BROWN-GITLER SPECTRA AT BP(2)

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Received 30 December 1986
Revised 14 April 1987

Brown-Gitler spectra for the homology theory associated to BP(2) are constructed. Complexes adapted to the new Brown-Gitler spectra are constructed and a spectral sequence converging to stable maps into these spectra is examined.

AMS (MOS) Subj. Class.: Primary 54P42, 55N35

Brown-Gitler spectra BP(2)

In [11], Brown and Gitler constructed certain spectra B(k) with the property that $H^*B(k) = \mathbb{A}/\mathbb{Z}_k$ for certain ideals $I_k$ over the Steenrod algebra $\mathbb{A}$. In [6], Goerss, Jones and Mahowald constructed what they called generalized Brown-Gitler spectra at BP(1), written $BP(1)^k$, with the property that $H^*BP(1)^k = \mathbb{A}/I_k + J_1$ where $J_1$ is the ideal with the property that $H^*BP(1) = \mathbb{A}/J_1$. This paper is devoted to constructing Brown-Gitler spectra at BP(2) with $H^*BP(2)^k = \mathbb{A}/I_k + J_2$ where $H^*BP(2) = \mathbb{A}/J_2$.

Although the original Brown-Gitler spectra have been put to a variety of uses in homotopy theory, the applications of generalized Brown-Gitler spectra have been restricted to splittings of $bo \wedge bo$ (see [9]), $BP(1) \wedge BP(1)$ (see [7]), and $BP(1) \wedge BP(n)$ (see [8]). In fact if $J_n$ is an ideal satisfying $H^*BP(n) = \mathbb{A}/J_n$ (one should interpret $BP(-1)$ as $H^p/p$, $BP(0)$ as $H^p/\mathbb{A}$, and $J_{-1}$ as the empty set), then $H^*(BP(n) \wedge BP(n)) \equiv \bigoplus_{k=0}^{\Sigma 2(p-1)^k} H^*(BP(n) \otimes \mathbb{A}/I_k + J_n$ and one hopes to be able to realize $A/I_k + J_n$ as the mod $p$ cohomology of a spectrum $BP(n)^k$ and then realize this as a splitting of spectra. This is the motivation for constructing $BP(2)^k$.

Having fixed $k$, the first task is to construct an acyclic resolution of $A/I_k + J_n$.

\begin{align*}
A/I_k + J_n & \leftarrow D_0^0 \leftarrow D_0^1 \leftarrow D_2^0 \leftarrow \cdots.
\end{align*}

Brown and Gitler [1] used the lambda algebra to produce the corresponding resolution; Goerss, Jones and Mahowald [6] used a modification of the lambda algebra. The appropriate modification for BP(2) is apparent in [6] and little will be said here about the details needed to get the acyclic resolution,

\begin{align*}
A/I_k + J_2 & \leftarrow D_0^0 \leftarrow D_1^0 \leftarrow D_2^0 \leftarrow \cdots.
\end{align*}
Along with this, one wants to have the resolution realized by maps of spectra,

\[ K^n_0 \xrightarrow{d_0} \Sigma K^n_1 \xrightarrow{d_1} \Sigma^2 K^n_2 \rightarrow \ldots. \]

Since in [1] the resolution was free over the Steenrod algebra, they got the maps of spectra for free. This was not the case in either [6] or this paper. In our resolution the first three terms are direct sums of suspensions of \( H^*\text{BP}(2) \), \( H^*\text{BP}(1) \) and \( H^*\mathbb{H}Z_p \) respectively. The only new constructions needed are maps \( S': \text{BP}(2) \rightarrow \Sigma^{21p^{-1}}\text{BP}(1) \) with the property that in cohomology \( S'^*(1) = \chi(P') \), where 1 and \( \chi(P') \) represent their respective equivalence classes in \( A/J_1 \) and \( A/J_2 \). This is a consequence of the fact that the Adams spectral sequence for \([\text{BP}(2), \text{BP}(1)]\) collapses to its \( E_2 \) term, as proved in [8].

Conceptually, the next step would be to construct a tower of spectra with the following properties.

