq p, \text{ where } C'(\theta_p) = p. \tag{3.18}$$

By (3.16) and (3.17), we obtain,

$$\Phi(y) = y\Psi(y) - C(\Psi(y)), \quad \text{for each } 0 \leq y \leq p_0, \tag{3.19}$$

and

$$\Phi'(y) = \Psi(y) \quad \text{for each } 0 \leq y \leq p_0. \tag{3.20}$$

3.1. A verification lemma

With the help of Φ in (3.16), the formal Hamilton–Jacobi–Bellman (HJB) equation (see [11]) for the BCP can be written as

$$\min\left\{\frac{\sigma^2}{2}\mathcal{V}''(x) - \Phi(\mathcal{V}'(x)) - \gamma x\mathcal{V}'(x) - \delta\mathcal{V}(x) + \beta\gamma x, \ \mathcal{V}'(x), \ p - \mathcal{V}'(x)\right\} = 0, \tag{3.21}$$

for almost every $x \in [0, \infty)$. The following verification lemma enables us to sort out an optimal strategy.

Lemma 3.5 (*Verification Lemma*)**.** *Let $p > 0$ and $\mathcal{V}$ be a $\mathcal{C}^2$-function which satisfies the HJB equation in (3.21) together with the boundary condition*

$$\mathcal{V}'(0) = 0. \tag{3.22}$$

Then

$$\mathrm{V}_p(x) \geq \mathcal{V}(x), \quad \text{for all } x \geq 0,$$

where $\mathrm{V}_p(\cdot)$ is the value function defined in (3.5).

Remark 3.6. Since $\mathcal{V}$ satisfies (3.21), $\mathcal{V}$ may depend on p, but we do not make it explicit in our notation for the clarity of the presentation.

Proof. We apply the generalized Itô's Lemma (see p. 285 of [22,13]) to $\mathcal{V}(X_x(T))e^{-\delta T}$ where X_x satisfies (3.1) and $T > 0$. We also need a localization procedure, hence we introduce the

sequence of stopping times $\{\tau_N : N \geq 1\}$ by

$$\tau_N = \inf\{t > 0 : X_x(t) \geq N\}$$
$$= +\infty, \quad \text{if the above set is empty.} \tag{3.23}$$

Since, $U(\cdot)$ is nondecreasing, by (3.1), it follows that $0 \leq X_x(t) \leq X_x(t-)$ for all $t \geq 0$. Hence, $0 \leq X_x(t) \leq N$ for all $0 \leq t \leq \tau_N$.

$$\mathcal{V}(X_x(T \wedge \tau_N))\mathrm{e}^{-\delta(T \wedge \tau_N)}$$

$$= \mathcal{V}(x) + \sigma \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathcal{V}'(X_x(s-))\mathrm{d}W(s) + \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathcal{V}'(X_x(s-))\mathrm{d}L(s)$$

$$- \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathcal{V}'(X_x(s-))\mathrm{d}U(s) + \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \left(\frac{\sigma^2}{2} \mathcal{V}''(X_x(s-)) \right.$$

$$\left. - u(s)\mathcal{V}'(X_x(s-)) - \gamma X_x(s-)\mathcal{V}'(X_x(s-)) - \delta \mathcal{V}(X_x(s-)) \right) \mathrm{d}s$$

$$+ \sum_{0 < s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} \left[\Delta \mathcal{V}(X_x(s)) + \mathcal{V}'(X_x(s-))\Delta U(s) \right], \tag{3.24}$$

where $\Delta \mathcal{V}(X_x(s)) \doteq \mathcal{V}(X_x(s)) - \mathcal{V}(X_x(s-))$ and $\Delta U(s) \doteq U(s) - U(s-)$. Since, $0 \leq \mathcal{V}'(x) \leq p$, notice that

$$|\Delta \mathcal{V}(X_x(s))| \leq p|X_x(s) - X_x(s-)| = p[U(s) - U(s-)].$$

Therefore, $\sum_{0 < s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} |\Delta \mathcal{V}(X_x(s))| \leq p\, U(T \wedge \tau_N) < +\infty$. Similarly,

$$0 \leq \sum_{0 < s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} |\mathcal{V}'(X_x(s-))|\Delta U(s) \leq p\, U(T \wedge \tau_N) < \infty.$$

Hence, we can write,

$$- \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathcal{V}'(X_x(s-))\mathrm{d}U(s) + \sum_{0 < s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} \left[\Delta \mathcal{V}(X_x(s)) + \mathcal{V}'(X_x(s-))\Delta U(s) \right]$$

$$= - \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathcal{V}'(X_x(s-))\mathrm{d}U^c(s) + \sum_{0 \leq s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} \Delta \mathcal{V}(X_x(s))$$

$$\geq -p \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathrm{d}U^c(s) - p \sum_{0 \leq s \leq T \wedge \tau_N} \mathrm{e}^{-\delta s} \Delta U(s) = -p \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathrm{d}U(s), \tag{3.25}$$

where $U^c(\cdot)$ is the continuous part of the process $U(\cdot)$. Combining (3.24) with (3.25) and then using (3.3), (3.16) and (3.21) and taking expected value, we obtain

$$E\left(\mathrm{e}^{-\delta(T \wedge \tau_N)} \mathcal{V}(X_x(T \wedge \tau_N)) \right) \geq \mathcal{V}(x) - E \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \left[\beta \gamma X_x(s-) + C(u(s)) \right] \mathrm{d}s$$

$$- pE \int_0^{T \wedge \tau_N} \mathrm{e}^{-\delta s} \mathrm{d}U(s). \tag{3.26}$$

By (3.21), we also obtain

$$E\left[\mathrm{e}^{-\delta(T \wedge \tau_N)} |\mathcal{V}(X_x(T \wedge \tau_N))| \right] \leq E\left[(\mathcal{V}(0) + pX_x(T \wedge \tau_N)) \mathrm{e}^{-\delta(T \wedge \tau_N)} \right]. \tag{3.27}$$

We intend to estimate $E\left[X_x(T \wedge \tau_N)e^{-\delta(T\wedge\tau_N)}\right]$. Notice that

$$0 \le E\left[X_x(T \wedge \tau_N)e^{-\delta(T\wedge\tau_N)}\right] \le \left[E(X_x(T \wedge \tau_N)^2)\right]^{\frac{1}{2}} \left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}}. \qquad (3.28)$$

To estimate $E(X_x(T \wedge \tau_N)^2)$, we can apply the generalized Itô's Lemma to $X_x(T \wedge \tau_N)^2$ and follow a similar computation as in the derivation of (3.25) and eliminate the negative terms to obtain

$$E(X_x(T \wedge \tau_N)^2) \le C(1 + T), \qquad (3.29)$$

where $C > 0$ is a constant independent of $T > 0$. The derivation of (3.29) is also very similar to the calculations in Lemma 2.1 of [13] (see the estimate (2.9) in [13]) and we omit the details. One can verify this calculation easily using (3.24) with $\mathcal{V}(x)$ replaced by x^2.

Now, (3.28) combined with (3.29) yields

$$0 \le E\left[X_x(T \wedge \tau_N)e^{-\delta(T\wedge\tau_N)}\right] \le [C(1 + T)]^{\frac{1}{2}} \left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}}.$$

Combining this with (3.26) and (3.27), we obtain

$$E\left[|\mathcal{V}(0)|e^{-\delta(T\wedge\tau_N)}\right] + p\sqrt{C(1+T)}\left[E(e^{-2\delta(T\wedge\tau_N)})\right]^{\frac{1}{2}}$$
$$+ E \int_0^{T\wedge\tau_N} e^{-\delta s}\left[(\beta\gamma X_x(s-) + C(u(s)))\mathrm{d}s + p\mathrm{d}U(s)\right] \ge \mathcal{V}(x).$$

Next, first letting N go to infinity, and then taking the limit as $T \to \infty$, we obtain

$$\tilde{J}(x, u, U) = E \int_0^\infty e^{-\delta s}\left[(\beta\gamma X_x(s-) + C(u(s)))\mathrm{d}s + p\mathrm{d}U(s)\right] \ge \mathcal{V}(x).$$

for each admissible policy (X_x, u, U). Taking the infimum over all admissible policies (X_x, u, U), we get

$$V_p(x) \ge \mathcal{V}(x), \quad \text{for all } x \ge 0.$$

This completes the proof.   □

3.2. An optimal control policy

First we describe our candidate for an optimal control policy for the BCP in detail and then prove its optimality in the next theorem (Theorem 3.8). The constant p_0 defined in (3.14) turned out to be the threshold point for the suggested optimal strategy in the following sense: When $0 < p < p_0$, the state space of the optimal state process is a finite interval (after a possible initial jump). When $p \ge p_0$, optimal strategy does not allow any rejections (i.e. $U^* \equiv 0$). Thus the state process is independent of p and the state space is the infinite interval $[0, \infty)$. Furthermore, when $p \ge p_0$, the value function $V_p(\cdot)$ satisfies $V_p(x) = V_{p_0}(x)$ for all x. Now we describe our candidate policy which is shown to be optimal in Theorem 3.8.

Definition 3.7 (*Optimal Policy*)**.** For $0 < p < p_0$, the optimal state process $X^*_{p,x}(\cdot)$ is a reflecting diffusion process on $[0, b^*_p]$ for some $b^*_p > 0$ (as in Theorem 3.10) and it satisfies

$$X^*_{p,x}(t) = x - \int_0^t u^*_p(X^*_{p,x}(s))\mathrm{d}s - \gamma \int_0^t X^*_{p,x}(s)\mathrm{d}s + \sigma W(t) + L^*_p(t) - U^*_p(t). \quad (3.30)$$

Here $L_p^*(\cdot)$ is the local-time process of $X_{p,x}^*(\cdot)$ at the origin. The feedback-type optimal drift control is given by $u_p^*(X_{p,x}^*(\cdot))$ where $u_p^*(\cdot)$ is a Lipschitz continuous function described in (3.39). Without any ambiguity, we refer to this feedback-type drift control by $u_p^*(\cdot)$. The optimal rejection policy $U_p^*(\cdot)$ satisfies $U_p^*(t) = (x - b_p^*)^+ + U_{b_p^*}^*(t)$ for all $t \geq 0$, where $U_{b_p^*}^*(\cdot)$ is the local-time process of $X_{p,x}^*(\cdot)$ at $b_p^* > 0$. Note that $X_{p,x}^*(\cdot)$ makes an initial jump to b_p^* if $x > b_p^*$. We simply identify this policy by $(X_{p,x}^*, u_p^*, U_p^*)$ for $0 < p < p_0$.

For $p \geq p_0$, the same admissible control is optimal for all the values of p and hence $V_p(x) = V_{p_0}(x)$, for all $x \geq 0$. Thus, we denote the optimal state process by $X_x^*(\cdot)$ and it is a reflecting diffusion on $[0, \infty)$ which satisfies

$$X_x^*(t) = x - \int_0^t u_p^*(X_x^*(s))\mathrm{d}s - \gamma \int_0^t X_x^*(s)\mathrm{d}s + \sigma W(t) + L_p^*(t), \tag{3.31}$$

with the same notation for the processes as in (3.30). The feedback-type optimal drift is given by $u_{p_0}^*(X_x^*(\cdot))$ where $u_{p_0}^*(\cdot)$ is a Lipschitz continuous function described in (3.45). Hence for all $p \geq p_0$, we take $u_p^* = u_{p_0}^*$ for the optimal drift function. In this case, the optimal rejection process is identically zero and hence X_x^* corresponds to a queue-length process with infinite buffer capacity. Accordingly, we denote this policy by $(X_x^*, u_p^*, 0)$.

Now we state the main theorem of this section.

Theorem 3.8. (a) *For each $p > 0$, the value function $V_p(\cdot)$ is a convex $\mathcal{C}^2$-function which satisfies the HJB equation in (3.21) together with (3.22). When $p \geq p_0$, $V_p(x) = V_{p_0}(x)$ for all $x \geq 0$. Furthermore the feedback-type optimal drift $u_p^*(\cdot)$ in (3.30) and (3.31) satisfies the condition*

$$u_p^*(x) = \Psi(V_p'(x)), \quad \text{for all } x \geq 0 \text{ and for each } p > 0, \tag{3.32}$$

where Ψ is as given in (3.17).

(b) *When $0 < p < p_0$, the policy $(X_{p,x}^*, u_{p,x}^*, U_{p,x}^*)$ described in (3.30) is optimal and b_p^* represents the optimal buffer size. It also satisfies*

$$b_p^* = \inf\{x > 0 : V_p'(x) = p\}. \tag{3.33}$$

If $p \geq p_0$, the policy $(X_x^*, u_p^*, 0)$ described in (3.31) is optimal. Here the state process X_x^* corresponds to a infinite buffer capacity.

Remark 3.9. When $0 < p < p_0$, b_p^* is finite and the value function $V_p(\cdot)$ also satisfies $V_p''(b_p^*) = 0$. In this case, from our optimal policy we have $V_p(x) = V_p(b_p^*) + p(x - b_p^*)$ when $x > b_p^*$. Since $V_p(\cdot)$ is convex and $V_p'(b_p^*) = p$, from (3.33) it follows that b_p^* is unique.

