Determinantal representations of closed orbits

Mao-Ting Chien\textsuperscript{a,\textdagger,1}, Hiroshi Nakazato\textsuperscript{b,2}

\textsuperscript{a} Department of Mathematics, Soochow University, Taipei 11102, Taiwan
\textsuperscript{b} Department of Mathematical Sciences, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

\textbf{ABSTRACT}

In this paper, we show that the orbit of a point mass under a central force $f(r) = -\alpha r^{-2} - \beta r^{-3}$ is realized as the hyperbolic curve $F_A(1, x, y) = 0$ associated with a nilpotent matrix $A$. On the contrary, we show that the orbit of motion of particles of infinitesimal mass in the gravitational field described by Schwarzschild geodesic metric is transcendental. In this case, the transcendental orbit has no determinantal representations.

1. Introduction

The orbit of Mercury is so much close to the Sun that ancient astronomers had difficulties to record the observation of Mercury. It was not until the 1960s that the prevailing theory held that the planet was tidally locked to the Sun and did not rotate at all. Ptolemaeus presented a model for the motion of Mercury based on his observations (cf. \cite{21}). In 19 century, Le Verrier found a discrepancy in the perihelion precession of Mercury predicted by Newton’s law of universal gravitation (cf. \cite{18}). In the system of Newton’s classical mechanics, the complicated motion of Mercury is viewed as a typical example of a many-body problem under the universal gravitation

\[ f(m, M, r) = -GMm \frac{1}{r^2}, \]
where $M$, $m$ are masses of objects and $G$ is the universal constant. Newton (cf. [4,22]) succeeded in explaining the Saros cycle concerning the periodic occurring of the lunar eclipses and solar eclipses, but he failed to provide a total picture of the complicated motion of the Moon. He provided an alternative mathematical way to produce variable orbits. In classical mechanics, a method explained that the perihelion precession of a planet is given by a central force

$$f(r) = -\alpha r^{-2} - \beta r^{-3}$$

under the one-body problem setting which perturbs the force $f(r) = -\alpha r^{-2}$. We treat the model (1.1) by a matrix method. Einstein [10] gave a general relative theoretic explanation of the perihelion precession. He was able to explain the observed result from more accurate measurements of the precession of the perihelion of Mercury. Schwarzschild [25] developed a spherical symmetric model of the space-time corresponding to a mass concentrated to one point. Hagihara [12] gave a rigid solution of a point mass moving in the Schwarzschild space-time by using elliptic functions. Hagihara’s trajectory orbit of a point mass coincides with the classical mechanical orbit under a central force $f(r) = -\alpha r^{-2} - \beta r^{-4}$.

Based on Bertrand’s classical works [2,3], Koenigs [17] showed that the central force under which every orbit is algebraic for every initial condition has to be the form $f(r) = \alpha r$ or $f(r) = \alpha r^{-2}$. However, the central force $f(r) = -r^{-3}$ may provide many algebraic orbits under some suitable initial conditions. In the previous paper [8], we developed matrix theoretic results related with the central force $f(r) = -r^{-3}$ inspired by Newton’s work. It gives a necessary and sufficient condition for the orbit of a point mass under the central force $f(r) = -r^{-3}$ to be algebraic in terms of the initial condition on the angular momentum. It is interesting to study closed or algebraic curves which are realized as the orbits of a point mass under some central forces with suitable initial conditions. In [7], we presented an explicit form for the central force that describes the orbit of a roulette curve, and interpreted the orbit of the roulette curve as an algebraic curve associated with a matrix. Such curves provide naive mathematical model for the perihelion precession of planets. In this paper, we treat the pericenter precession of a point mass via matrix method. Our main tool is the numerical range of a matrix and its related objects.