**Theorem A.** For any \( n \) between \(-1\) and 2 there is a tower of spectra

\[ X_{q+1} \xrightarrow{p_q} X_q \rightarrow \ldots \rightarrow X_1 \xrightarrow{p_0} X_0 \]

with the property that

(a) there are maps \( e_1: X_q \rightarrow \Sigma K^n_{q+1} \) where \( K^n_{q+1} \) are the spectra mentioned in the previous paragraph,

(b) \( X_0 = K^n_0 = \text{BP}(n) \), \( e_0 = d_0 \),

(c) \( K^n_{q+1} \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K^n_{q+1} \),

is a cofiber sequence (defining \( i_{q+1} \)),

(d) \( e q i_q = d_q, i_0 = Id \),

(e) For any CW complex \( Z \), the induced map of homology theories \( e_q^*: (X_q)_m Z \rightarrow (K^n_{q+1})_m Z \) is zero for \( m \leq 2p(k+1) - 1 \) and \( q \geq n \).

This was proved in [6] for \(-1 \leq n \leq 1\) (one should identify \( \text{BP}(-1) \) with \( \mathbb{H}Z/p \) and \( \text{BP}(0) \) with \( \mathbb{H}Z_p^\ast \)). Given Theorem A, the \( \text{BP}(2)^k \) are defined to be the homotopy inverse limit of the \( X_q \) tower in the theorem when \( n = 2 \).

**Theorem B.** For each \( k \geq 0 \) and prime \( p \), there is a \( p \)-complete spectrum \( \text{BP}(2)^k \) and a map \( w: \text{BP}(2)^k \rightarrow \text{BP}(2) \) such that

(a) \( H^*\text{BP}(2)^k = A/\{\beta, \beta P^1 \beta P^{p+1}, \chi P^j | j > k\} \cong A/I_k + J_2 \)

(b) \( \{\text{Im } w_\ast: \text{BP}(2)^k_m Z \rightarrow \text{BP}(2)_m Z \} \subseteq \bigcap L_{j > k} \ker S'_j \).

If \( Z \) is a CW complex with \( \mathbb{H}Z_p^\ast Z \) an \( \mathbb{F}_p \) vector space and \( m \leq 2p(k+1) - 1 \) then \( \text{Im } w_\ast \) is all of \( \bigcap L_{j > k} \ker S'_j \).

**Partial Proof.** To prove part (a), apply \( H^* \) to the tower of fibrations in Theorem A. The spectral sequence associated to the unravelled exact couple converges to
\[ H^* \text{holim}_q X_q = H^* \text{BP}(2)^k \] and has \( E^1_{i, s} = H^* K_i^q \cong D_i^q \). Since \( 0 \leftarrow A/I_k + J_1 \leftarrow D_0 \leftarrow D_1 \leftarrow \cdots \) was an acyclic resolution, the \( E^2 \) term of this spectral sequence has \( E^2_{ij} = \text{Coker} H^* d_0 \) and \( E^2_{ij} = 0 \) for \( i \geq 1 \). Thus \( H^* \text{BP}(2)^k \) is as claimed in (a).

The map \( w : \text{BP}(2)^k \to \text{BP}(2) \) is just
\[ \text{BP}(2)^k = \text{holim}_q X_q \to X_0 = \text{BP}(2) \]
which can be factored as
\[ \text{BP}(2)^k \to X_1 \to \text{BP}(2). \]
Since
\[ \text{BP}(2)^k \to X_1 \to \bigcup_{j > k} \Sigma^{2j(p-1)+1} \text{BP}(1) = \Sigma K_i^q \]
is a cofibration we get the first part of assertion (b). The proof of the last part of (b) requires some knowledge of how Theorem A is proved and will appear in Section 3.