Proof. First we consider $0 < p < p_0$. We assume that there exists a point $b_p^* > 0$ and an increasing function $\mathcal{Y}_p$ such that

$$\frac{\sigma^2}{2}\mathcal{Y}_p'(x) - \Phi(\mathcal{Y}_p(x)) - \gamma x \mathcal{Y}_p(x) + \beta \gamma x = \frac{\sigma^2}{2}\mathcal{Y}_p'(0) + \delta \int_0^x \mathcal{Y}_p(u)\mathrm{d}u, \tag{3.34}$$

for $0 < x < b_p^*$, together with the boundary conditions

$$\mathcal{Y}_p(0) = 0, \qquad \mathcal{Y}_p(b_p^*) = p, \qquad \mathcal{Y}_p'(b_p^*) = 0, \quad \text{and} \quad 0 \leq \mathcal{Y}_p(x) < p$$

$$\text{when } 0 \leq x < b_p^*. \tag{3.35}$$

We will verify the existence of such a $b_p^* > 0$ and the function $\mathcal{Y}_p$ in Theorem 3.10. Next introduce

$$\mathcal{V}_p(x) = \begin{cases} \dfrac{\sigma^2}{2\delta}\mathcal{Y}_p'(0) + \displaystyle\int_0^x \mathcal{Y}_p(u)\mathrm{d}u & \text{for all } 0 \leq x \leq b_p^*, \\ \mathcal{V}_p(b_p^*) + p(x - b_p^*) & \text{for all } x > b_p^*. \end{cases} \tag{3.36}$$

Since $\mathcal{Y}_p(\cdot)$ is an increasing $\mathcal{C}^1$-function on $[0, b_p^*]$, $\mathcal{V}_p(\cdot)$ is a convex $\mathcal{C}^2$-function on $[0, \infty)$. Furthermore, $\mathcal{V}_p(\cdot)$ satisfies

$$\frac{\sigma^2}{2}\mathcal{V}_p''(x) - \Phi(\mathcal{V}_p'(x)) - \gamma x\mathcal{V}_p'(x) - \delta\mathcal{V}_p(x) + \beta\gamma x = 0 \quad \text{for } 0 \leq x \leq b_p^*. \tag{3.37}$$

Evaluating (3.37) at $x = b_p^*$ and using (3.35) we obtain

$$\delta \mathcal{V}_p(b_p^*) = \beta\gamma b_p^* - p\gamma b_p^* - \Phi(p).$$

A direct computation using this identity and the fact that $p < p_0$ yields

$$\frac{\sigma^2}{2}\mathcal{V}_p''(x) - \Phi(\mathcal{V}_p'(x)) - \gamma x\mathcal{V}_p'(x) - \delta\mathcal{V}_p(x) + \beta\gamma x > 0 \quad \text{for } x > b_p^*. \tag{3.38}$$

Hence, (3.35), (3.37) and (3.38) implies that $\mathcal{V}_p$ satisfies all the assumptions of the verification lemma (Lemma 3.5). Therefore, we conclude that $\mathrm{V}_p(x) \geq \mathcal{V}_p(x)$ for all $x \geq 0$. To show that $\mathrm{V}_p(x)$ is indeed equal to $\mathcal{V}_p(x)$ for all $x \geq 0$, we verify that the proposed policy $(X_{p,x}^*, u_p^*, U_p^*)$ in (3.30) (with appropriately defined $u_p^*(\cdot)$) is an admissible policy and the cost $\widetilde{J}_p(x, u_p^*, U_p^*)$ from this policy (as defined in (3.4)) is equal to $\mathcal{V}_p(x)$ for each $x \geq 0$. Thus, it will follow that $\mathrm{V}_p(x) \leq \mathcal{V}_p(x)$ and consequently, $\mathrm{V}_p(x) = \mathcal{V}_p(x)$ for all $x \geq 0$.

For each $0 < p < p_0$, introduce

$$u_p^*(x) = \Psi(\mathcal{V}_p'(x)), \quad \text{for all } x \geq 0, \tag{3.39}$$

where $\Psi(\cdot)$ is given in (3.17). By (3.36) and (3.39), $u_p^*(\cdot)$ is a Lipschitz continuous function. Thus, $u_p^*(\cdot)$ takes values in $[0, \theta_p]$ where $C'(\theta_p) = p$. This interval $[0, \theta_p]$ is contained in the control set $\mathcal{D}$ by the assumption (3.15). Let $b_p^* > 0$ be as in (3.34) and (3.35). We consider the policy $(X_{p,x}^*, u_p^*, U_p^*)$ with $u_p^*(\cdot)$ defined in (3.39). Since, $u_p^*(\cdot)$ is a Lipschitz continuous function and $X_{p,x}^*$ is a reflecting diffusion on $[0, b_p^*]$, it is evident that $(X_{p,x}^*, u_p^*, U_p^*)$ is an admissible policy. Note that if $x > b_p^*$, the state process makes an initial jump to b_p^* as explained in the discussion below (3.30). For simplicity, we consider that $X_{p,x}^*(0) = x$ is in $[0, b_p^*]$, and apply Itô's Lemma to $\mathcal{V}_p(X_{p,x}^*(T))\mathrm{e}^{-\delta T}$. We use (3.18) and (3.37), $\mathcal{V}_p'(0) = 0$ and $\mathcal{V}_p'(b_p^*) = p$ to obtain

$$E[\mathcal{V}_p(X_{p,x}^*(T))\mathrm{e}^{-\delta T}] = \mathcal{V}_p(x) - E\int_0^T \mathrm{e}^{-\delta s}[\beta\gamma X_{p,x}^*(s) + C(u_p^*(X_{p,x}^*(s)))]\mathrm{d}s$$

$$- pE\int_0^T \mathrm{e}^{-\delta s}\mathrm{d}U_p^*(s).$$

Here $U_p^*(\cdot)$ is the local-time process of $X_{p,x}^*(\cdot)$ at $b_p^* > 0$. Since $\mathcal{V}_p$ is bounded on $[0, b_p^*]$, by letting $T \to \infty$, we obtain

$$\mathcal{V}_p(x) = \widetilde{J}_p(x, u_p^*, U_p^*), \tag{3.40}$$

where $\widetilde{J}_p(\cdot)$ is as given in (3.4). When $x > b_p^*$, there is an initial jump to b_p^* using the rejection process U_p^*. Hence,

$$\widetilde{J}_p(x, u_p^*, U_p^*) = p(x - b_p^*) + \widetilde{J}_p(b_p^*, u_p^*, U_p^*) = p(x - b_p^*) + \mathcal{V}_p(b_p^*) = \mathcal{V}_p(x), \quad (3.41)$$

by (3.36). Hence we have $V_p(x) \le \mathcal{V}_p(x)$ (which implies that $V_p(x) = \mathcal{V}_p(x)$) and therefore, $(X_{p,x}^*, u_p^*, U_p^*)$ is an optimal policy for $0 < p < p_0$. The conclusions (3.32) and (3.33) both follow directly from (3.35) and (3.39). This completes the proof of both parts of Theorem 3.8, when $0 < p < p_0$.

To prove the theorem for $p \ge p_0$, we assume the existence of an increasing function $\mathcal{Y}_0$ which satisfies

$$\frac{\sigma^2}{2}\mathcal{Y}_0'(x) - \Phi(\mathcal{Y}_0(x)) - \gamma x \mathcal{Y}_0(x) + \beta \gamma x$$

$$= \frac{\sigma^2}{2}\mathcal{Y}_0'(0) + \delta \int_0^x \mathcal{Y}_0(u) \mathrm{d}u, \quad \text{for all } x \ge 0, \quad (3.42)$$

together with the boundary conditions

$$\mathcal{Y}_0(0) = 0, \quad 0 \le \mathcal{Y}_0(x) < p_0 \quad \text{for all } x \ge 0 \quad \text{and} \quad \lim_{x \to \infty} \mathcal{Y}_0(x) = p_0. \quad (3.43)$$

We will also verify the existence of such a function $\mathcal{Y}_0$ in Theorem 3.10. Introduce

$$\mathcal{V}_0(x) = \frac{\sigma^2}{2\delta}\mathcal{Y}_0'(0) + \int_0^x \mathcal{Y}_0(u) \mathrm{d}u \quad \text{for all } x \ge 0. \quad (3.44)$$

Since $\mathcal{Y}_0(\cdot)$ is an increasing $\mathcal{C}^1$-function, $\mathcal{V}_0(\cdot)$ is a convex $\mathcal{C}^2$-function. We take any $p \ge p_0$. Then a direct computation using (3.42) and (3.43) verifies that $\mathcal{V}_0$ satisfies all the assumptions of the verification lemma (Lemma 3.5). Hence, we obtain $V_p(x) \ge \mathcal{V}_0(x)$ for all $x \ge 0$. Now we prove that $V_p(x) = \mathcal{V}_0(x)$ for all $x \ge 0$. For each $p \ge p_0$, we introduce

$$u_p^*(x) = \Psi(\mathcal{V}_0'(x)), \quad \text{for all } x \ge 0, \quad (3.45)$$

where $\Psi(\cdot)$ is given in (3.17). Notice that for $p \ge p_0$, $u_p^*(x) = u_{p_0}^*(x)$ for all $x \ge 0$, since $\mathcal{V}_0$ defined in (3.44) depends only on p_0. We intend to show that $(X_x^*, u_p^*, 0)$ is an admissible policy for all $p \ge p_0$, and $\widetilde{J}_p(x, u_p^*, 0) = \mathcal{V}_0(x)$ for all $x \ge 0$. Note that $u_p^*(\cdot)$ is a Lipschitz continuous function and $X_{p,x}^*$ is a reflecting diffusion on $[0, +\infty)$ with a reflecting barrier at the origin. By (3.17) and (3.45), $u_p^*(\cdot)$ take values in $[0, \theta_0]$, where $C'(\theta_0) = p_0$. Notice that $[0, \theta_0]$ is contained in the control space $\mathcal{D}$ by (3.15). Therefore, $(X_x^*, u_p^*, 0)$ is an admissible policy.

Now X_x^* satisfies (3.31) with optimal drift $u_p^*(\cdot)$ defined in (3.45). Hence we apply Itô's Lemma to $\mathcal{V}_0(X_x^*(T))\mathrm{e}^{-\delta T}$ to obtain

$$E[\mathcal{V}_0(X_x^*(T))\mathrm{e}^{-\delta T}] = \mathcal{V}_0(x) - E \int_0^T \mathrm{e}^{-\delta s}[\beta \gamma X_x^*(s) + C(u_p^*(X_x^*(s)))]\mathrm{d}s.$$

To verify $\lim_{T \to \infty} E[\mathcal{V}_0(X_x^*(T))\mathrm{e}^{-\delta T}] = 0$, by (3.43), it suffices to show that

$$\lim_{T \to \infty} E[X_x^*(T)\mathrm{e}^{-\delta T}] = 0.$$

For this, we again apply Itô's Lemma to $[X_x^*(T)]^2$, using (3.31) and eliminate the negative terms to get the estimate $E[X_x^*(T)]^2 \le C(1 + T)$, where $C > 0$ is a constant independent of $T > 0$

(see (3.29) for a similar calculation). This yields $\lim_{T \to \infty} E[X_x^*(T) e^{-\delta T}] = 0$. Hence, using a similar approach as used in deriving (3.40), we obtain

$$\mathcal{V}_0(x) = \widetilde{J}_p(X_x^*, u_p^*, 0), \quad \text{for all } x \geq 0, \ p \geq p_0,$$

and $(X_x^*, u_p^*, 0)$ is an optimal policy for each $p \geq p_0$. Furthermore, the feedback-type drift control u_p^* is given by $u_p^*(x) \equiv u_{p_0}^*(x) = \Psi(\mathcal{V}_0'(x)) = \Psi(\mathrm{V}_{p_0}'(x))$, for all $x \geq 0$. Since, $\mathcal{V}_0(\cdot)$ is a $\mathcal{C}^2$-function, the proof of Theorem 3.8 for the case $p \geq p_0$ is also complete. $\square$

Proof of Theorem 2.6. The proof of Theorem 3.8 given above directly shows the existence of such b_p^* which satisfies (2.15) and (2.16). Since, $\mathcal{V}_p(x) \equiv \mathrm{V}_p(x)$ for all $x \geq 0$, where $\mathrm{V}_p(\cdot)$ is the value function defined in (3.5), the pair $(b_p^*, \mathcal{V}_p(\cdot))$ is unique. $\square$

It remains to verify the existence of a function $\mathcal{Y}_p(\cdot)$ which satisfies (3.34) and (3.35) and a function $\mathcal{Y}_0(\cdot)$ which satisfies (3.42) and (3.43). We address this issue in the next subsection.

3.3. A parametrization method

Our aim here is to establish the existence of a function $\mathcal{Y}_p(\cdot)$ which satisfies (3.34)–(3.35) and another function $\mathcal{Y}_0(\cdot)$ which satisfies (3.42) and (3.43). This will be achieved in the following theorem and it will complete the proof of Theorem 3.8.

Theorem 3.10. (i) *For each p in $(0, p_0)$, there exists a point $b_p^* > 0$ and an increasing function $\mathcal{Y}_p(\cdot)$ which satisfies (3.34) and (3.35).*
(ii) *There also exists an increasing function $\mathcal{Y}_0(\cdot)$ defined on $[0, \infty)$, which satisfies (3.42) and (3.43).*

The proof of this theorem will be given at the end of this section, since it needs several results about the behavior of a parametric family of solutions to the differential equation in (3.47) below.
First we extend the function Φ defined in (3.16) to negative real axis by setting

$$\Phi(y) = 0, \quad \text{for all } y \leq 0. \tag{3.46}$$

Then, by the assumptions on the cost function C (Assumption 2.4), (3.19) and (3.20), it is clear that Φ' is a Lipschitz continuous function on $\mathbb{R}$. For our purposes, only the behavior of Φ on the interval $[0, p_0]$ is crucial.
Next we consider the following parametric family of differential equations:

$$\begin{cases} \sigma^2 \mathcal{Y}_r'(x) - 2\Phi(\mathcal{Y}_r(x)) - 2\gamma x \mathcal{Y}_r(x) + 2\beta \gamma x = \sigma^2 r + 2\delta \int_0^x \mathcal{Y}_r(u)\mathrm{d}u \\ \mathcal{Y}_r(0) = 0, \qquad \mathcal{Y}_r'(0) = r. \end{cases} \tag{3.47}$$

We differentiate the above equation and use (3.20) to obtain

$$\sigma^2 \mathcal{Y}_r''(x) - 2\Psi(\mathcal{Y}_r(x))\mathcal{Y}_r'(x) - 2\gamma x \mathcal{Y}_r'(x) - 2(\gamma + \delta)\mathcal{Y}_r(x) + 2\beta\gamma = 0. \tag{3.48}$$

Since Ψ is a $\mathcal{C}^1$-function, this second-order non-linear differential equation with the initial data $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r$ has a unique solution which is valid on the interval $[0, \omega_r)$ where ω_r is the explosion point for $\mathcal{Y}_r$ (see [16]), and $0 < \omega_r \leq +\infty$. Consequently, (3.47) has a unique solution $\mathcal{Y}_r$ which is valid on $[0, \omega_r)$. Furthermore, this solution $\mathcal{Y}_r(x)$ is jointly continuous in (r, x) (see chapter 5 of [16]) and we will use this fact in our analysis of (3.47).
Our next proposition describes the properties of the solution $\mathcal{Y}_r$. See Fig. 2 for the solution profile.
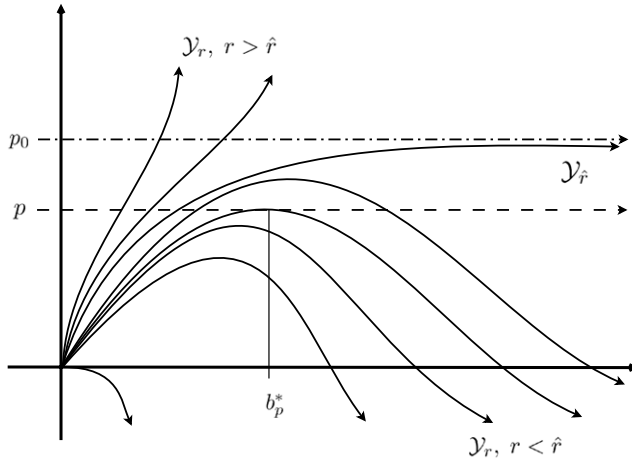
Fig. 2. Nature of the family of solutions $\mathcal{Y}_r$.