The numerical range of an $n \times n$ complex matrix $A$ is defined as

$$W(A) = \{ \xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1 \}.$$ 

Denote by $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$, two Hermitian parts of $A$. For any $0 \leq \theta \leq 2\pi$,

$$\Re(e^{-i\theta} A) = (e^{-i\theta} A + e^{i\theta} A^*)/2 = \cos \theta \Re(A) + \sin \theta \Im(A).$$

From Kippenhahn’s view point, the numerical range $W(A)$ is the convex hull of the real part of the dual curve, called boundary generating curve, of the algebraic curve $F_A(1, x, y) = 0$ defined by a ternary form

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)).$$ (1.2)

The ternary form $F_A(t, x, y)$ completely determines the numerical range of $A$ (cf. [16]). A real homogeneous polynomial $p(x) = p(x_1, x_2, \ldots, x_m)$ of degree $n$ is hyperbolic with respect to a vector $e = (e_1, e_2, \ldots, e_m)$ if $p(e) \neq 0$ and, for all vectors $w \in \mathbb{R}^m$, the univariate polynomial $t \mapsto p(w - te)$ has all real roots. It is clear that the ternary form (1.2) associated with a matrix is hyperbolic with respect to $(1, 0, 0)$. Fiedler–Lax [11,19] conjectured that if $p(x_0, x_1, x_2)$ is a ternary form of degree $n$ which is hyperbolic with respect to $(1, 0, 0)$ and $p(1, 0, 0) = 1$, then there exists a pair of $n \times n$ Hermitian matrices $H_1, H_2$ satisfying

$$p(x_0, x_1, x_2) = \det(x_0I_n + x_1H_1 + x_2H_2).$$
Recently, the Fiedler–Lax conjecture is affirmatively solved by Helton and Vinnikov [13] (see also [20, 24, 26]). In Section 2, we deal with a pericenter precession model caused by \( f(r) = -\alpha r^2 - \beta r^3 \). The orbit of a point mass under this force \( f(r) \) with some initial conditions is realized as the hyperbolic curve \( F_A(1, x, y) = 0 \) associated with a nilpotent Toeplitz matrix \( A \). The graph of the velocity \((x'(t), y'(t))\), known as the hodograph, can be obtained as the boundary generating curve of \( W(A) \) rotated with angle \( \pi/2 \) around the origin. In Section 3, we show that the orbit of particles in Schwarzschild geodesic metric is transcendental which cannot be achieved by determinantal representation of a matrix.

2. Universal gravitation perturbation

Let \( n \geq 3 \). Consider a real Toeplitz nilpotent matrix

\[
T(a_1, a_2, \ldots, a_{n-1}) = \begin{pmatrix}
0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\
0 & 0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} \\
0 & 0 & 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & a_2 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & a_1 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix},
\]

where \( a_1, a_2, \ldots, a_{n-1} \) are real numbers. For an even number \( n = 2m \geq 4 \) and real numbers \( b_0, b_1, b_2, \ldots, b_{m-1} \), we define an \( n \times n \) matrix

\[
A(b_0, b_1, b_2, \ldots, b_{m-1}) = T(b_{m-1}, b_{m-2}, \ldots, b_2, b_1, b_0, b_1, b_2, \ldots, b_{m-2}, b_{m-1}).
\]

In particular, for an integer \( 1 \leq s \leq m - 1 \) and real numbers \( b_0, b_s \), we denote the \( n \times n \) matrix

\[
A(b_0, b_s : n, s) = A(b_0, 0, \ldots, 0, b_s, 0, \ldots, 0),
\]

that is, \( A(b_0, b_s : n, s) = A(b_0, b_1, \ldots, b_{m-1}) \) where \( b_j = 0 \) for all \( 0 \leq j \leq m - 1 \) but \( j = 0, s \). In [5], the so called c-numerical range of the matrix (2.1) is studied and can be reduced to the convex hull of the classical numerical ranges of matrices of the same type. In [6], the matrix (2.1) produces flat portions on the boundary of its numerical range.

For a positive real number \( 0 < K < \infty \) and two positive real numbers \( a, b \), we consider an analytic curve

\[
r = \frac{1}{a + b \cos(K\theta)}.
\]

If \( K \) is rational, this curve is a rational algebraic curve. It is easy to see that the equation

\[
r = \frac{1}{a + b \cos \theta},
\]

represents a parabola

\[
2ax + a^2y^2 - 1 = 0
\]
in case \( a = b > 0 \), a non-parabolic conic curve

\[
(a^2 - b^2)(x + \frac{b}{a^2 - b^2})^2 + a^2y^2 = \frac{a^2}{a^2 - b^2}
\]

in case \( 0 < a \neq b \). In either case, the origin \((0, 0)\) is a focus of the conic.