Finally taking the tower of fibrations created in Theorem A, smashing with any spectrum \( Y \) and taking stable homotopy gives a spectral sequence converging to \( \text{BP}(n)^k \otimes Y \) with
\[ E^1_{i, s}(Y \wedge \text{BP}(n)) \cong (K_i^q)_{s}, Y = \pi_{s-i}(K_i \wedge Y). \]
When \( n = 0 \) or \(-1\) and \( Y \) is the suspension spectrum of a space then part (e) of Theorem A implies that
\[ E^1_{i, s}(Y \wedge \text{BP}(n)) \cong E^\infty_{s, i}(Y \wedge \text{BP}(n)) \quad \text{for} \quad t-s \leq 2p(k+1)-2. \]
When \( n = 1 \) this becomes \( E^2_{i, s}(Y \wedge \text{BP}(n)) \cong E^\infty_{s, i}(Y \wedge \text{BP}(n)) \) for \( t-s \leq 2p(k+1)-2. \)
For Brown–Gitler spectra at \( \text{BP}(2) \) we get

**Corollary C.** If \( Y \) is the suspension spectrum of a space then in the spectral sequence converging to \( \text{BP}(2)^k \otimes Y \),
\[ E^3_{s, i}(Y \wedge \text{BP}(2)) \cong E^\infty_{s, i}(Y \wedge \text{BP}(2)) \quad \text{for all} \quad t-s \leq 2p(k+1)-2. \]

The dual to this situation involves applying the functor \([ Y, \_ ]\) to the tower of Theorem A to get a spectral sequence converging to \([ Y, \text{BP}(n)^k ]^{s-t} \) with
\[ F_{s, i}^t(Y, \text{RP}(n)) \cong [ Y, K_i^q ]^{s-t}. \]
In order to get the collapsing result analogous to Corollary C, [6] introduced the notion of spacelike spectra.

**Definition.** Let \( Y \) be a finite CW spectrum. \( Y \) is said to be **spacelike of dimension** \( n \) if there is a spectrum \( T \) and a map \( f : T \to Y \) so that

(i) \( T = \Sigma^n DZ \) where \( Z \) is a finite CW complex, and

(ii) \( f^*: H^* Y \to H^* T \) is injective.
Corollary D. If \( Y \) is spacelike of dimension \( n \), then in the spectral sequence converging to \( [Y, \text{BP}(2)^k]^{s-t} \), for all \( s - t \geq n - 2p(k+1)+2 \)

\[
E_1^{s,t} \cong E_\infty^{s,t} \quad \text{and} \quad E_\infty^{s,t} \cong E_\infty^{s,t} \quad \text{for} \quad s \geq 3.
\]

Although conceptually the construction of the \( X_n \) tower comes after realizing the \( D^n \) resolution, performing the construction requires the use of some complicated machinery and while constructing the tower one must simultaneously construct maps from the spectra needed for the \( \text{BP}(2)^k \) construction to those that were used in the construction of the original Brown–Gitler spectra.

In Section 1 the acyclic resolution of \( M(2, k) \) is given and realized. Section 2 contains the definition and construction of adapted complexes, the complicated machinery referred to in the previous paragraph. Section 3 gives the proof of Theorem A and Theorem B while Section 4 concludes with a proof of Corollary D.

Throughout this paper we are working with a fixed prime \( p \) and a fixed positive integer \( k \). Homology is always with mod \( p \) coefficients unless otherwise specified.

In this paper, references to earlier results about Brown–Gitler spectra and their construction are almost always to [6]. This is for convenience since that is where the results are stated in the form we need. In fact all results about Brown–Gitler spectra at \( \text{BP}(-1) = \mathbb{H}Z/p \) are due to [1] when \( p = 2 \) and [3] when \( p \) is odd. Brown–Gitler spectra at \( \text{BP}(0) = \mathbb{H}Z; \) are due to Mahowald [9] when \( p = 2 \) and Kane [7] for \( p \) odd and also to Goerss [4] and Shimamoto [10].

1. Acyclic resolutions

In this section we collect all the definitions and results from [6] regarding the construction and realization of the acyclic resolutions that we need. Their work made extensive use of the \( \Lambda \) algebra which we will not do, since their work is sufficient to construct the acyclic resolution we need and the \( \Lambda \) algebra’s work is done once they have been constructed. Throughout this section, if the prime you have chosen is \( p = 2 \), replace \( P' \) by \( Sq^n \) and \( \beta \) by \( Sq^n \).