Proposition 3.11. *For the family of solutions $(\mathcal{Y}_r(\cdot))_{r>0}$, the following properties hold:*

(i) *if $r_1 > r_2 > 0$ then $\mathcal{Y}_{r_1}(x) > \mathcal{Y}_{r_2}(x)$ for all $0 < x < \omega_{r_1} \wedge \omega_{r_2}$.*
   Furthermore, $\mathcal{Y}_{r_1}(x) > (r_1 - r_2)x + \mathcal{Y}_{r_1}(x)$ on this interval $(0, \omega_{r_1} \wedge \omega_{r_2})$.
(ii) *If $\mathcal{Y}_r'(\xi) = 0$ for some $\xi > 0$, then $\mathcal{Y}_r(\xi) \neq p_0$ where $p_0 > 0$ is given in (3.14).*
   Furthermore, if $x = \xi > 0$ is the local maximum for $\mathcal{Y}_r$ then $\mathcal{Y}_r(\xi) < p_0$. Also, $\mathcal{Y}_r$
   cannot have any local minima.
(iii) *There exist $r_0 > 0$ such that for each $r > r_0$, $\mathcal{Y}_r$ does not have any local maxima and $\mathcal{Y}_r(x)$*
   is increasing to ∞ as x increases to ω_r.
(iv) *For each $r > 0$, $\mathcal{Y}_r$ has a positive local maximum on $(0, \infty)$ if and only if $\mathcal{Y}_r(z) = 0$ for*
   some $z > 0$.

Proof. Let $r_1 > r_2 > 0$. Since $\mathcal{Y}_{r_1}(0) = \mathcal{Y}_{r_2}(0) = 0$ and $\mathcal{Y}_{r_1}'(0) = r_1 > r_2 = \mathcal{Y}_{r_2}'(0)$, it follows that $\mathcal{Y}_{r_1}(x) > \mathcal{Y}_{r_2}(x)$ for all x in an interval $(0, \delta)$ for some $\delta > 0$. Now suppose $\mathcal{Y}_{r_2}(z) \geq \mathcal{Y}_{r_1}(z)$ for some $z \geq 0$, then there is a point $c \geq \delta > 0$ such that $\mathcal{Y}_{r_2}(c) = \mathcal{Y}_{r_1}(c)$ and $\mathcal{Y}_{r_2}(x) < \mathcal{Y}_{r_1}(x)$ when $0 < x < c$. Then using (3.47),

$$\sigma^2[\mathcal{Y}_{r_1}'(x) - \mathcal{Y}_{r_2}'(x)] = \sigma^2(r_1 - r_2) + 2\gamma x(\mathcal{Y}_{r_1}(x) - \mathcal{Y}_{r_2}(x))$$
$$+ 2\left(\Phi(\mathcal{Y}_{r_1}(x)) - \Phi(\mathcal{Y}_{r_2}(x))\right) + 2\delta \int_0^x [\mathcal{Y}_{r_1}(u) - \mathcal{Y}_{r_2}(u)]\mathrm{d}u.$$

Since Φ is an increasing function, this implies that $\mathcal{Y}_{r_1}'(x) - \mathcal{Y}_{r_2}'(x) > (r_1 - r_2)$ for each x in $(0, c)$. Hence, $\mathcal{Y}_{r_1}(c) = \mathcal{Y}_{r_2}(c)$ is impossible and the same argument implies that $\mathcal{Y}_{r_1}'(x) - \mathcal{Y}_{r_2}'(x) > (r_1 - r_2)$ for all x in $(0, \omega_{r_1} \wedge \omega_{r_2})$. Consequently $\mathcal{Y}_{r_1}(x) > (r_1 - r_2)x + \mathcal{Y}_{r_1}(x)$ on this interval $(0, \omega_{r_1} \wedge \omega_{r_2})$. This completes the proof of part (i).

For part (ii), let $\xi > 0$ be a point which satisfies $\mathcal{Y}_r'(\xi) = 0$. Suppose that $\mathcal{Y}_r(\xi) = p_0$ where p_0 is given in (3.45). Now let

$$x_0 = \inf\{\xi > 0 : \mathcal{Y}_r(\xi) = p_0 \text{ and } \mathcal{Y}_r'(\xi) = 0\}.$$

Then $x_0 > 0$, $\mathcal{Y}_r(x_0) = p_0$ and $\mathcal{Y}_r'(x_0) = 0$. The function $\mathcal{Y}_r$ also satisfies (3.48) with the same initial data $\mathcal{Y}(x_0) = p_0$ and $\mathcal{Y}'(x_0) = 0$. Since Ψ is a C^1-function, this initial value problem has a

unique solution in an interval $(x_0 - \delta, \ x_0 + \delta)$ for some $\delta > 0$ where $x_0 > \delta$. Hence, $\mathcal{Y}_r(x) \equiv p_0$ on $(x_0 - \delta, \ x_0 + \delta)$ and this contradicts with the definition of x_0. Consequently $\mathcal{Y}_r(\xi) \neq p_0$ if $\mathcal{Y}_r'(\xi) = 0$.

Next, if $\mathcal{Y}_r'(\xi) = 0$, by (3.48) we obtain,

$$\frac{\sigma^2}{2}\mathcal{Y}_r''(\xi) = (\delta + \gamma)(\mathcal{Y}_r(\xi) - p_0). \tag{3.49}$$

Hence if $x = \xi$ is a local maximum, then $\mathcal{Y}_r''(\xi) \leq 0$ and by (3.49) we obtain $\mathcal{Y}_r(\xi) \leq p_0$. Since $\mathcal{Y}_r'(\xi) = 0$, we know that $\mathcal{Y}_r(\xi) \neq p_0$ and consequently, $\mathcal{Y}_r(\xi) < p_0$. If $x = \xi > 0$ is a local minimum then $\mathcal{Y}_r'(\xi) = 0$ and $\mathcal{Y}_r''(\xi) \geq 0$. Then by (3.49), $\mathcal{Y}_r(\xi) \geq p_0$. Since $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r > 0$, it follows that $\mathcal{Y}_r$ is strictly increasing in an interval $(0, \delta)$ for some $\delta > 0$. These two facts imply the existence of a local maximum at $x = z$ where $0 < z < \xi$ and $\mathcal{Y}_r(z) > p_0$. This is a contradiction. Hence $\mathcal{Y}_r$ cannot have any local minima. This completes the proof of part (ii).

To prove part (iii), we pick $r_1 > 0$, then by the initial conditions in (3.47), $\mathcal{Y}_{r_1}(x) > 0$ for all x in $(0, 2\delta_{r_1})$ for some $\delta_{r_1} > 0$. For $r > r_1$, using (3.47) and part (i) of this proposition, we obtain

$$\sigma^2 \mathcal{Y}_r'(x) > \sigma^2 r - 2\beta\gamma x \quad \text{for } 0 < x < 2\delta_{r_1}.$$

Next, we pick $r_0 > r_1$ such that $(\sigma^2 r_0 - 2\beta\gamma\delta_{r_1})\delta_{r_1} > \sigma^2 p_0$. Hence $\sigma^2 \mathcal{Y}_{r_0}'(x) > \sigma^2 r_0 - 2\beta\gamma\delta_{r_1}$ when $0 < x < \delta_{r_1}$ and consequently for $r > r_0$, $\sigma^2 \mathcal{Y}_r(\delta_{r_1}) > \sigma^2 \mathcal{Y}_{r_0}(\delta_{r_1}) \geq (\sigma^2 r_0 - 2\beta\gamma\delta_{r_1})\delta_{r_1} > \sigma^2 p_0 > 0$. By part (ii), $\mathcal{Y}_r$ cannot have any local maxima when $\mathcal{Y}_r(x) > p_0$ and therefore, we conclude that $\mathcal{Y}_r(\cdot)$ is an increasing function when $x > \delta_{r_1}$.

Now if $\lim_{x \to \omega_r} \mathcal{Y}_r(x) = \lambda_0$ exists and if λ_0 is finite, by integrating (3.47), it is easy to observe that ω_r is infinite. Then again using (3.47), we obtain

$$\lim_{x \to \infty} \frac{\mathcal{Y}_r'(x)}{x} = \frac{2}{\sigma^2}(\delta + \gamma)(\lambda_0 - p_0). \tag{3.50}$$

Clearly $\lambda_0 > p_0$, thus the above limit is positive and $\lim_{x \to \infty} \mathcal{Y}_r(x) = +\infty$. This is a contradiction and hence $\lambda_0 = +\infty$. Thus $\mathcal{Y}_r$ is increasing to $+\infty$ as x increases to ω_r. This completes part (iii).

Now let $x = \xi > 0$ be the first local maximum of $\mathcal{Y}_r$ on $(0, +\infty)$. Then $p_0 > \mathcal{Y}_r(\xi) > 0$, $\mathcal{Y}_r'(\xi) = 0$ and $0 < \mathcal{Y}_r(x) < \mathcal{Y}_r(\xi)$ when $0 < x < \xi$. By (3.49), $\mathcal{Y}_r''(\xi) < 0$ and by part (ii), $\mathcal{Y}_r$ does not have any local minima. Therefore $\mathcal{Y}_r$ is decreasing when $x > \xi$. Suppose that $\lim_{x \to \omega_r} \mathcal{Y}_r(x)$ is finite. Then we can use (3.50) and the argument above to conclude that $\omega_r \equiv +\infty$ and $\lim_{x \to \infty} \frac{\mathcal{Y}_r'(x)}{x} < 0$. Thus $\lim_{x \to \infty} \mathcal{Y}_r(x) = -\infty$ and this is a contradiction. Hence $\lim_{x \to \omega_r} \mathcal{Y}_r(x) = -\infty$ and as a consequence, $\mathcal{Y}_r(z) = 0$ for some $z > \xi$.

Conversely, if $\mathcal{Y}_r(z) = 0$ for $z > 0$, since $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r > 0$ it is clear that there is a local maximum at a point $\xi > 0$ where $0 < \xi < z$ and $\mathcal{Y}_r(\xi) > 0$. This completes the proof of the proposition. $\quad\square$

Remark 3.12. One reason that $p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$ is a critical value in the analysis of the parametric family of solutions to (3.47) is that the constant function $\mathcal{Y}(x) = p_0$ is the only constant solution to (3.48). But note that, it does not satisfy (3.47).

Proposition 3.13. *There exists $\hat{r} > 0$ which satisfies the following conditions:*

(i) *If $0 < r < \hat{r}$ then there exists $z_r > 0$ such that $\mathcal{Y}_r(z_r) = 0$ and the set $\{x > 0 : \mathcal{Y}_r(x) > 0\}$ is equal to the open interval $(0, z_r)$. Furthermore, let*

$$H(r) = \max_{x>0} \mathcal{Y}_r(x). \tag{3.51}$$

Then $H(r)$ is finite, $H(r) = \max_{0<x<z_r} \mathcal{Y}_r(x)$ and $0 < H(r) < p_0$.

(ii) *When $r = \hat{r}$, $\mathcal{Y}_{\hat{r}}$ is strictly increasing, $\omega_{\hat{r}} \equiv +\infty$ and $\lim_{x\to+\infty} \mathcal{Y}_{\hat{r}}(x) = p_0$.*

(iii) *If $r > \hat{r}$, $\mathcal{Y}_r$ increases to $+\infty$ when x increases to ω_r.*

Proof. First we consider the solution $\mathcal{Y}_0$ to (3.47) which corresponds to $r = 0$. Using (3.48), the fact that $\Psi(0) = 0$, and the initial conditions $\mathcal{Y}_0(0) = \mathcal{Y}_0'(0) = 0$, we obtain $\sigma^2 \mathcal{Y}_0''(0) = -2\beta\gamma < 0$. Hence, there exists an $\epsilon_0 > 0$ such that $\mathcal{Y}_0$ is strictly concave on $(-2\epsilon_0, 2\epsilon_0)$ and $\mathcal{Y}_0$ has a local maximum at $x = 0$. Consequently, $\mathcal{Y}_0(\epsilon_0) < 0$. Since $\mathcal{Y}_r(x)$ is jointly continuous in (r, x) and using part (i) of Proposition 3.11, we can find $\eta_0 > 0$ such that $\mathcal{Y}_r(\epsilon_0) < 0$ for all $0 \le r < \eta_0$. Thus, for each such r in $(0, \eta_0)$, $\mathcal{Y}_r$ has a positive local maximum ξ_r in $(0, \epsilon_0)$ and a zero at z_r in $(0, \epsilon_0)$ where $0 < \xi_r < z_r < \epsilon_0$.

Introduce

$$\hat{r} = \sup\{r > 0 : \mathcal{Y}_r(x) = 0 \text{ for some } x > 0\}. \tag{3.52}$$

The interval $(0, \eta_0)$ is in the above set and thus $\hat{r}$ is well defined. Let r_0 be as in part (iii) of the Proposition 3.11. Then clearly $\hat{r} \le r_0$. Consequently $0 < \eta_0 \le \hat{r} \le r_0 < +\infty$. Next, by parts (i) and (iv) of Proposition 3.11, it clearly follows that for each $0 < r < \hat{r}$, $\mathcal{Y}_r(x) = 0$ for some $x > 0$. We let

$$z_r = \inf\{x > 0 : \mathcal{Y}_r(x) = 0\}.$$

By part (ii) of Proposition 3.11, each $\mathcal{Y}_r$ can have at most one local maximum and then we can deduce that $H(r)$ is finite, $H(r) = \max_{0<x<z_r} \mathcal{Y}_r(x)$ and $0 < H(r) < p_0$. This completes part (i).