In the following, we show that the curve (2.2) is realized as the orbit of a point mass under a central force, and can be interpreted as the curve \( F_A(1, x, y) = 0 \) associated with a matrix of the form (2.1) if \( 0 < K < 1 \) is a rational number.

**Theorem 2.1.** Let \( a, b \) and \( K \) be positive real numbers. Then the analytic curve (2.2) is realized as the orbit of a point mass under a central force \( f(r) = -\alpha r^{-2} - \beta r^{-3} \) for some constants \( \alpha > 0 \) and \( \beta \) with suitable initial conditions. Furthermore, for rational \( 0 < K = s/m < 1 \) with mutually prime positive integers \( s \) and \( m \), the curve (2.2) is realized as the hyperbolic curve \( F_A(1, x, y) = 0 \) associated with a nilpotent Toeplitz matrix \( A = A(b_0, b_5 : 2m, s) \) given in (2.1) with \( b_0 = -2a, b_5 = -b \).

**Proof.** The motion of a point mass under a central force

\[
f(r) = -\frac{\alpha}{r^2} - \frac{\beta}{r^3}
\]

is described by

\[
\begin{align*}
x''(u) &= \frac{x(u)}{\sqrt{x(u)^2 + y(u)^2}} f(x, y) = -\alpha \frac{x(u)}{(x(u)^2 + y(u)^2)^{3/2}} - \beta \frac{x(u)}{(x(u)^2 + y(u)^2)^2} \\
y''(u) &= \frac{y(u)}{\sqrt{x(u)^2 + y(u)^2}} f(x, y) = -\alpha \frac{y(u)}{(x(u)^2 + y(u)^2)^{3/2}} - \beta \frac{y(u)}{(x(u)^2 + y(u)^2)^2}.
\end{align*}
\]

The potential function \( V(r) \) corresponding to the force (1.1) is given by

\[
V(r) = -\int f(r) \, dr = -\frac{\alpha}{r} - \frac{\beta}{2r^2}.
\]

We assume the angular momentum \( M = x(u)y'(u) - y(u)x'(u) > 0 \). Then the total energy of a point mass becomes

\[
E_0 = \frac{1}{2} (x''(u) + y''(u)) + V(r) = \frac{1}{2} \left( \frac{M^2}{r^4} \left( \frac{dr}{d\theta} \right)^2 + \frac{M^2}{r^2} \right) - \frac{\alpha}{r} - \frac{\beta}{2r^2}. \tag{2.3}
\]

From (2.3), we obtain that the orbit \( r = r(\theta) \) under the central force \( f(r) \) becomes

\[
\frac{d\theta}{dr} = \frac{M}{r \sqrt{2E_0r^2 + 2\alpha r + \beta - M^2}}. \tag{2.4}
\]

On the other hand, the curve (2.2) is represented as

\[
\theta = \frac{1}{K} \arccos \left( \frac{1}{b} \left( \frac{1}{r} - a \right) \right),
\]

and thus

\[
\frac{d\theta}{dr} = \frac{1}{Kr \sqrt{-a^2r^2 + b^2r^2 + 2ar - 1}}. \tag{2.5}
\]
By choosing
\[
\alpha = M^2 K^2 a, \\
\beta = (1 - K^2)M^2, \\
2E_0 = -M^2 (a^2 - b^2)K^2
\]
the right hand side of (2.4) is exactly the same as the right hand side of (2.5). Therefore, the curve (2.2) is realized as the orbit of a point mass under the central force (1.1) satisfying conditions (2.6)–(2.8).