For \( n \geq 0 \), let \( E_n \) be the Hopf subalgebra of the Steenrod algebra \( A \), generated by the Milnor elements \( Q_0, Q_1, \ldots, Q_n \). let \( E_n \) be the augmentation ideal and \( J_n \) the left \( A \) ideal generated by \( E_n \). Notice that \( H^*\text{BP}(n) = A/J_n \). For \( n \geq 1 \), let \( M_n(k) = A/I_k + J_n \) (where \( J_{-1} = \emptyset \)). Theorem 2.6 of [6] gives an acyclic resolution of \( M_n(k) \) by \( A \) modules

\[
\cdots \rightarrow D_{q+1}^n \xrightarrow{d_q^n} D_q^n \rightarrow \cdots \rightarrow D_1^n \xrightarrow{d_0^n} D_0^n \rightarrow M_n(k) \rightarrow 0
\]

for \( n = 0 \) and 1. Their proof works equally well for \( n = 2 \). One should notice that
$D_0^2 = H^*\text{BP}(2)$, $D_1^2 = \bigvee_{j > k} \Sigma^q H^*\text{BP}(1)$ and $D_2^2$ is a sum of $H^*\text{BP}(0)$'s while for $n \geq 3$, $D_n^2$ is a free module and a direct summand of $D_n^{-1}$ which is also free. In general, $D_n^2$ is a direct sum of copies of $H^*\text{BP}(\max(-1, n - m))$ and the set indexing the copies is a subset of the set indexing copies of $D_m^{n-1}$ if $r > 0$. Thus there are quotient maps $\theta_q^*: D_n^{-1} \rightarrow D_n$ so that

$$
\begin{array}{ccc}
D_{q+1}^{-1} & \xrightarrow{d_q^*} & D_n^{-1} \\
\downarrow & & \downarrow \\
D_{q+1}^{-n} & \xrightarrow{\theta_q^*} & D_n^{-n}
\end{array}
$$

commutes and for $q \geq 2$, $\theta_q^*$ is a projection onto a direct summand. We can choose spectra $K_q^n$ such that $H^*K_q^n \equiv D_q^n$ and get the following:

**Proposition 1.1.** For $q \geq 0$, $0 \leq n \leq 2$, there exist maps $\theta_q: K_q^n \rightarrow K_q^{n-1}$ and $d_q: K_q^n \rightarrow \Sigma K_q^{n+1}$ so that

(i) $H^*(\theta_q): D_n^{-1} \rightarrow D_q^n$ is just the $\theta_q^*$ mentioned above,

(ii) $H^*(d_q): D_n^{-n} \rightarrow D_q^n$ is the differential $d_q^*$ in the acyclic resolution,

(iii) $K_q^n \xrightarrow{d_q} \Sigma K_q^{n+1} \xrightarrow{\theta_q} K_q^{n-1}$ commutes,

(iv) if $q \geq n + 1$ there is a map $s_q: K_q^{-1} \rightarrow K_q^n$ so that $s_q \theta_q$ is the identity.

**Proof.** Cases $n = 0$ and 1 were done in Proposition 2.7 of [6]. For $n = 2$, parts (i) and (iv) follow immediately from the structure of the $D_q^n$'s mentioned earlier in this paragraph. Since for $q \geq 2$, $K_q^2$ is a wedge of Eilenberg–MacLane spectra, the only thing that needs to be done is the construction of $d_0$ and $d_1$. To get $d_0$, we want to realize $d_q^*$. Looking at $d_0$'s effect on just one factor (see [6]), we have

$$
\Sigma 2^{(p-1)/j} H^*\text{BP}(1) \rightarrow \bigoplus_{i > k} \Sigma 2^{2(p-1)/i} H^*\text{BP}(1) \xrightarrow{d_q^*} H^*\text{BP}(2)
$$