Since $0 < H(r) < p_0$ for each $r < \hat{r}$ and $\mathcal{Y}_r(x)$ is jointly continuous in (r, x), it follows that $0 < \mathcal{Y}_{\hat{r}}(x) \le p_0$, for all $x \in (0, \omega_{\hat{r}})$. Suppose that there is a $\xi > 0$ with $\mathcal{Y}_{\hat{r}}'(\xi) = 0$, then $\mathcal{Y}_{\hat{r}}(\xi) < p_0$ by part (ii) of Proposition 3.11. Now using (3.49), we have $\mathcal{Y}_r''(\xi) < 0$ and $x = \xi$ is a strict local maximum for $\mathcal{Y}_r$. Therefore we can employ the joint continuity of $\mathcal{Y}_r(x)$ in (r, x) and the monotonicity of $\mathcal{Y}_r$ in r as in part (i) of the Proposition 3.11 to conclude that for some $r > \hat{r}$, $\mathcal{Y}_r$ also has a local maximum in a neighborhood of ξ when $|r - \hat{r}|$ is sufficiently small. Using part (iv) of Proposition 3.11, it follows that for each such $r > \hat{r}$, $\mathcal{Y}_r(x) = 0$ for some x. This contradicts with the definition of $\hat{r}$ in (3.52). Hence $\mathcal{Y}_{\hat{r}}'(x) \ne 0$ for all $x \ge 0$ and $\mathcal{Y}_{\hat{r}}$ is a C^2-function. But, $\mathcal{Y}_{\hat{r}}'(0) = \hat{r} > 0$ and consequently $\mathcal{Y}_{\hat{r}}'(x) > 0$ for all $0 < x < \omega_{\hat{r}}$. Thus $\mathcal{Y}_{\hat{r}}$ is an increasing function which satisfies $0 < \mathcal{Y}_{\hat{r}}(x) \le p_0$ and (3.47). If $\mathcal{Y}_{\hat{r}}(x_1) = p_0$ for some x_1, then it is a local maximum and $\mathcal{Y}_{\hat{r}}'(x_1) = 0$. Then by the uniqueness of the solutions to the differential equation (3.48), it follows that $\mathcal{Y}_{\hat{r}}(x) = p_0$ for all x which is a contradiction. Hence $0 < \mathcal{Y}_{\hat{r}}(x) < p_0$ for all x. By integrating (3.47) it is evident that $\mathcal{Y}_{\hat{r}}(x)$ is finite for each x and thus $\omega_{\hat{r}} \equiv +\infty$. Now let $\lambda_0 = \lim_{x\to\infty} \mathcal{Y}_{\hat{r}}(x)$. Then $0 < \lambda_0 \le p_0$. By (3.50), $\lim_{x\to\infty} \frac{\mathcal{Y}_{\hat{r}}'(x)}{x} = \frac{2}{\sigma^2}(\delta + \gamma)(\lambda_0 - p_0)$. Since $\mathcal{Y}_{\hat{r}}'(x) > 0$ for all x, it follows that $\lambda_0 \ge p_0$. Hence $\lambda_0 = p_0$ and thus part (ii) follows.

When $r > \hat{r}$, the definition of $\hat{r}$ and part (iv) of Proposition 3.11 implies that $\mathcal{Y}_r$ cannot have any local maxima. Also, if $\mathcal{Y}_r'(\xi) = 0$ for some $\xi > 0$, since $\mathcal{Y}_r$ does not have any positive local

maxima, Eq. (3.49) and part (ii) of Proposition 3.11 implies that $\mathcal{Y}_r''(\xi) > 0$ and hence $x = \xi$ is a strict local minimum. But $\mathcal{Y}_r(0) = 0$ and $\mathcal{Y}_r'(0) = r > 0$, therefore $\mathcal{Y}_r$ must have a positive local maximum at some point in $(0, \xi)$ and this is a contradiction. Consequently, $\mathcal{Y}_r'(x) > 0$ for all $0 < x < \omega_r$. Suppose $\lim_{x \to \omega_r} \mathcal{Y}_r(x)$ is finite, say λ_0, then $0 < \mathcal{Y}_r(x) < \lambda_0$ for all $0 < x < \omega_r$. Thus by integrating (3.47), we obtain $\omega_r = +\infty$ and (3.50) holds. But $\mathcal{Y}_r(x) > (r - \hat{r})x + \mathcal{Y}_{\hat{r}}(x)$ for each x, by part (i) of the Proposition 3.11. Consequently $\lim_{x \to \infty} \mathcal{Y}_r(x) = +\infty$ and hence $\lambda_0 = +\infty$ and this is a contradiction.

Therefore, we conclude that $\lim_{x \to \omega_r} \mathcal{Y}_r(x) = +\infty$. This completes the proof. □

Proposition 3.14. *Let the point $\hat{r}$ and the function H be as in Proposition 3.13. Then*

(i) *H is a continuous strictly increasing function defined on $(0, \hat{r})$ and it takes all the values in the interval $(0, p_0)$.*

(ii) *$\lim_{r \to 0+} H(r) = 0$ and $\lim_{r \to \hat{r}-} H(r) = p_0$.*

Proof. Part (i) of Proposition 3.13 implies that $H(r)$ is finite and $0 < H(r) < p_0$ for each r in $(0, \hat{r})$. Also there is a point ξ_r such that $0 < \xi_r < z_r$ and $H(r) = \mathcal{Y}_r(\xi_r)$. By part (ii) of Proposition 3.13, we have $\mathcal{Y}_r(\xi_r) < \mathcal{Y}_{\hat{r}}(\xi_r) < p_0$. Therefore, by (3.49), $\mathcal{Y}_r''(\xi_r) < 0$ and $x = \xi_r$ is a strict local maximum. By part (ii) of Proposition 3.11, $\mathcal{Y}_r$ cannot have any local minima and therefore this local maximum point $x = \xi_r$ is unique (see Fig. 2). Since $\mathcal{Y}_r(x)$ is jointly continuous in (r, x) and using part (i) of Proposition 3.11, it evidently follows that $H(\cdot)$ is a continuous strictly increasing function on $(0, \hat{r})$. This proves part (i).

When $r = 0$, the function $\mathcal{Y}_0$ has a strict local maximum $x = 0$ and is concave in a neighborhood of $x = 0$ as we have noticed in the proof of part (i) of Proposition 3.13. Thus, we can pick a $\delta_0 > 0$ such that $\mathcal{Y}_0(x) < 0$ on $(0, 2\delta_0)$. In particular, $\mathcal{Y}_0(\delta_0) < 0$. For a given $\epsilon > 0$, using part (i) of Proposition 3.11 and joint continuity of $\mathcal{Y}_r(x)$ in both r and x, we can find $\eta_0 > 0$ such that $\mathcal{Y}_r(\delta_0) < 0$ and $|\mathcal{Y}_0(x) - \mathcal{Y}_r(x)| < \epsilon$ for all x in $[0, \delta_0]$ and for all r in $[0, \eta_0)$. Thus $0 < H(r) < \epsilon$ for each $0 < r < \eta_0$. Consequently $\lim_{r \to 0+} H(r) = 0$. The fact that $\lim_{r \to \hat{r}} H(r) = p_0$ can also be proved by combining the joint continuity of $\mathcal{Y}_r(x)$, the monotonicity property of $\mathcal{Y}_r$ as in part (i) of Proposition 3.11 and the fact that $\lim_{x \to \infty} \mathcal{Y}_{\hat{r}}(x) = p_0$. This completes the proof. □

Proof of Theorem 3.10. Let $0 < p < p_0$. By the previous proposition, there exists a unique r_p in $(0, \hat{r})$ and a unique point ξ_{r_p} such that

$$p = H(r_p) = \mathcal{Y}_{r_p}(\xi_{r_p}).$$

Furthermore $\mathcal{Y}_{r_p}'(x) > 0$ when $0 < x < \xi_{r_p}$. We relabel the point ξ_{r_p} by b_p^* and the function $\mathcal{Y}_{r_p}$ by $\mathcal{Y}_p$ on the interval $[0, b_p^*]$. Then the point $b_p^* > 0$ and the function $\mathcal{Y}_p(\cdot)$ satisfies (3.34) and (3.35).

For part (ii), consider $\hat{r} > 0$ given in (3.52) and the associated function $\mathcal{Y}_{\hat{r}}(\cdot)$ as described in Proposition 3.13. We simply relabel this function as $\mathcal{Y}_0(\cdot)$. Then clearly $\mathcal{Y}_0$ satisfies (3.42) and (3.43). This completes the proof. □

Remark 3.15. A similar parametrization method was used in [13]. However, in [13] the HJB equation (corresponding to a long-run average cost problem) can be considered as a first-order non-linear differential equation (in terms of the derivative) — see Theorems 3.1 and 4.2 of [13]. For the infinite horizon discounted cost minimization problem considered in this article, the situation is much more difficult and we have a truly second-order non-linear differential equation (see (2.15) and (2.16)) for the value function. Hence, the parametrization method used here is

more involved than in [13]. In fact, solving the infinite horizon discounted cost minimization problem is, in some sense, more general than the long-run average cost minimization problem since it is possible to obtain optimal controls for the latter from those of the former problem by letting the discount factor δ tend to zero. For such an approach, see [30].

4. Asymptotic optimality

In this section we provide the proof of our main result, Theorem 2.8. This proof involves showing that the policy proposed in Definition 2.7 is asymptotically optimal, using Theorem 3.8 from Section 3. The proof of Theorem 2.8 and other weak convergence results leading to this proof are given in Section 4.2. These proofs also use properties of the "regulator maps" discussed first in Section 4.1.

4.1. Regulator maps

Definition 4.1 (*Generalized Regulator Maps*)**.** Let $u : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous, non-negative function and $\gamma > 0$ be a constant. Then

One-sided generalized regulator mapping is a mapping

$$(\phi^{u,\gamma}, \psi^{u,\gamma}) : \mathcal{D}([0, \infty), \mathbb{R}) \to \mathcal{D}([0, \infty), [0, \infty) \times [0, \infty))$$

such that for any given $w \in \mathcal{D}([0, \infty), \mathbb{R})$ as in Definition 4.1, $(\tilde{q}, \tilde{\ell}) \equiv (\phi^{u,\gamma}, \psi^{u,\gamma})(w)$ satisfies

(i) $\tilde{q}(t) = w(t) - \int_0^t [u(\tilde{q}(s)) + \gamma \tilde{q}(s)] ds + \tilde{\ell}(t) \geq 0, \ \forall t \geq 0,$

(ii) $\tilde{\ell}(\cdot)$ is nondecreasing, $\tilde{\ell}(0) = 0$ and $\int_0^\infty \tilde{q}(t) d\tilde{\ell}(t) = 0.$

Two-sided generalized regulator mapping is defined for any real $b \in (0, \infty)$ as a mapping

$$(\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma}) : \mathcal{D}([0, \infty), \mathbb{R}) \to \mathcal{D}([0, \infty), [0, b] \times [0, \infty) \times [0, \infty))$$

such that for any given $w \in \mathcal{D}([0, \infty), \mathbb{R})$ with $0 \leq w(0) \leq b$ and $(\tilde{q}, \tilde{\ell}, \tilde{k}) \equiv (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w)$ satisfies

(i) $\tilde{q}(t) = w(t) - \int_0^t [u(\tilde{q}(s)) + \gamma \tilde{q}(s)] ds + \tilde{\ell}(t) - \tilde{k}(t) \in [0, b], \ \forall t \geq 0,$

(ii) $\tilde{\ell}(\cdot), \tilde{k}(\cdot)$ are both nondecreasing, $\tilde{\ell}(0) = \tilde{k}(0) = 0, \ \int_0^\infty \tilde{q}(t) d\tilde{\ell}(t) = \int_0^\infty (b - \tilde{q}(t))^+ d\tilde{k} = 0.$

The argument for the existence and uniqueness of the two types of generalized regulator mappings can be found in Proposition 4.1(i) of [29] and Lemma 4.1(i) of [23]. Let $w \in \mathcal{D}([0, \infty), \mathbb{R})$ be as in Definition 4.1. We introduce the unique solutions $v(\cdot)$ and $v_b(\cdot)$ to the following integral equations:

$$v(t) = w(t) - \int_0^t [u(\phi(v)(s)) + \gamma \phi(v)(s)] ds, \quad t \geq 0,$$

$$v_b(t) = w(t) - \int_0^t [u(\phi_b(v_b)(s)) + \gamma \phi_b(v_b)(s)] ds, \quad t \geq 0, \tag{4.1}$$

where (ϕ, ψ) is the conventional one-sided regulator map (or the Skorokhod map) on $[0, \infty)$ and $(\phi_b, \psi_{1,b}, \psi_{2,b})$ is the two-sided regulator map (or the Skorokhod map) on $[0, b]$ (see [25,17]). Observe that these conventional one-sided and two-sided regulator maps can be obtained from

Definition 4.1 by setting $u \equiv 0$ and $\gamma = 0$. Now define the maps $\mathcal{M}^{u,\gamma}(w)$, $\mathcal{M}_b^{u,\gamma}(w)$ from $\mathcal{D}([0, \infty), \mathbb{R})$ to $\mathcal{D}([0, \infty), \mathbb{R})$ as follows:

$$\mathcal{M}^{u,\gamma}(w(\cdot)) \equiv v(\cdot), \qquad \mathcal{M}_b^{u,\gamma}(w(\cdot)) \equiv v_b(\cdot).$$

As shown in [29], the explicit forms of the generalized regulator mappings in Definition 4.1 can be given in terms of the conventional regulator maps as:

$$(\phi^{u,\gamma}, \psi^{u,\gamma})(w) = (\phi, \psi)(\mathcal{M}^{u,\gamma}(w)),$$
$$(\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w) = (\phi_b, \psi_{1,b}, \psi_{2,b})(\mathcal{M}_b^{u,\gamma}(w)), \qquad (4.2)$$

where $w \in \mathcal{D}([0, \infty), \mathbb{R})$ is as given in Definition 4.1 The properties of the two-sided regulator map described below are generalizations of the work of [29].

The following proposition provides some properties of the regulator maps described above. Most of the properties are described in Proposition 4.1 of [29], but we state it in a form that is convenient for our proofs, and a short outline of the proofs of these provided along the lines of those in [29].