Suppose that 0 < K = s/m < 1 is rational. Consider the 2m × 2m matrix \( A(b_0, b_s : n, s) \) in (2.1) with \( n = 2m \). Then, by [5], the eigenvalues of the Hermitian matrix \( H_n(\theta : b_0, b_s) = \Re(e^{-i\theta}A(b_0, b_s : n, s)) \) are
\[
\lambda_q(\theta : b_0, b_s) = (-1)^q b_0 + (-1)^q b_s \cos \left( \frac{s\theta}{m} + \frac{qs\pi}{m} \right),
\]
where \( q = 0, 1, 2, n - 1 \), and the associative curve
\[
F_A(b_0, b_s, n, s)(1, x, y) = 0
\]
is parametrized as
\[
\left\{ \left( -\frac{\cos \theta}{\lambda_0(\theta : b_0, b_s)}, -\frac{\sin \theta}{\lambda_0(\theta : b_0, b_s)} \right) : 0 \leq \theta \leq n\pi \right\}
\]
except for a finite number of real points at infinity. For positive real numbers \( a, b \), we set \( b_0 = -2a, b_s = -b \). Then the parametrized curve (2.9) is expressed as
\[
x = \frac{1}{a + b \cos((s/m)\theta)}, \quad y = \frac{1}{a + b \cos(\phi)}
\]
in polar coordinates. This proves the second assertion. □

Further, we give a necessary and sufficient condition for the curve (2.2) to be realized as the curve \( F_A(1, x, y) = 0 \) for some matrix \( A \).

**Theorem 2.2.** Let \( a, b \) and \( K \) be positive real numbers. Then the analytic curve (2.2) is realized as the hyperbolic curve \( F_A(1, x, y) = 0 \) for some matrix \( A \) if and only if \( 0 < K \leq 1 \) and \( K \) is rational.

**Proof.** If \( K = 1 \) then the parametrized curve (2.2) is a conic curve with a focus at (0.0). Its dual curve is a circle. Hence, the curve (2.2) is represented as
\[
F_A(t, x, y) = \det(tl_2 + x/2(A + A^*) - yi/2(A - A^*))
\]
for some \( 2 \times 2 \) real matrix
\[
A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}.
\]
Combining this case with Theorem 2.1, the sufficiency is proved.

For necessity. We assume that \( K \) is an irrational positive real number. Setting \( L = 1/K \), \( L \) is also an irrational number. Then the curve (2.2) is parametrized by
\[
x = \frac{\cos(L\phi)}{a + b \cos(\phi)}, \quad y = \frac{\sin(L\phi)}{a + b \cos(\phi)},
\]
(2.10)
\(-\infty < \phi < \infty\). For \(n = 1, 2, 3, \ldots\), choose \(0 < \phi_n < \pi/2\) so that \(\cos(\phi_n) = n/(n + 1)\). Then the points \((x_{n,m}, y_{n,m})\), \(n, m = 1, 2, 3, \ldots\), with

\[
x_{n,m} = \frac{\cos(L\phi_n + 2mL\pi)}{a + b \cos(\phi_n)}, \quad y_{n,m} = \frac{\sin(L\phi_n + 2mL\pi)}{a + b \cos(\phi_n)}
\]

lying on the curve (2.10) are distinct, and lie also on the circle

\[
x^2 + y^2 = \frac{1}{(a + b \cos(\phi_n))^2}.
\]

Suppose that the curve (2.2), the same as (2.10), is an algebraic curve. Then the Bezout theorem implies that the circle (2.11) is a component of the curve (2.10) for every \(n = 1, 2, 3, \ldots\). However this is impossible since the circles (2.11) are mutually distinct for different natural numbers \(n\). Therefore, if \(K\) is irrational then the curve (2.2) is transcendental, and thus can’t be represented as an algebraic hyperbolic curve \(F_A(1, x, y) = 0\) for any matrix \(A\).