This is well defined since the formula

$$
P_i^j Q_{i-1}^j = Q_i^j P_{i-p^j}^j
$$

implies that $P^j Q_0$ and $P^j Q_1$ are in $E_2 \cdot A$ so that $Q_0 \chi P_i^j$ and $Q_i \chi P_i^j$ are zero in $H^*\text{BP}(2) \equiv A/A \cdot E_2$. Since, as proved in [8], the Adams spectral sequence for
[BP(2), BP(1)] collapses to its \( E_2 \) term, this element of \( E^{0,2(p-1)}_2(\text{BP}(1), \text{BP}(2)) \) gives a map \( S^j = \text{BP}(2) \to \Sigma^{2(p-1)j} \text{BP}(1) \) making
\[
\begin{array}{ccc}
\text{BP}(2) & \xrightarrow{S^j} & \Sigma^{2(p-1)j} \text{BP}(1) \\
\downarrow p & & \downarrow p \\
\mathbb{Z}/p & \xrightarrow{\chi^p} & \Sigma^{2(p-1)j} \mathbb{Z}/p
\end{array}
\]
commute. The map \( d_1 \) is handled in the same way. \( \square \)

2. Adapted complexes

Let \( B \) a subalgebra of \( A \) and \( E \) a ring spectrum with \( H^*E = A/B = A/\overline{A} \). Write \( N(k) \) for \( A/A \{ \overline{w}_k \} \) where \( w_k = \{ \chi^P^i \mid i > k \} \) and let \( 1 : E \to H \) be the generator of \( H^*E \) over \( A \). If \( Z \) is a finite CW complex and \( h' \in E_{m}Z \) with \( m < 2p(k+1) - 1 \) and \( h : \Sigma^m DZ \to E \) is the dual of \( h' \), then we say that \((Z, h')\) is adapted to \( N(k) \) if
\[
A\{ \overline{B}, w_k \} \to A \xrightarrow{h' \cdot 1^*} H^* \Sigma^m DZ
\]
is exact. The degree of the adapted complex is the degree \( m \) of \( h' \) in \( E_{m}Z \). The aim of this section is to prove the following theorem.

**Theorem 2.1.** For each \( k \geq 0 \), there is a finite CW complex \( Z_k \) and \( h'_k \in \text{BP}(2)_mZ_k \) with \( m = 2pk + 3 \) so that \((Z_k, h'_k)\) is adapted to \( M_2(k) \). Also \( h'_k \) can be chosen so that \( d_{0*}h'_k = 0 \) where \( d_0 \) is as in Proposition 1.1.

Let \( R = \{ \chi^P^i \mid Q_2Q_0P^i \neq 0, I \text{ admissible, } i_1 \geq k \} \). If \( \varepsilon : A \to M_2(k) \) is the projection map, then \( \varepsilon(R) \) forms a vector space basis for \( M_2(k) \). Lemma 3.6 of [6] reduces the construction of an adapted complex for \( M_2(k) \) to proving the next proposition.

**Proposition 2.2.** For each \( P \in R \) there is a CW complex \( Z_P \) and \( \overline{h}_P \in \mathbb{Z}/p_{2pk+3}Z_P \) with
(1) \( \chi(P)^*\overline{h}_P \neq 0 \),
(2) \( \overline{h}_P = 1_P^*h'_P \) for some \( h'_P \in \text{BP}(2)_*Z_P \) where \( 1_P \) is the reduction.

\[
\text{BP}(2)_*Z_P \longrightarrow \mathbb{Z}/p_{2pk+3}Z_P.
\]

This will suffice since their lemma tells us that \( Z = \bigvee_{P \in R} Z_P \) and \( h' = \bigoplus_{P \in R} h'_P \)
will be adapted to \( M_2(k) \).

**Proof of Proposition 2.2.** Take any \( \chi^P^i \in R_i \), so \( Q_2Q_0P^i \neq 0 \), \( I = (i_1, \varepsilon_1, i_2, \varepsilon_2, \ldots, i_m, \varepsilon_m) \) is admissible and \( i_1 \geq k \). \( Q_2Q_0P^i \neq 0 \), implies \( \varepsilon_1 = 0 = \varepsilon_2 \). Set
\[
eq 2i_1 - \sum_{j=2}^{m} 2(p-1)i_j - \sum_{j=3}^{m} \varepsilon_j, \quad t = \sum_{i=3}^{m} \varepsilon_i \quad \text{and} \quad s = \frac{1}{2}(e-t).
\]
The $Z$ associated to $\chi P^I$ will be a subcomplex of
\[
C(s, t + 3) = \bigwedge_{s} C P^\infty \wedge \bigwedge_{t+3} BZ/p,
\]
the $s$-fold smash product of $C P^\infty$ smashed with the $(t+3)$-fold smash product of $BZ/p$. Then $H^{2s+t-3}C(s, t + 3)$ is the first nonzero group and is generated by a single nonzero element $c$.