Proposition 4.2. *Let w and w_n, $n \geq 1$ be as in Definition 4.1, and let $\gamma_n > 0$, $b_n > 0$, $\gamma > 0$, $b > 0$ be such that $\gamma_n \to \gamma$ and $b_n \to b$ as $n \to \infty$. Also assume that the function u and the sequence of functions $\{u_n\}$ are non-negative uniformly Lipschitz continuous (with the same Lipschitz constant κ_u) and satisfies $\|u_n - u\|_\infty \equiv \sup_{x \in \mathbb{R}} |u_n(x) - u(x)| \to 0$, as $n \to \infty$. Then for some universal constant $\tilde{c} > 0$, the following holds for all $T > 0$:*
(a) *There exists $n_0 \geq 1$ such that for $n \geq n_0$,*

$$\|\phi^{u_n,\gamma_n}(w)\|_T \leq \tilde{c}\|w\|_T.$$

$$\|\psi_{2,b_n}^{u_n,\gamma_n}(w)\|_T \leq \tilde{c}\left(\|w\|_T + \sup_{0 \leq t \leq T} |\psi_{2,b_n}(w)(t) - \psi_{2,b_n}(w)(t-)|\right).$$

(b) *If for some $C_1 > 0$, $\sup_{n \geq 1} \|w_n(t) - w_n(t-)\|_T < C_1$ then $\lim_{n \to 0} \|w_n - w\|_T = 0$ implies*

$$\lim_{n \to 0} \|\phi^{u_n,\gamma_n}(w_n) - \phi^{u,\gamma}(w)\|_T \vee \|\psi^{u_n,\gamma_n}(w_n) - \psi^{u,\gamma}(w)\|_T = 0,$$
$$\lim_{n \to 0} \|\phi_{b_n}^{u_n,\gamma_n}(w_n) - \phi_b^{u,\gamma}(w)\|_T \vee \|\psi_{1,b_n}^{u_n,\gamma_n}(w_n) - \psi_{1,b}^{u,\gamma}(w)\|_T \vee \|\psi_{2,b_n}^{u_n,\gamma_n}(w_n)$$
$$- \psi_{2,b}^{u,\gamma}(w)\|_T = 0.$$

In other words, part (b) states that for $n \to \infty$, if $w_n \to w$ uniformly on compacts (u.o.c.), $u_n \to u$ uniformly on $\mathbb{R}$, $\gamma_n \to \gamma$ and $b_n \to b$, then $(\phi^{u_n,\gamma_n}, \psi^{u_n,\gamma_n})(w_n) \to (\phi^{u,\gamma}, \psi^{u,\gamma})(w)$ u.o.c., and if in addition $\sup_{n \geq 1} \|w_n(t) - w_n(t-)\|_T < C_1$ for all $T > 0$, then $(\phi_{b_n}^{u_n,\gamma_n}, \psi_{1,b_n}^{u_n,\gamma_n}, \psi_{2,b_n}^{u_n,\gamma_n})(w_n) \to (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(w)$ u.o.c.

Proof. First note that from the definition of $\mathcal{M}^{u_n,\gamma_n}$ and $\mathcal{M}_{b_n}^{u_n,\gamma_n}$ in (4.1) and the fact that $u_n \geq 0$, it follows that

$$\mathcal{M}^{u_n,\gamma_n}(w)(t) \leq w(t), \quad \text{and} \quad \mathcal{M}_{b_n}^{u_n,\gamma_n}(w)(t) \leq w(t), \quad \forall\, t \geq 0.$$

Hence, by a suitable monotonicity property of the conventional regulator maps in (4.2) (see [17]) and using the fact that the difference between $w(\cdot)$ and the functions on the left side of the inequalities are nondecreasing, we obtain that for all $n \geq 1$,

$$0 \leq \phi^{u_n,\gamma_n}(w) \equiv \phi(\mathcal{M}^{u_n,\gamma_n}(w)) \leq \phi(w),$$
$$0 \leq \psi_{2,b_n}^{u_n,\gamma_n}(w) \equiv \psi_{2,b_n}(\mathcal{M}_{b_n}^{u_n,\gamma_n}(w)) \leq \psi_{2,b_n}(w).$$

The first part of (a) follows from the Lipschitz continuity of the conventional regulator map $\phi(\cdot)$. For the second part of (a), let $Osc(x, [0, T]) = \sup_{0 \leq t_1 < t_2 \leq T} |x(t_2) - x(t_1)|$, for any $x \in \mathcal{D}([0, \infty), \mathbb{R})$. Then, following the proof of Proposition 4.1(ii)(c) in [29], we get that

$$Osc\left(\psi_{2,b_n}^{u_n,\gamma_n}(w), [0, T]\right) \leq \kappa \left(Osc(w, [0, T]) + \sup_{0 \leq t \leq T} |\psi_{2,b_n}(w)(t) - \psi_{2,b_n}(w)(t-)|\right),$$

for some constant $\kappa > 0$ (possibly depending on b, but not on n) and for $n \geq n_0$. Note that $\psi_{2,b_n}^{u_n,\gamma_n}(w)$ is a nondecreasing function with $\psi_{2,b_n}^{u_n,\gamma_n}(w)(0) = 0$ which implies that $Osc\left(\psi_{2,b_n}^{u_n,\gamma_n}(w), [0, T]\right) = \psi_{2,b_n}^{u_n,\gamma_n}(w)(T)$. Since $Osc(x, [0, T]) \leq 2\|x\|_T$, for any $x \in \mathcal{D}([0, \infty), \mathbb{R})$, the proof of the second part of (a) follows from the display above.

For part (b), let $v \equiv \mathcal{M}^{u,\gamma}(w)$, $v_n \equiv \mathcal{M}^{u_n,\gamma_n}(w_n)$ satisfy the first equation of (4.1). Straightforward calculations yield that for all $t \in [0, T]$,

$$|v_n(t) - v(t)| \leq |w_n(t) - w(t)| + \int_0^t |u_n(\phi(v_n)(s)) - u(\phi(v)(s))|ds$$

$$+ \int_0^t |\gamma_n\phi(v_n)(s) - \gamma\phi(v)(s)|ds$$

$$\leq |w_n(t) - w(t)|$$

$$+ \int_0^t |u_n(\phi(v_n)(s)) - u(\phi(v_n)(s))|ds + \int_0^t |u(\phi(v_n)(s)) - u(\phi(v)(s))|ds$$

$$+ \int_0^t \gamma_n|\phi(v_n)(s) - \phi(v)(s)|ds + |\gamma_n - \gamma| \int_0^t |\phi(v)(s)|ds. \tag{4.3}$$

Hence, using the Lipschitz continuity of the conventional regulator map ϕ (with respect to the uniform norm on compacts, with Lipschitz constant 2) and the Lipschitz continuity of u with Lipschitz constant κ_u, we obtain

$$\|v_n - v\|_t \leq \|w_n - w\|_T + T\|u_n - u\|_\infty + \int_0^t \kappa_u|\phi(v_n)(s) - \phi(v)(s)|ds$$

$$+ c_1 \int_0^t |\phi(v_n)(s) - \phi(v)(s)|ds + T\|\phi(v)\|_T |\gamma_n - \gamma|$$

$$\leq \left[\|w_n - w\|_T + T\|u_n - u\|_\infty + T\|\phi(v)\|_T|\gamma_n - \gamma|\right]$$

$$+ 2(\kappa_u + c_1) \int_0^t \|v_n - v\|_s ds, \tag{4.4}$$

where $c_1 = \sup_{n \geq 1}\{\gamma_n\}$. Thus, by Gronwall's inequality, we have for all $T > 0$

$$\|v_n - v\|_T \leq \left[\|w_n - w\|_T + T \|u_n - u\|_\infty + T\|\phi(v)\|_T|\gamma_n - \gamma|\right] e^{-2(\kappa_u+c_1)T}. \tag{4.5}$$

Hence if $\|w_n - w\|_T \to 0$ as $n \to \infty$, then $\|\mathcal{M}^{u_n,\gamma_n}(w_n) - \mathcal{M}^{u,\gamma}(w)\|_T \equiv \| v_n - v \|_T \to 0$ as $n \to \infty$. Thus, by the continuity of the maps ϕ, ψ along with the representations of regulator maps given in (4.2) concludes the proof of the first part of (b).

For the second half of part (b), we need some more properties of the two-sided regulator maps. First, notice that for any real numbers x, y, z we have $|x^+ \wedge z - y^+ \wedge z| \leq |x - y|$, where a^+ represents $\max(0, a)$. Using this, and the explicit form of ϕ_b in [17] (e.g., using (1.11)–(1.13)

in [17]), it follows that for any $x \in \mathcal{D}([0, \infty), \mathbb{R})$,

$$|\phi_{b_n}(x)(t) - \phi_b(x)(t)| \leq |b_n - b| \quad \text{for all } t \geq 0. \tag{4.6}$$

Using (4.6) and calculations similar to (4.3)–(4.5), we get

$$\|\mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n) - \mathcal{M}_b^{u, \gamma}(w)\|_T \leq \Big[\|w_n - w\|_T + T\|u_n - u\|_\infty + T\|\phi_b(v)\|_T|\gamma_n - \gamma|$$
$$+ (\kappa_u + c_1)T|b_n - b|\Big]e^{-\kappa_b(\kappa_u + c_1)T}, \tag{4.7}$$

where $\kappa_b > 0$ is the Lipschitz constant for the Lipschitz continuous map ϕ_b (see Theorem 14.8.1 of [31]). This implies that if $\|w_n - w\|_T \to 0$ as $n \to \infty$, then

$$\|\mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n) - \mathcal{M}_b^{u, \gamma}(w)\|_T \to 0, \quad \text{as } n \to \infty. \tag{4.8}$$

Since $\lim_{n \to \infty} |b_n - b| = 0$ and $\lim_{n \to \infty} \|w_n - w\|_T = 0$, using (4.2), (4.6) and (4.8), we obtain

$$\|\phi_{b_n}^{u_n, \gamma_n}(w_n) - \phi_b^{u, \gamma}(w)\|_T$$
$$\leq |b_n - b| + \|\phi_b\left(\mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n)\right) - \phi_b\left(\mathcal{M}_b^{u, \gamma}(w)\right)\|_T \to 0, \tag{4.9}$$

as $n \to \infty$.

Let $\delta > 0$ be a constant, $y \in \mathcal{D}([0, \infty), \mathbb{R})$ and $\delta y \in \mathcal{D}([0, \infty), \mathbb{R})$ be the function $(\delta y)(t) = \delta y(t)$ for $t \geq 0$. Observe that from the scaling properties of two-sided regulator maps (follow from the characterizing properties of these maps) it follows that

$$\big(\phi_{\delta b}, \psi_{1, \delta b}, \psi_{2, \delta b}\big)(\delta y) = \big(\delta \phi_b, \delta \psi_{1, b}, \delta \psi_{2, b}\big)(y), \tag{4.10}$$

and consequently, for any $y \in \mathcal{D}([0, \infty), \mathbb{R})$,

$$\|\psi_{2, b}(y) - \psi_{2, \delta b}(\delta y)\|_T \leq |\delta - 1| \|\psi_{2, b}(y)\|_T.$$

Hence, by choosing $\delta_n = b/b_n$, from (4.2), we obtain

$$\|\psi_{2, b_n}^{u_n, \gamma_n}(w_n) - \psi_{2, b}^{u, \gamma}(w)\|_T \leq \|\psi_{2, b_n}\left(\mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n)\right) - \psi_{2, b_n \delta_n}\left(\delta_n \mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n)\right)\|_T$$
$$+ \|\psi_{2, b_n \delta_n}\left(\delta_n \mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n)\right) - \psi_{2, b}\left(\mathcal{M}_b^{u, \gamma}(w)\right)\|_T$$
$$\leq |\delta_n - 1| \|\psi_{2, b_n}^{u_n, \gamma_n}(w_n)\|_T$$
$$+ \|\psi_{2, b}\left(\delta_n \mathcal{M}_{b_n}^{u_n, \gamma_n}(w_n)\right) - \psi_{2, b}\left(\mathcal{M}_b^{u, \gamma}(w)\right)\|_T$$
$$\to 0, \quad \text{since } \delta_n \to 1 \text{ as } n \to \infty. \tag{4.11}$$

In the estimation of the right hand side, we have also used $\psi_{2, b_n}^{u_n, \gamma_n}(w_n) = \psi_{2, b_n}\left(\mathcal{M}_{b_n}^{u_n, \gamma_n}\right)(w_n)$, the estimate in the second half of part (a) and the assumed properties on "jump sizes" of w_n; hence, the first term converges to zero. The second term converges to zero using (4.8), the fact that $\delta_n \to 1$ and the continuity of the map $\psi_{2, b}$. Now, from the definition of the two-sided regulator maps, it follows that as $n \to \infty$

$$\|\psi_{1, b_n}^{u_n, \gamma_n}(w_n) - \psi_{1, b}^{u, \gamma}(w)\|_T \leq \|w_n - w\|_T + \|\phi_{b_n}^{u_n, \gamma_n}(w_n) - \phi_b^{u, \gamma}(w)\|_T$$
$$+ \|\psi_{2, b_n}^{u_n, \gamma_n}(w_n) - \psi_{2, b}^{u, \gamma}(w)\|_T \to 0, \tag{4.12}$$

when $\|w_n - w\|_T \to 0$, using (4.9) and (4.11). The proof of the second half of part (b) follows from (4.9), (4.11) and (4.12). $\quad \square$

4.2. Weak convergence analysis.

In this section, we prove the main theorem and other necessary results involving the processes introduced in Sections 2 and 3. We begin this section by giving alternative representations of such processes using the generalized regulator maps and define few other associated processes.

Using the results in Section 3 and the definition of the regulator processes in Definition 4.1, the solution of the BCP can be expressed as follows:

$$(X_x^*, L^*, U^*) = (X_{p,x}^*, L_p^*, U_p^*) = (\phi_{b^*}^{u^*,\gamma}, \psi_{1,b^*}^{u^*,\gamma}, \psi_{2,b^*}^{u^*,\gamma})(W_x), \tag{4.13}$$

where $W_x = x + \sigma W$ is a Brownian motion starting from $x \geq 0$ with zero drift and variance $\sigma^2 = 2\lambda$ and W is a standard Brownian motion as in (3.1). When the reference to value of the parameter p is not important, we simply identify $(X_x^*, L^*, u^*, U^*, b^*)$ as $(X_{p,x}^*, L_p^*, u_p^*, U_p^*, b_p^*)$. We first state a general result about alternative representations of our discounted cost functions (see Lemma 4.2 of [29] for a similar result). Here the scaled processes are the ones defined in Section 2.3.