Now we assume that \(1 < K = m/s\) for some mutually prime positive integers \(m > s\). By using a parameter \(\tau = \tan(\theta/(2s))\), the functions \(\cos(K\theta), \cos(\theta), \sin(\theta)\) become

\[
\cos(K\theta) = \Re \left( \left( \frac{1 - \tau^2}{1 + \tau^2} + i \frac{2\tau}{1 + \tau^2} \right)^m \right),
\]

\[
\cos(\theta) = \Re \left( \left( \frac{1 - \tau^2}{1 + \tau^2} + i \frac{2\tau}{1 + \tau^2} \right)^s \right),
\]

\[
\sin(\theta) = \Im \left( \left( \frac{1 - \tau^2}{1 + \tau^2} + i \frac{2\tau}{1 + \tau^2} \right)^s \right).
\]

Then any point \((x, y)\) of (2.2) satisfies

\[
L_1(x, \tau) = -x(a + b \cos(K\theta))(1 + \tau^2)^m + \cos(\theta)(1 + \tau^2)^m = 0,
\]

\[
L_2(y, \tau) = -y(a + b \cos(K\theta))(1 + \tau^2)^m + \sin(\theta)(1 + \tau^2)^m = 0.
\]

By eliminating \(\tau\) from these two equations, we get a polynomial \(F(1, x, y)\) of degree \(2m\). For a generic angle \(\theta\), the line \(y = x \tan(\theta)\) has only \(2m\) intersections with the curve (2.2). Hence the homogeneous polynomial \(F(t, x, y)\) is not hyperbolic. This proves the necessity. \(\square\)

Note that we may use a criterion obtained in [14, 15] to find conditions on \(a, b\) and \(K\) for the positive definite determinantal representation of the rational curve (2.2).

**Remark 1.** We are capable of choosing \(\alpha, \beta, M, E_0\) that satisfy conditions (2.6)–(2.8) in Theorem 2.1. For instance, we assume the initial conditions

\[
x(0) = \frac{1}{a + b}, \quad y(0) = 0, \quad x'(0) = 0, \quad y'(0) = a + b,
\]

and hence \(M = 1\). We choose \(\alpha, \beta\) by (2.6), (2.7), that is, \(\alpha = K^2a > 0\) and \(\beta = 1 - K^2\). Then the total energy \(E_0\) is given by

\[
E_0 = \frac{1}{2}(a + b)^2 + V \left( \frac{1}{a + b} \right) = -\frac{1}{2}(a^2 - b^2)K^2
\]

which coincides the value given in (2.8).
Remark 2. A concrete example would be useful to illustrate the non-hyperbolicity for \( K > 1 \) in Theorem 2.2. Consider \( K = 5/4, a = 25/9, b = 20/9 \). Then the curve (2.2) is expressed as \( F(1, x, y) = 0 \) by the following non-hyperbolic ternary form of degree 10:

\[
F(t, x, y) = 43046721x^2t^8 - 1753755300x^4t^6 + 19970043750x^6t^4 - 2916000000x^7t^3
- 59135062500x^8t^2 - 15300000000x^9t + 25400390625x^{10} + 43046721y^2t^8
- 3507510600x^2y^2t^6 + 59910131250x^3y^2t^4 + 26244000000x^4y^2t^3
- 236540250000x^5y^2t^2 + 122400000000x^6y^2t + 137001953125x^8y^2
- 1753755300y^4t^6 + 59910131250x^2y^4t^4 + 14580000000x^3y^4t^3
- 354810375000x^4y^4t^2 + 214200000000x^5y^4t + 214003906250x^6y^4 + 19970043750y^6t^4 - 1458000000000x^6y^4t^3
- 236540250000x^7y^4t^2 + 298003906250x^8y^6 + 59135062500y^8t^2 - 76500000000xy^8t
+ 119001953125x^2y^8 + 25800390625y^{10}.
\]