Lemma 4.3. *Let* $\widetilde{J}_p(x, u, U)$ *and* $J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ *be as defined in* (3.4) *and* (2.14) *, respectively.*

(a) *For any admissible policy* (u, U) *for the BCP defined in Definition* 3.1*, we have*

$$\widetilde{J}_p(x, u, U) = E\left(\int_0^\infty \delta e^{-\delta t}\left\{\beta\gamma\int_0^t X_x(s)\mathrm{d}s + \int_0^t C(u(s))\mathrm{d}s + p\, U(t)\right\}\mathrm{d}t\right).$$

(b) *For any admissible control* $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ *for the queueing system (see Definition* 2.2*), we have*

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) = \liminf_{n\to\infty} E\left(\int_0^\infty \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n(s)\mathrm{d}s \right.\right.$$

$$\left.\left. + \int_0^t C(u_n(\hat{Q}_n(s)))\mathrm{d}s + p\hat{U}_n(t)\right\}\mathrm{d}t\right).$$

Proof. Note that for all $t \geq 0$,

$$e^{-\delta t} = \int_t^\infty \delta e^{-\delta s}\mathrm{d}s = \int_{\mathbb{R}} I_{[t,\infty)}(s)\delta e^{-\delta s}\mathrm{d}s. \tag{4.14}$$

From (4.14) and the non-negativity of all the integrands below, we can interchange the order of integration using Fubini–Tonelli's theorem, and consequently we obtain

$$\int_0^\infty e^{-\delta t}[\{\beta\gamma X_x(t) + C(u(t))\}\mathrm{d}t + p\mathrm{d}U(t)]$$

$$= \int_0^\infty\int_0^\infty I_{[t,\infty)}(s)(\delta e^{-\delta s})[\{\beta\gamma X_x(t) + C(u(t))\}\mathrm{d}t + p\mathrm{d}U(t)]\mathrm{d}s$$

$$= \int_0^\infty \delta e^{-\delta s}\left[\int_0^s \{\beta\gamma X_x(t) + C(u(t))\}\mathrm{d}t + p\, U(s)\right]\mathrm{d}s.$$

This proves part (a). Similar calculation yields part (b) as well. □

Next we define the following time-change processes: For each $n \geq 1$ and $t \geq 0$, we let

$$\tau_n^A(t) \equiv \int_0^t \bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s))\mathrm{d}s, \qquad \tau_n^S(t) \equiv \int_0^t \bar{\mu}_n(\sqrt{n}\hat{Q}_n(s))\mathrm{d}s,$$

$$\tau_n^R(t) \equiv \int_0^t \gamma_n \sqrt{n}\, \hat{Q}_n(s)\mathrm{d}s, \tag{4.15}$$

where $\bar{\lambda}_n(x) = \lambda_n(x)I_{\{x < \sqrt{n}b\}}$, $\bar{\mu}_n(x) = \mu_n(x)I_{\{x > 0\}}$ are as in Section 2. Also define

$$\hat{M}_n^A(t) \equiv \hat{Y}_n^A(\tau_n^A(t)), \qquad \hat{M}_n^S(t) \equiv \hat{Y}_n^S(\tau_n^S(t)), \quad \text{and} \quad \hat{M}_n^R(t) \equiv \hat{Y}_n^R(\tau_n^R(t)). \tag{4.16}$$

Then from (2.13), we have the following alternative representation of $\hat{W}_n$:

$$\hat{W}_n(t) = \hat{M}_n^A(t) - \hat{M}_n^S(t) - \hat{M}_n^R(t), \quad n \geq 1, t \geq 0. \tag{4.17}$$

Using the existence, uniqueness and other properties of the generalized regulator maps in Definition 4.1, we obtain that for any admissible control $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$, the associated processes in the queueing system have the following representation. For $n \geq 1$, recalling that $b_n = \frac{\lfloor \sqrt{n}b \rfloor}{\sqrt{n}}$, we have

$$(\hat{Q}_n, \hat{L}_n, \hat{U}_n) = \left(\phi_{b_n}^{u_n, n\gamma_n}, \psi_{1,b_n}^{u_n, n\gamma_n}, \psi_{2,b_n}^{u_n, n\gamma_n} \right)(\hat{W}_n), \quad \text{if } b < \infty,$$

$$(\hat{Q}_n, \hat{L}_n) = (\phi^{u_n, n\gamma_n}, \psi^{u_n, n\gamma_n})(\hat{W}_n), \quad \text{if } b = \infty. \tag{4.18}$$

We also define the following *fluid scaled* version of the processes: For $n \geq 1$, $t \geq 0$, let

$$\bar{Q}_n(t) \doteq \frac{1}{n}Q_n(nt) = \frac{1}{\sqrt{n}}\hat{Q}_n(t), \qquad \bar{L}_n(t) \doteq \frac{1}{n}L_n(nt) = \frac{1}{\sqrt{n}}\hat{L}_n(t),$$

$$\bar{U}_n(t) \doteq \frac{1}{n}U_n(nt) = \frac{1}{\sqrt{n}}\hat{U}_n(t), \quad \text{and} \quad \bar{W}_n(t) = \frac{1}{\sqrt{n}}\hat{W}_n(t). \tag{4.19}$$

For each $n \geq 1$ and $x \geq 0$, we let $\bar{u}_n(x) = \frac{u_n(\sqrt{n}x)}{\sqrt{n}}$. By Definition 2.2, we deduce that

$$\|\bar{u}_n\|_\infty = \sup_{x \geq 0} |\bar{u}_n(x)| \to 0, \quad \text{as } n \to 0. \tag{4.20}$$

Hence, from (4.19) and (2.12), we have

$$\bar{Q}_n(t) = \frac{1}{\sqrt{n}}\hat{Q}_n(t)$$

$$= \bar{W}_n(t) - \int_0^t [\bar{u}_n(\bar{Q}_n(s)) + (n\gamma_n)\bar{Q}_n(s)]\mathrm{d}s + \bar{L}_n(t) - \bar{U}_n(t), \quad t \geq 0. \tag{4.21}$$

From the properties of the regulator maps in Definition 4.1 and (4.19), it follows that

$$(\bar{Q}_n, \bar{L}_n, \bar{U}_n) = \left(\phi_{b_n}^{\bar{u}_n, n\gamma_n}, \psi_{1,b_n}^{\bar{u}_n, n\gamma_n}, \psi_{2,b_n}^{\bar{u}_n, n\gamma_n} \right)(\bar{W}_n), \quad \text{if } b < \infty,$$

$$(\bar{Q}_n, \bar{L}_n) = (\phi^{\bar{u}_n, n\gamma_n}, \psi^{\bar{u}_n, n\gamma_n})(\bar{W}_n), \quad \text{if } b = \infty. \tag{4.22}$$

The following representation also follows from (4.15) and (4.22):

$$\bar{Q}_n(t) = \bar{W}_n(t) + [\tau_n^A(t) - \tau_n^S(t) - \tau_n^R(t)] + \bar{L}_n(t) - \bar{U}_n(t),$$
$$\text{for all } t \geq 0, \ n \geq 1. \tag{4.23}$$

Proposition 4.4. *Let* $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ *be an admissible control policy (as in Definition 2.2) for the queueing system. Let* $\tau_n = (\tau_n^A, \tau_n^S, \tau_n^R), n \geq 1$ *and* $\tau = (\lambda e, \lambda e, 0)$, *where* $\tau_n^A, \tau_n^S, \tau_n^R$ *are as in* (4.15), $e(t) \equiv t, t \geq 0$ *is the identity function and 0 denotes the function that is identically zero. Then,*

(a) $\lim_{n \to \infty} \sup_{t \in [0,T]} \|\tau_n(t) - \tau(t)\| = 0$ *a.s. as* $n \to \infty$, *for all* $T > 0$.

(b) $\hat{W}_n \Rightarrow W_0$ *as* $n \to \infty$, *where* W_0 *is a Brownian motion starting from zero and has infinitesimal mean and variance 0 and* 2λ, *respectively.*

(c) *If* $b < \infty$, *we let* $(X_0, L, U) \doteq (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(W_0)$. *In the case of* $b = \infty$, *we define* $(X_0, L, U) \doteq (\phi^{u,\gamma}(W_0), \psi^{u,\gamma}(W_0), 0)$. *Then in both cases,*

$$(\hat{Q}_n, \hat{L}_n, U_n) \Rightarrow (X_0, L, U) \quad as \ n \to \infty, \tag{4.24}$$

and (X_0, u, U) *is admissible for the BCP with the initial value* $x = 0$ *(see Definition 3.1).*

(d) *There exists a constant* $\bar{c} > 0$, *such that for all* $n \geq 1$ *and* $T > 0$

$$E\left[\sup_{0 \leq t \leq T} |\hat{W}_n(t)|^2\right] \leq \bar{c}(T^2 + T).$$

Proof. We begin by proving part (a). As we show below, the proofs of parts (b), (c) and (d) follow from part (a). The main steps for the proof of part (a) are : we first bound the time-change processes τ_n using the functional strong law of large numbers (see (4.27) and (4.28) below). Then, this bound together with the Martingale structure of $\bar{W}_n$ implies that $\bar{W}_n \to 0$ almost surely, u.o.c. (see (4.29)). With the help of the properties of the generalized regulator maps, we complete the proof of part (a) (see (4.32)–(4.34) below).

Fix $T > 0$. Note that from (2.1), we have

$$Q_n(t) \leq Y_n^A\left(\int_0^t \bar{\lambda}_n(Q_n(s))ds\right), \quad \text{for all } n \geq 1, t \geq 0,$$

where Y_n^A is as defined in (2.1). Hence, by (4.19) and (4.15), we obtain

$$0 \leq \bar{Q}_n(t) \leq \frac{Y_n^A(n\tau_n^A(t))}{n}, \quad \text{for all } n \geq 1, t \geq 0. \tag{4.25}$$

Note that by the functional law of large numbers for Poisson process (with intensity 1), it follows that for large n ($n \geq n_0 \equiv n_0(\omega)$),

$$t - 1 \leq \frac{Y_n^A(nt)}{n} \leq t + 1, \quad \text{for all } t \in [0, cT], \tag{4.26}$$

where c is as in (2.5). Observe that $\tau_n^A(t) \leq ct \leq cT$ for all $t \in [0, T]$. Hence, by (4.25) and (4.26) we derive the following bound for $n \geq n_0$,

$$0 \leq \bar{Q}_n(t) \leq c(t + 1), \quad \text{for all } t \in [0, T].$$

Since $\sup_{n \geq 1}\{n\gamma_n\} < \infty$ (by Assumption 2.1), we get

$$\tau_n^R(t) = (n\gamma_n)\int_0^t \bar{Q}_n(s)ds \leq c_1(t + 1)^2 \quad \text{for all } t \in [0, T], \tag{4.27}$$

where $c_1 > 0$ is a generic constant which is independent of n and T. Using (2.5) and (4.15), we also obtain

$$\tau_n^A(t) \le ct, \quad \text{and} \quad \tau_n^S(t) \le ct, \quad \text{for all } t \in [0, T]. \tag{4.28}$$

By the functional strong law of large numbers for any sequence of unit intensity independent Poisson processes $\{Y_n\}$, we have

$$\sup_{0 \le t \le T} \left| \frac{Y_n(nt)}{n} - t \right| \to 0 \quad \text{a.s., as } n \to \infty.$$

The bounds in (4.27)–(4.28) together with (4.16) yields that for all $T > 0$, the following almost sure convergence results hold.

$$0 \le \frac{\sup\limits_{0 \le t \le T} |\hat{M}_n^A(t)|}{\sqrt{n}} \le \sup_{0 \le t \le cT} \left| \frac{Y_n^A(nt)}{n} - t \right| \to 0 \quad \text{a.s., as } n \to \infty,$$

$$0 \le \frac{\sup\limits_{0 \le t \le T} |\hat{M}_n^S(t)|}{\sqrt{n}} \le \sup_{0 \le t \le cT} \left| \frac{Y_n^S(nt)}{n} - t \right| \to 0 \quad \text{a.s., as } n \to \infty,$$

$$0 \le \frac{\sup\limits_{0 \le t \le T} |\hat{M}_n^R(t)|}{\sqrt{n}} \le \sup_{0 \le t \le c_1 T(T+1)} \left| \frac{Y_n^R(nt)}{n} - t \right| \to 0 \quad \text{a.s., as } n \to \infty.$$

Consequently, by (4.17) and (4.19) we have

$$\bar{W}_n = \frac{1}{\sqrt{n}} [\hat{M}_n^A - \hat{M}_n^S - \hat{M}_n^R] \to 0 \quad \text{a.s., as } n \to \infty, \tag{4.29}$$

and this convergence is uniform on compact sets. Note that from the definition of the $\bar{W}_n$ (which involves the sum of the three fluid scaled time-changed Poisson processes), it follows that

$$\sup_{0 \le t \le T} |\bar{W}_n(t) - \bar{W}_n(t-)| \le \frac{3}{n} \quad \text{for all } t \ge 0, n \ge 1. \tag{4.30}$$

Hence, we can use (4.29), (4.20) and (4.22) together with the continuity properties of the generalized regulator maps established in Proposition 4.2(b) to conclude that

$$\bar{Q}_n \to 0, \qquad \bar{L}_n \to 0, \quad \text{and} \quad \bar{U}_n \to 0 \quad \text{a.s., as } n \to \infty, \tag{4.31}$$

and this convergence is uniform on compact sets. Note that $b = \infty$ will correspond to representations of the above processes using one-sided generalized maps in (4.22) and $\bar{U}_n \equiv 0$. Hence

$$\sup_{0 \le t \le T} \left| \tau_n^A(t) - \lambda t \right|$$

$$\le \sup_{0 \le t \le T} \left[\int_0^t |\bar{\lambda}_n(\sqrt{n}\hat{Q}_n(s)) - \lambda_n(\sqrt{n}\hat{Q}_n(s))| ds + \int_0^t |\lambda_n(\sqrt{n}\hat{Q}_n(s)) - \lambda| ds \right]$$

$$\le \lambda_n(\sqrt{n}b) \int_0^t I_{\{\hat{Q}_n(s) \ge b\}} ds + T \left[\sup_{x \ge 0} |\lambda_n(x) - \lambda| \right]$$

$$\le \bar{U}_n(T) + T \left[\sup_{x \ge 0} |\lambda_n(x) - \lambda| \right] \to 0 \quad \text{a.s., as } n \to \infty,$$

by (2.11), (4.31) and (2.2). This proves that

$$\tau_n^A \to \lambda e \quad \text{a.s., as } n \to \infty, \tag{4.32}$$

uniformly on compact sets. A similar argument can be used to prove that

$$\tau_n^S \to \lambda e \quad \text{a.s., as } n \to \infty, \tag{4.33}$$

uniformly on compact sets. Also observe that from (4.23), one has

$$\tau_n^R(t) = \bar{W}_n(t) - \bar{Q}_n(t) + [\tau_n^A(t) - \tau_n^S(t)] + \bar{L}_n(t) - \bar{U}_n(t).$$

Hence, (4.29) and (4.31)–(4.33) together yield

$$\tau_n^R \to 0 \quad \text{a.s., as } n \to \infty, \tag{4.34}$$

and this convergence is uniform on compact sets. This completes the proof of part (a).