By using Henrion’s method in [15], we can construct 10 × 10 real symmetric matrices \( S_1, S_2, S_3 \) satisfying

\[
\det(tS_1 + xS_2 + yS_3) = cF(t, x, y),
\]

where

\[
S_1 = \begin{pmatrix}
9 & 0 & -54 & 0 & 0 & 0 & 54 & 0 & -9 & 0 \\
0 & 189 & 0 & -378 & 0 & -324 & 0 & 234 & 0 & -9 \\
-54 & 0 & 1080 & 0 & -324 & 0 & -1224 & 0 & 234 & 0 \\
-378 & 0 & 1944 & 0 & 1044 & 0 & -1224 & 0 & 54 & 0 \\
0 & 0 & -324 & 0 & 1044 & 0 & 1044 & 0 & -324 & 0 \\
0 & 0 & 1044 & 0 & 1044 & 0 & 1044 & 0 & -324 & 0 \\
54 & 0 & -1224 & 0 & 1044 & 0 & 1944 & 0 & -378 & 0 \\
1224 & 0 & -324 & 0 & 1044 & 0 & 1044 & 0 & -324 & 0 \\
0 & 234 & 0 & -1224 & 0 & -324 & 0 & 1044 & 0 & -378 & 0 \\
0 & -9 & 0 & 54 & 0 & 0 & 0 & -54 & 0 & 9
\end{pmatrix}
\]

\[
S_2 = \begin{pmatrix}
-45 & 0 & 270 & 0 & 0 & 0 & -270 & 0 & 45 & 0 \\
0 & -505 & 0 & 4450 & 0 & -4220 & 0 & 1070 & 0 & 5 \\
270 & 0 & -200 & 0 & -4220 & 0 & 5720 & 0 & -770 & 0 \\
0 & 4450 & 0 & -30920 & 0 & 29420 & 0 & -6920 & 0 & -30 \\
0 & 0 & -4220 & 0 & 29420 & 0 & -33620 & 0 & 4420 & 0 \\
0 & -4220 & 0 & 29420 & 0 & -33620 & 0 & 4420 & 0 & 0 \\
-270 & 0 & 5720 & 0 & -33620 & 0 & 28120 & 0 & -3950 & 0 \\
0 & 1070 & 0 & -6920 & 0 & 4420 & 0 & 2200 & 0 & 30 \\
45 & 0 & -770 & 0 & 4420 & 0 & -3950 & 0 & 1055 & 0 \\
0 & 5 & 0 & -30 & 0 & 0 & 0 & 30 & 0 & -5
\end{pmatrix}
\]
\[ S_3 = \begin{pmatrix} 0 & -55 & 0 & -320 & 0 & 730 & 0 & -280 & 0 & 5 \\ -55 & 0 & -320 & 0 & 730 & 0 & -280 & 0 & 5 & 0 \\ 0 & -320 & 0 & 11680 & 0 & -17680 & 0 & 6080 & 0 & -80 \\ -320 & 0 & 11680 & 0 & -17680 & 0 & 6080 & 0 & -80 & 0 \\ 0 & 730 & 0 & -17680 & 0 & 50180 & 0 & -20480 & 0 & 530 \\ 730 & 0 & -17680 & 0 & 50180 & 0 & -20480 & 0 & 530 & 0 \\ 0 & -280 & 0 & 6080 & 0 & -20480 & 0 & 8480 & 0 & -520 \\ -280 & 0 & 6080 & 0 & -20480 & 0 & 8480 & 0 & -520 & 0 \\ 0 & 5 & 0 & -80 & 0 & 530 & 0 & -520 & 0 & 145 \\ 5 & 0 & -80 & 0 & 530 & 0 & -520 & 0 & 145 & 0 \end{pmatrix}, \]

and \( c = -2^{62} \times 5^2 \). The symmetric matrix \( S_1 \) has non-negative eigenvalues, but it is non-invertible. The ternary form \( F(t, x, y) \) has a rather good property resembling the hyperbolicity with respect to \((1, 0, 0)\). For every \( 0 \leq \theta \leq 2\pi \), \( F(t, \cos \theta, \sin \theta) \) is a polynomial in \( t \) of degree 8 with the leading coefficient 43046721. The equation \( F(t, \cos \theta, \sin \theta) = 0 \) in \( t \) has 8 real roots. However, the point \((t, x, y) = (1, 0, 0)\) is an isolated singular point of the curve \( F(1, x, y) = 0 \). This means that the form \( F(t, x, y) = 0 \) is not hyperbolic with respect to \((1, 0, 0)\). So the curve (2.2) in this case cannot be realized as \( F_B(1, x, y) = 0 \) for any matrix \( B \). The non-hyperbolic curve \( F(1, x, y) = 0 \) is displayed in Figure 1 at which \((0, 0)\) is an isolated singular point of the curve.