For part (b), observe that from the functional central limit theorem for Poisson processes:

$$(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R) \Rightarrow (W^A, W^S, W^R) \quad \text{as } n \to \infty, \tag{4.35}$$

where W^A, W^S, W^R are three independent standard Brownian motions with mean 0 and variance t. We can use (4.16), part (a) above and the random-time-change theorem (see Section 14 of [7]) to obtain

$$
\begin{aligned}
(\hat{M}_n^A(\cdot), \hat{M}_n^S(\cdot), \hat{M}_n^R(\cdot)) &\equiv (\hat{Y}_n^A(\tau_n^A(\cdot)), \hat{Y}_n^S(\tau_n^S(\cdot)), \hat{Y}_n^R(\tau_n^R(\cdot))) \\
&\Rightarrow (W^A(\lambda\cdot), W^S(\lambda\cdot), 0),
\end{aligned}
\tag{4.36}
$$

as $n \to \infty$. Here, we also use the continuity of the weak limit of $(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^S)$ and the sum (and the difference) is a continuous map on the space of continuous functions. Hence, from (4.36) and the continuous mapping theorem, we obtain

$$\hat{W}_n(\cdot) = \hat{M}_n^A(\cdot) - \hat{M}_n^S(\cdot) - \hat{M}_n^R(\cdot) \Rightarrow W^A(\lambda\cdot) - W^S(\lambda\cdot) \quad \text{as } n \to \infty.$$

Notice that, if we define $W_0(\cdot) \doteq W^A(\lambda\cdot) - W^S(\lambda\cdot)$, then by the independence of W^A and W^S, W_0 is a Brownian motion starting from 0 and has mean 0, variance $2\lambda t$. The proof of (b) is now complete.

To prove part (c), note that from part (b) we have

$$\hat{W}_n \Rightarrow W_0 \quad \text{as } n \to \infty.$$

The weak limit above is continuous and the space of all continuous functions is separable. Hence, by Skorokhod representation theorem (Theorem 6.7 in [7]), one can assume that the above convergence takes place almost surely between $\{\hat{W}_n'\}$, W_0' defined on some common probability space and $(\{W_n'\}, W_0')$ has the same law as $(\{W_n\}, W_0)$. Denoting these new elements by $(\{W_n\}, W_0)$ again (to simplify notation), we have the following convergence uniformly on compact sets.

$$W_n \to W_0 \quad \text{a.s., as } n \to \infty.$$

In the case of $b < \infty$, note that being a sum of three diffusion scaled Poisson processes, we have

$$\sup_{0 \le t \le T} |\hat{W}_n(t) - \hat{W}_n(t-)| \le \frac{3}{\sqrt{n}} \quad \text{for all } t \ge 0, n \ge 1. \tag{4.37}$$

Hence, by (4.18) and Proposition 4.2(b) we obtain,

$$(\hat{Q}_n, \hat{L}_n, \hat{U}_n) = (\phi_{b_n}^{u_n, n\gamma_n}, \psi_{1,b_n}^{u_n, n\gamma_n}, \psi_{2,b_n}^{u_n, n\gamma_n})(\hat{W}_n) \rightarrow (\phi_b^{u,\gamma}, \psi_{1,b}^{u,\gamma}, \psi_{2,b}^{u,\gamma})(W_0)$$
$$\doteq (X_0, L, U) \quad \text{a.s.,}$$

as $n \rightarrow \infty$. When $b = \infty$, with the same reasoning, we have

$$(\hat{Q}_n, \hat{L}_n) = (\phi^{u_n, n\gamma_n}, \psi^{u_n, n\gamma_n})(\hat{W}_n) \rightarrow (\phi^{u,\gamma}, \psi^{u,\gamma})(W_0)$$
$$\doteq (X_0, L) \quad \text{a.s., as } n \rightarrow \infty. \tag{4.38}$$

Both of these convergence results hold uniformly on compact sets. Therefore, we can conclude that for each $b \in (0, \infty]$

$$(\hat{Q}_n, \hat{L}_n, U_n) \Rightarrow (X_0, L, U) \quad \text{as } n \rightarrow \infty,$$

with the convention that $\hat{U}_n = U \equiv 0$ if $b = \infty$. By the properties of the regulator maps in Definition 4.1 and the properties of W_0 in part (b), it is clear that the weak limit (X_0, L, U) satisfies the properties of the corresponding processes of the BCP (see (3.1)). Hence we conclude that the limit (X_0, u, U) is admissible for the BCP as required in Definition 3.1, and the proof of part (c) is complete.

Now we prove part (d). First observe that $\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R$ defined in (2.10) are scaled compensated Poisson processes, and hence these processes are martingales. So, by Doob's maximal inequality (Corollary 2.17 of Chapter 2 of [10]), we get for $T > 0$,

$$E\left[\sup_{0 \le t \le T} |\hat{Y}_n^A(t)|\right]^2 \le 4E\left[|\hat{Y}_n^A(T)|^2\right] = 4T.$$

Hence by (4.16) and (4.28), for all $T > 0$ the following estimate holds.

$$E\left[\sup_{0 \le t \le T} |\hat{M}_n^A(t)|\right]^2 \le 4cT. \tag{4.39}$$

Similar calculations involving $\hat{Y}_n^S, \hat{Y}_n^R$, with (4.16), (4.28) and (4.27) yield

$$E\left[\sup_{0 \le t \le T} |\hat{M}_n^S(t)|\right]^2 \le 4cT, \quad \text{and} \quad E\left[\sup_{0 \le t \le T} |\hat{M}_n^R(t)|\right]^2 \le 2c_1(T + 1)^2. \tag{4.40}$$

Hence, from the definition of $\hat{W}_n$ in (4.17) together with (4.39), (4.40) and the fact that $(a + b + c) \le 3(a^2 + b^2 + c^2)$, we obtain that

$$E\left[\sup_{0 \le t \le T} |\hat{W}_n(t)|\right]^2 \le 3\left(4cT + 4cT + 2c_1(T + 1)^2\right) \le C(T + 1)^2, \quad \text{for all } T > 0,$$

where $C > 0$ is a generic constant independent of n and T. This completes the proof of part (d), and that of the proposition. $\quad \square$

Theorem 4.5. *Let $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ be a proposed candidate for optimal policy as given in Definition 2.7. Then,*

(a) $(\hat{W}_n^*, \hat{Q}_n^*, \hat{L}_n^*, U_n^*) \Rightarrow (W_0, X_0^*, L^*, U^*)$ as $n \to \infty$ where W_0 is a standard Brownian motion starting from zero and (X_0^*, L^*, U^*) are the processes associated with the solution of the BCP with W_0 and the initial point $x = 0$, as in (3.30). Here if $b^* = \infty$, then $\hat{U}_n^* = U^* \equiv 0$ and the processes X_0^* and L^* are as described in (3.31).

(b) $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = V_p(0)$ where $V_p(x)$ represents the value function defined in (3.5).

Remark 4.6. 1. Note that any sequence $\{b_n\}$ such that $\lim_{n\to\infty} b_n = b^*$ defines an asymptotically optimal policy, since the proof of Proposition 4.2 and the consequent results hold for any convergent sequence $\{b_n\}$.

2. In part (a) of the above theorem, for $(\hat{W}_n^*, \hat{Q}_n^*, \hat{L}_n^*, U_n^*)$, we use an additional superscript $*$ to our notation of the queueing system processes to emphasize that these processes are obtained by using the proposed policy in Definition 2.7. Also, in part (b), for $(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$, $J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*)$ turned out to be the limit of the right side of (2.14) (instead of the lim inf in (2.14)).

Proof. Part (a) follows directly from part(c) of Proposition 4.4. We now prove part (b) using part (a). The proof is different for the different values of the cost parameter p, and is described separately in two cases.

Case I: $0 < p < p_0$. This case leads to an optimal finite buffer size $b^* < \infty$ as in Theorem 3.8. Note that by Assumption 2.1 and continuous mapping theorem (for the map $\eta(x)(t) = \int_0^t x(s)ds, \ t \geq 0, \ x \in \mathcal{D}([0, \infty), [0, \infty))$ and under uniform convergence on compacts), we obtain

$$\beta(n\gamma_n) \int_0^\cdot \hat{Q}_n^*(s)ds \Rightarrow \beta\gamma \int_0^\cdot X_0^*(s)ds \quad \text{a.s., as } n \to \infty, \tag{4.41}$$

uniformly on compact sets. Also note that since $b^* < \infty$, $0 \leq \int_0^t \hat{Q}_n^*(s)ds \leq b^*t$ for all $t \geq 0$ and (4.41) implies that for each $t \geq 0$,

$$r_1^n(t) \equiv \beta(n\gamma_n)E\left[\int_0^t \hat{Q}_n^*(s)ds\right] \to \beta\gamma E\left[\int_0^t X_0^*(s)ds\right] \quad \text{as } n \to \infty. \tag{4.42}$$

Also, using Cauchy–Schwarz inequality, we derive for $c_1 = \max_{n\geq 1}\{n\gamma_n\} < \infty$, that

$$0 \leq \int_0^\infty \delta e^{-\delta t}\left[r_1^n(t)\right]^2 dt \leq [\beta^2 c_1^2 b^{*2}]\int_0^\infty \delta e^{-\delta t}t^2 dt < \infty,$$

and this bound on the right side does not involve n. Hence, we have established a sufficient condition (see (3.18) of Section 3 of [7]) for the uniform integrability to conclude (from (4.42)) that

$$\lim_{n\to\infty} E\int_0^\infty \delta e^{-\delta t}\left\{\beta(n\gamma_n)\int_0^t \hat{Q}_n^*(s)ds\right\}dt = \lim_{n\to\infty}\int_0^\infty \delta e^{-\delta t}r_1^n(t)dt$$

$$= \int_0^\infty \delta e^{-\delta t}E\left[\beta\gamma \int_0^t X_0^*(s)ds\right]dt$$

$$= E\int_0^\infty \delta e^{-\delta t}\left\{\beta\gamma \int_0^t X_0^*(s)ds\right\}dt. \tag{4.43}$$

Also combining the admissibility of the proposed control, the fact that $u_n^* = u^*$ and the properties of the cost function $C(\cdot)$ in Assumption 2.4 and part (a) above together with the continuous

mapping theorem (as in (4.41)) we obtain

$$\int_0^{\cdot} C(u_n^*(\hat{Q}_n^*(s)))\mathrm{d}s \Rightarrow \int_0^{\cdot} C(u^*(X_0^*(s)))\mathrm{d}s \quad \text{as } n \to \infty. \tag{4.44}$$

For $t \geq 0$, let $r_2^n(t) \equiv E\left[\int_0^t C(u^*(X_0^*(s)))\mathrm{d}s\right]$. Then from the fact that $0 \leq X_0^*(s) \leq b^* < \infty$ for all $s \geq 0$, it similarly follows that

$$0 \leq \int_0^{\infty} \delta e^{-\delta t} \left[r_2^n(t)\right]^2 \mathrm{d}t \leq [c_2]^2 \int_0^{\infty} \delta e^{-\delta t} t^2 \mathrm{d}t < \infty, \tag{4.45}$$

and this upper bound is also independent of n. Here $c_2 = \sup_{y \in [0, \bar{u}]} C(y)$, where $\bar{u} = \sup_{x \in [0, b^*]} u^*(x)$. Following the same argument as in (4.43), we obtain

$$\lim_{n \to \infty} E \int_0^{\infty} \delta e^{-\delta t} \left\{\int_0^t C(u_n^*(\hat{Q}_n^*(s)))\mathrm{d}s\right\} \mathrm{d}t$$

$$= E \int_0^{\infty} \delta e^{-\delta t} \left\{\int_0^t C(u^*(X_0^*(s)))\mathrm{d}s\right\} \mathrm{d}t. \tag{4.46}$$

Note that $u_n^* \equiv u^* \geq 0$. Hence, using Assumption 2.1, (4.18), nondecreasing nature of $\hat{U}_n^*$ and the second bound in part (a) of Proposition 4.2(a), we have

$$0 \leq \hat{U}_n^*(t) = \|\hat{U}_n^*\|_t = \left\|\left|\psi_{2,b_n}^{u^*, n\gamma_n}\left(\hat{W}_n^*\right)\right|\right\|_t \tag{4.47}$$

$$\leq \tilde{c}\left(\|\hat{W}_n^*\|_T + \sup_{0 \leq t \leq T} |\psi_{2,b_n}(\hat{W}_n^*)(t) - \psi_{2,b_n}(\hat{W}_n^*)(t-)|\right),$$

$$\text{for all } t \geq 0, n \geq n_0. \tag{4.48}$$

Note that for $\hat{W}_n^*$, the estimate in (4.37) holds. Hence, we have (see display (97) in the proof of Proposition 4.3 in [29] for a similar estimate)

$$\sup_{0 \leq t \leq T} |\psi_{2,b_n}(\hat{W}_n^*)(t) - \psi_{2,b_n}(\hat{W}_n^*)(t-)| \leq \frac{3}{\sqrt{n}} \quad \text{for all } t \geq 0, n \geq 1.$$

Therefore, using (4.47) and part (d) of Proposition 4.4, we have for each $n \geq n_0, t \geq 0$,

$$E[\hat{U}_n^*(t)]^2 \leq 2\tilde{c}^2 \left(E\left[\sup_{0 \leq s \leq t} |\hat{W}_n^*(s)|\right]^2 + \frac{9}{n}\right) \leq [2\tilde{c}^2]\left(\bar{c}(t^2 + t) + 9\right). \tag{4.49}$$

With this upper bound and following the same approach as we used in establishing the convergence in (4.43) and (4.46), we obtain

$$r_3^n(t) \equiv pE[\hat{U}_n^*(t)] \to pE[U^*(t)], \quad \text{for all } t \geq 0 \text{ and } 0 < p < p_0. \tag{4.50}$$

Also, by (4.49) we have

$$\int_0^{\infty} \delta e^{-\delta t} \left[r_3^n(t)\right]^2 \mathrm{d}t \leq p^2[4\tilde{c}^2 \bar{c}] \int_0^{\infty} \delta e^{-\delta t} (t^2 + t)\mathrm{d}t < \infty$$

and this upper bound is free of n. Thus, using a similar calculation as in (4.43) and (4.46) above, we obtain

$$\lim_{n \to \infty} E \int_0^\infty \delta e^{-\delta t} \{p\, \hat{U}_n^*(t)\} dt = E \int_0^\infty \delta e^{-\delta t} \{p\, U^*(t)\} dt. \tag{4.51}$$

Using (4.43), (4.46), (4.51), definition of the cost function in (2.14), (3.4) and (3.5), the Lemma 4.3 and the fact that W_0, the weak limit of $\{\hat{X}_n^*\}$ is a standard Brownian motion starting at $x = 0$, we derive

$$J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \widetilde{J}_p(0, u^*, U^*) = V_p(0). \tag{4.52}$$

This completes the proof for the case $0 < p < p_0$.