### 3. Schwarzschild geodesic curve

In the previous paper [7], we gave an example of transcendental closed curve related with classical dynamics. In this section, we provide a closed transcendental curve related with general relativity. At first we examine the exact solutions of the time-like geodesic in the Schwarzschild universe following Kraniotis and Whitehouse’s paper (cf. [18], also [9,12]). A model of the motion of a planet according to general relativity is given by a timelike geodesic in a Schwarzschild space-time surrounding the Sun. We assume a zero cosmological constant \( \Lambda = 0 \), a special case in [18]. The metric in the Schwarzschild space-time with \( \Lambda = 0 \) is given by
with respect to the spherical polar coordinates \((r, \theta, \phi)\) in the space and the time parameter \(t\), where \(G\) denotes the Newton’s gravitational constant, \(c\) the velocity of light and \(M_5\) the mass of the Sun. The Ricci tensor and the scalar curvature of this space-time vanish (cf. \([1, 9, 23]\)). We assume that a point mass with negligible mass moves in the equatorial plane \(\theta = \pi/2\). Then the orbit of the point mass \(r = r(\phi)\) has a period slightly greater than \(2\pi\). The analytic function \(r(\phi)\) on the real line is extended to an elliptic function with a real half period \(\omega\) and a pure imaginary half period \(\omega’\) in case the treated model is a generalization of Mercury’s motion and the related elliptic function has a positive discriminant \(\Delta = g_2^2 - 27g_3^2\) for some real coefficients \(g_2, g_3\). The expression of Schwarzschild geodesics by elliptic functions was found by Hagihara \([12]\). The orbit \(r = r(\phi)\) in Weierstrass representation of the Schwarzschild geodesic equation is given by

\[
r = \frac{\alpha_5}{4 \wp(\phi + \omega’ + \epsilon : g_2, g_3) + 1/3}\quad (3.1)
\]

for some real integral constant \(\epsilon\) and Weierstrass \(\wp\)-function, where the Schwarzschild radius \(\alpha_5\) is a physical constant of the Sun given by \(\alpha_5 = 2.953\) km. The cubic polynomial

\[
4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3), \quad e_1 > e_2 > e_3, \quad e_1 + e_2 + e_3 = 0
\]
determines the half-periods \(\omega, \omega’\) by the equations

\[
\omega = \int_{e_1}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} = \int_{e_3}^{e_2} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}},
\]

\[
\omega’ = i \int_{-\infty}^{e_2} \frac{dx}{\sqrt{-4x^3 + g_2x + g_3}} = i \int_{e_1}^{e_2} \frac{dx}{\sqrt{-4x^3 + g_2x + g_3}}.
\]

Conversely, the real constants \(e_2, e_3\) are realized as \(\wp(\omega’) = e_3, \wp(\omega’ + \omega) = e_2\) on the line \(\wp(z) = \omega’/i\). Observed physical values of the half-periods \(\omega, \omega’\) for Mercury are respectively given by

\[
\omega = 3.141592904646, \quad \omega’ = 20.40864976i,
\]

while \(\pi\) is approximately \(3.1415926535898\) and the corresponding coefficients \(e_2, e_3\) are given by

\[
e_2 = -0.083333317275, \quad e_3 = -0.08333322754
\]

(cf. \([18]\)). The ratio of the minimum distance \(r_p\) of the planet Mercury from the Sun (perihelion) and the maximum distance \(r_A\) (aphelion) are related with \(e_2, e_3\) as

\[
\frac{r_p}{r_A} = \frac{1 - e}{1 + e} = \frac{4e_3 + 1/3}{4e_2 + 1/3} \quad (3.2)
\]

where \(e\) is the eccentricity of the orbit. In this model with \(\Lambda = 0\), the rate \(2(\omega - \pi)\) of the perihelion precession of a planet (in one revolution) caused by general relativistic effect is predicted by the ratio \(r_p : r_A = 1 - e : 1 + e\) as we can see it from the Eq. (3.2). If \(\omega/\pi\) is irrational, then Bezout’s theorem implies the transcendence of the trajectory of a point mass. We assume a fictitious model when \(\omega/\pi\) is a rational number and hence the orbit is a closed analytic curve. Does this curve lie on an algebraic curve? We give a negative answer to this question. It turns out that Schwarzschild geodesic curve cannot be achieved by the dual curve of the boundary generating curve of a matrix numerical range.
Theorem 3.1. If \( \omega / \pi \) is a rational number, then the closed Schwarzschild geodesic (3.1) is a transcendental curve, that is, there is no non-zero real polynomial \( h(x, y) \) satisfying

\[
h \left( \frac{\alpha_S \cos \phi}{4 \mathcal{P}(\phi + \omega' : g_2, g_3) + 1/3}, \frac{\alpha_S \sin \phi}{4 \mathcal{P}(\phi + \omega' : g_2, g_3) + 1/3} \right) = 0 \tag{3.3}
\]

for \( \phi \in \mathbb{R} \).

Proof. We may assume that \( \epsilon = 0 \). Suppose that there is a non-zero real polynomial

\[
h(x, y) = \sum_{n, m} a_{n,m} x^n y^m
\]
satisfying (3.3). By normalization, we have that

\[
h \left( \frac{\cos \phi}{4 \mathcal{P}(\phi + \omega' : g_2, g_3) + 1/3}, \frac{\sin \phi}{4 \mathcal{P}(\phi + \omega' : g_2, g_3) + 1/3} \right) = 0,
\]

\( \phi \in \mathbb{R} \). By the symmetry property \( \mathcal{P}(-\phi + \omega') = \mathcal{P}(\phi + \omega') \), it follows that there is a non-zero real polynomial \( g(x, y) \) satisfying

\[
g(\cos \phi, \mathcal{P}(\phi + \omega' : g_2, g_3)) = 0, \tag{3.4}
\]

for \( \phi \in \mathbb{R} \), and hence for \( \phi \in \mathbb{C} \). We express (3.4) as

\[
c_m(\mathcal{P}(\phi + \omega' : g_2, g_3)) \cos^m \phi + c_{m-1}(\mathcal{P}(\phi + \omega' : g_2, g_3)) \cos^{m-1} \phi + \cdots + c_0(\mathcal{P}(\phi + \omega' : g_2, g_3)) = 0,
\]

for some polynomials \( c_m(y), \ldots, c_0(y) \) with \( c_m(y) \neq 0 \). We choose a real \( 0 \leq \phi_0 < 2\omega \) so that \( c_m(\mathcal{P}(\phi_0 + \omega')) \neq 0 \). Such a choice is possible since \( c_m(\mathcal{P}(z)) = 0 \) on the line \( \Im(z) = \omega'/i \) will result in \( c_m(z) = 0 \) on the whole plane \( \mathbb{C} \). Then the inequality

\[
|\cos^m(\phi_0 - 2k\omega')| \leq \sum_{\ell=0}^{m-1} \frac{|c_\ell(\mathcal{P}(\phi_0 + \omega'))| |c_m(\mathcal{P}(\phi_0 + \omega'))| |\cos^\ell(\phi_0 - 2k\omega')|}{C_m}
\]

holds for every positive integer \( k \) since \( \mathcal{P}(z) \) has a pure imaginary period \( 2\omega' \). The right-hand side of (3.5) is estimated from above by

\[
C_{m-1} e^{-i2k(m-1)\omega'},
\]

where \( C_{m-1} \) is a positive constant independent of \( k \). On the other hand, the left-hand side of (3.5) is estimated from below by

\[
C_m e^{-i2km\omega'}
\]

for some positive constant \( C_m \). Then we obtain the inequality

\[
0 < e^{-i2k\omega'} \leq \frac{C_{m-1}}{C_m}
\]

for large positive integer \( k \). This is impossible since \( \omega' \) has a positive imaginary part. \( \Box \)
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References