Case II: $p \geq p_0$. This case leads to the optimality of the infinite buffer size $b^* = \infty$ (see Theorem 3.8). Hence, the proof of this case is somewhat straightforward, since

$$\hat{U}_n^* = U^* \equiv 0. \tag{4.53}$$

Hence the convergence of the last component of the cost function (the one dealt with in (4.51)) follows trivially. Since, $u_n^* \equiv u^* \geq 0$, using Assumption 2.1, (4.18) and the first bound in part (a) of Proposition 4.2, we obtain

$$0 \leq \sup_{0 \leq s \leq t} |\hat{Q}_n^*(s)| = \|\hat{Q}_n^*\|_t = \left\| \phi^{u^*, n\gamma_n} \left(\hat{W}_n^* \right) \right\|_t \leq \tilde{c} \sup_{0 \leq s \leq t} |\hat{W}_n^*(s)|,$$
$$\text{for all } n \geq 1, t \geq 0, \tag{4.54}$$

for some $\tilde{c} > 0$. Hence, using part (d) of Proposition 4.4 and assumptions in Definition 2.2, we have for each $n \geq 1$,

$$\int_0^\infty \delta e^{-\delta t} E\left[\left\{ \beta(n\gamma_n) \int_0^t \hat{Q}_n^*(s) ds \right\}^2 \right] dt \leq \kappa \int_0^\infty \delta e^{-\delta t} t^2 E\left[\sup_{0 \leq s \leq t} |\hat{W}_n^*(s)|^2 \right] dt$$

$$\leq \kappa \int_0^\infty \delta e^{-\delta t} t^2 \bar{c}(t^2 + t) dt < \infty, \tag{4.55}$$

where $\kappa = [\tilde{c}^2 \beta^2 c_1^2]$. Notice that the upper bound above does not involve n. Since (4.41) holds in this case as well, the uniform square integrability in (4.55) provides the required uniform integrability with respect to the product measure $P \times \mu$, where $d\mu/dt = \delta e^{-\delta t}$, to conclude that

$$\lim_{n \to \infty} E \int_0^\infty \delta e^{-\delta t} \left\{ \beta(n\gamma_n) \int_0^t \hat{Q}_n^*(s) ds \right\} dt$$

$$= E \int_0^\infty \delta e^{-\delta t} \left\{ \beta(\gamma) \int_0^t X_0^*(s) ds \right\} dt. \tag{4.56}$$

Also note that the convergence results in (4.44) holds in this case as well. Recall that, by our definition of optimal drift $u^* = u_p^*$ in (3.32) of Theorem 3.8, we have

$$u^*(x) = \Psi(V_p'(x)) \quad \text{and} \quad 0 \leq V_p'(x) < p_0,$$

and Ψ is a nondecreasing function with $\Psi(p_0) = \theta_{p_0} < \infty$ (see (3.19) and the discussion above that). Hence,

$$0 \leq u_n^*(x) \equiv u^*(x) \leq \theta_{p_0}, \quad \text{for all } x \geq 0.$$

Let $c_2 \doteq \sup_{y \in [0, \theta_{p_0}]} C(y)$. Using the above bound, we can obtain the same bound as in (4.45). Hence using (4.46), we conclude that (arguing as in (4.46)) that

$$
\lim_{n \to \infty} E \int_0^\infty \delta e^{-\delta t} \left\{ \int_0^t C(u_n^*(\hat{Q}_n^*(s))) ds \right\} dt
$$

$$
= E \int_0^\infty \delta e^{-\delta t} \left\{ \int_0^t C(u^*(X_0^*(s))) ds \right\} dt. \tag{4.57}
$$

Using (4.53), (4.56), (4.57), the definition of the cost functional in (2.14), (3.4) and (3.5), the Lemma 4.3 and the fact that W_0, the weak limit of $\{\hat{X}_n^*\}$, is a Brownian motion starting at $x = 0$, we derive that

$$
J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = \widetilde{J}_p(0, u^*, U^*) = V_p(0). \tag{4.58}
$$

This completes the proof for $p \geq p_0$. $\quad\square$

Proof of Theorem 2.8. Theorem 4.5 proves that

$$
J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b) = V_p(0).
$$

Hence, it is enough to prove that if $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$ is any admissible policy satisfying Definition 2.2, then

$$
J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \geq V_p(0). \tag{4.59}
$$

Note that (4.59) holds trivially if $J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) = \infty$. Hence, we will assume that

$$
J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) < \infty \tag{4.60}
$$

and intend to verify (4.59). Using Assumption 2.4, (2.4), part (c) of Proposition 4.4 and the Skorokhod representation theorem, it follows that

$$
\left(\int_0^{\cdot} \hat{Q}_n(s) ds, \int_0^{\cdot} C(u_n(\hat{Q}_n(s))) ds \right) \to \left(\int_0^{\cdot} X_0(s) ds, \int_0^{\cdot} C(u(X_0(s))) ds \right)
$$

a.s., as $n \to \infty$, \tag{4.61}

uniformly on compact sets (see (4.41) and (4.44) for a similar argument). Using part (b) of Lemma 4.3, (4.24) and applying Fatou's lemma twice, we derive

$$
\begin{aligned}
J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) &= \liminf_{n \to \infty} E \int_0^\infty \delta e^{-\delta t} \left\{ \beta(n\gamma_n) \int_0^t \hat{Q}_n(s) ds \right. \\
&\quad + \left. \int_0^t C(u(\hat{Q}_n(s))) ds + p\, U_n(t) \right\} ds \\
&\geq E \int_0^\infty \delta e^{-\delta t} \left[\lim_{n \to \infty} \left\{ \beta(n\gamma_n) \int_0^t \hat{Q}_n(s) ds \right. \right. \\
&\quad + \left. \left. \int_0^t C(u_n(\hat{Q}_n(s))) ds + p\, U_n(t) \right\} \right] dt \\
&= E \int_0^\infty \delta e^{-\delta t} \left\{ \beta\gamma \int_0^t X_0(s) ds + \int_0^t C(u(X_0(s))) ds + p\, U(t) \right\} dt \quad (4.62)
\end{aligned}
$$

where X_0 and U are as defined in part (c) of Proposition 4.4. As shown in Proposition 4.4(c), (X_0, u, U) is an admissible control of the BCP (with W_0). Hence, using part (a) of Lemma 4.3 (3.5), we have

$$E \int_0^\infty \delta \mathrm{e}^{-\delta t} \left\{ \beta \gamma \int_0^t X_0(s) \mathrm{d}s + \int_0^t C(u(X_0(s))) \mathrm{d}s + pU(t) \right\} \mathrm{d}t$$
$$\geq \widetilde{J}_p(0, u, U) \geq V_p(0). \tag{4.63}$$

Thus we get from (4.62)–(4.63) that

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \geq V_p(0)$$

and the proof of the theorem is complete.　□

We now give a short proof of Corollary 2.9.

Proof of Corollary 2.9. Note that, from the proof of Theorem 4.5, it follows that the proposed policy actually achieves the limit, and hence the asymptotic cost of this policy defined using lim sup in (2.18) is the same as the cost in that theorem (see the limit calculations before (4.52) and (4.58) in the proof). Hence, we get that

$$I_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*) = J_p(\underset{\sim}{\lambda}^*, \underset{\sim}{\mu}^*, b^*).$$

But since $\liminf a_n \leq \limsup a_n$ for any sequence $\{a_n\}$, it follows that

$$J_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b) \leq I_p(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b),$$

for any admissible policy $(\underset{\sim}{\lambda}, \underset{\sim}{\mu}, b)$. Hence, the proof follows from the conclusion of Theorem 2.8.
　□

Remark 4.7 (*Numerically Computing the Optimal Buffer Size*). For the given cost structure in (2.14), when $0 < p < p_0$, we can compute the finite buffer size b^* numerically by an algorithm very similar to the one described in Section 5 of [13] (see also [19] for a different approach).

Acknowledgements

References

[1] B. Ata, Dynamic control of a multiclass queue with thin arrival streams, Oper. Res. 54 (5) (2006) 876–892.
[2] B. Ata, J.M. Harrison, L.A. Shepp, Drift rate control of a Brownian processing system, Ann. Appl. Probab. 15 (2) (2005) 1145–1160.
[3] B. Ata, S. Kumar, Heavy traffic analysis of open processing networks with complete resource pooling: asymptotic optimality of discrete review policies, Ann. Appl. Probab. 15 (1A) (2005) 331–391.
[4] B. Ata, S. Shneorson, Dynamic control of an M/M/1 service system with adjustable arrival and service rates, Manage. Sci. 52 (11) (2006) 1778–1791.
[5] S.L. Bell, R.J. Williams, Dynamic scheduling of a system with two parallel servers in heavy traffic with resource pooling: asymptotic optimality of a threshold policy, Ann. Appl. Probab. 11 (3) (2001) 608–649.

[6] S.L. Bell, R.J. Williams, Dynamic scheduling of a parallel server system in heavy traffic with complete resource pooling: asymptotic optimality of a threshold policy, Electron. J. Probab. 10 (33) (2005) 1044–1115 (electronic).

[7] P. Billingsley, Convergence of Probability Measures, second edition, in: Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999, A Wiley-Interscience Publication.

[8] A. Budhiraja, A.P. Ghosh, A large deviations approach to asymptotically optimal control of crisscross network in heavy traffic, Ann. Appl. Probab. 15 (3) (2005) 1887–1935.

[9] A. Budhiraja, A.P. Ghosh, Diffusion approximations for controlled stochastic networks: an asymptotic bound for the value function, Ann. Appl. Probab. 16 (4) (2006) 1962–2006.

[10] S.N. Ethier, T.G. Kurtz, Markov Processes: Characterization and Convergence, in: Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986.

[11] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, second edition, in: Stochastic Modelling and Applied Probability, vol. 25, Springer, New York, 2006.

[12] J.M. George, J.M. Harrison, Dynamic control of a queue with adjustable service rate, Oper. Res. 49 (5) (2001) 720–731.

[13] A.P. Ghosh, A. Weerasinghe, Optimal buffer size for a stochastic processing network in heavy traffic, Queueing Syst. 55 (3) (2007) 1572–9443.

[14] J.M. Harrison, L.M. Wein, Scheduling networks of queues: heavy traffic analysis of a simple open network, Queueing Syst. 5 (4) (1989) 265–279.

[15] J.M. Harrison, L.M. Wein, Scheduling networks of queues: heavy traffic analysis of a two-station closed network, Oper. Res. 38 (6) (1990) 1052–1064.

[16] P. Hartman, Ordinary Differential Equations, John Wiley and Sons Inc., 1964.

[17] L. Kruk, J. Lehoczky, K. Ramanan, S. Shreve, An explicit formula for the Skorokhod map on $[0, a]$, Ann. Probab. 35 (5) (2007) 1740–1768.

[18] S. Kumar, Two-server closed networks in heavy traffic: diffusion limits and asymptotic optimality, Ann. Appl. Probab. 10 (3) (2000) 930–961.

[19] S. Kumar, M. Muthuraman, A numerical method for solving singular stochastic control problems, Oper. Res. 52 (4) (2004) 56–582.

[20] T.G. Kurtz, Approximation of Population Processes, in: CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 36, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, 1981.

[21] H.J. Kushner, Heavy Traffic Analysis of Controlled Queueing and Communication Networks, in: Applications of Mathematics (New York), vol. 47, Springer-Verlag, New York, 2001. Stochastic Modelling and Applied Probability.

[22] P.A. Meyer, Un cours sur les intégrales stochastiques, in: Séminaire de Probabilités, X (Seconde partie: Théorie des intégrales stochastiques, Univ. Strasbourg, Strasbourg, année universitaire 1974/1975), in: Lecture Notes in Math., vol. 511, Springer, Berlin, 1976, pp. 245–400.

[23] J. Reed, A.R. Ward, A diffusion approximation for a generalized Jackson network with reneging. in: Proceedings of the 42nd Annual Allerton Conference on Communication, Control, and Computing, 2004.

[24] J. Reed, A.R. Ward, Approximating the GI/GI/1 + GI queue with a nonlinear drift diffusion: Hazard rate scaling in heavy traffic, Math. Oper. Res. 33 (3) (2008) 606–644.

[25] A.V. Skorokhod, Stochastic equations for diffusions in a bounded region, Theory Probab. Appl. 6 (1961) 264–274.

[26] A.L. Stolyar, Maxweight scheduling in a generalized switch: state space collapse and workload minimization in heavy traffic, Ann. Appl. Probab. 14 (1) (2004) 1–53.

[27] A.R. Ward, P.W. Glynn, A diffusion approximation for a Markovian queue with reneging, Queueing Syst. 43 (1–2) (2003) 103–128.

[28] A.R. Ward, P.W. Glynn, A diffusion approximation for a GI/GI/1 queue with balking or reneging, Queueing Syst. 50 (4) (2005) 371–400.

[29] A. Ward, S. Kumar, Asymptotically optimal admission control of a queue with impatient customers, Math. Oper. Res. 33 (1) (2008) 167–202.

[30] A. Weerasinghe, An Abelian limit approach to a singular ergodic control problem, SIAM J. Control Optim. 46 (2) (2007) 714–737.

[31] W. Whitt, Stochastic-process Limits, in: Springer Series in Operations Research, Springer-Verlag, New York, 2002.

[32] K. Yamada, Diffusion approximation for open state-dependent queueing networks in the heavy traffic situation, Ann. Appl. Probab. 5 (4) (1995) 958–982.