Let $pr$ be the vector space map
\[
pr: H^*C(s, t + 3) \to (H^*C(s, t+3)/\mathcal{E}_2) \cdot H^*C(s, t + 3)).
\]

Now $s$ and $t$ have been chosen so that $Q_3 Q_1 Q_0 P^I C \neq 0$. This implies that $P^I c$ is not in $\mathcal{E}_2 \cdot H^*C(s, t + 3)$ so that $v = pr(P^I C) \neq 0$. Dualizing we get
\[
(P^I)^*pr^*(v^*) \neq 0 \text{ in } H^*_C(s, t + 3).
\]

Let $X_j$ be a finite skeleton of $C(s, t + 3)$ containing the $2p(k + 1) + |Q_0| + |Q_1| + |Q_2|$ skeleton and such that $HZ_p^* X_j$ is an $F_p$ vector space. Then $HZ_p^* X_j \to HZ/p^* X_j$ is a monomorphism.

Let $j = 2p k - 2p_{i_1}$ and $Z_j = \Sigma^j X_k$. Let $c \in H^{j+2s+t+3}Z_j$ be the generator. Then we still have $pr(P^I C) \neq 0$ in $H^* Z_j/\mathcal{E}_2 \cdot H^*Z_j$. Setting $\vec{h}_j = pr^*(pr(P^I C))^* \in H_{2pk+j}Z_j$ we find that $(P^I)^* \vec{h}_j \neq 0$. All that we have left to show is that $\vec{h}_j$ is in the image of the reduction map $1_+: BP(2)_* Z_j \to H_* Z_j$. This follows from putting together some facts from [6] and [5]. In [5], Goerss proves that the Adams spectral sequence converging to $BP(2)_* C(s, t)$ collapses to its $E_2$ term. This ensures that in the required range of dimensions the Adams spectral sequence converging to $BP(2)_* Z_j$ collapses to its $E_2$ term. As pointed out in [6], one easily checks that this $E_2$ term satisfies
\[
\text{Ext}_{\mathcal{A}}^0(H^* BP(2) \otimes H^* Z_j, Z/p) = \text{Hom}_{E_2}^*(H^* Z_j, Z/p)
\]
\[
\cong (H^* Z_j/\mathcal{E}_2 \cdot H^*Z_j)^*
\]
and the composite.
\[
BP(2)_* Z_j \to E^0_\infty \cong (H^0 Z_j/\mathcal{E}_2 \cdot H^*Z_j)^* \to HZ/p_* X
\]
is the reduction induced by $1_+: BP(2) \to HZ/p$. From this we can conclude that $\vec{h}_j = pr^*(pr(P^I C))^*$ is in the image of this reduction map; so choose some $h_j'$ with $1_+ h_j' = \vec{h}_j$.

Having constructed the adapted complex mentioned in Theorem 2.1, all that remains to be proved of that theorem is that $d_{0*} h_j' = 0$. The fact that $P^I C$ generated a free $E_2$ module in $H^* Z_j$ means that $h_j'$ will generate a submodule of $BP(2)_* Z_j$ with trivial $p, v_1$ and $v_2$ actions. This means that if we split $BP(2)_* \wedge Z_j = B \vee KV$, where KV is the Eilenberg-MacLane spectrum for some graded vector space $V$, then $H^* B$ has no free summands as a module over $A$; then $h_j': S \to B \vee KV$ is given by $* \vee f$ for some $f: S \to KV$. Since, for any spectrum $X$, $[KV, X] = \text{Hom}_A(A_* , H_0 X)$
(the appropriate Adams spectral sequence collapses), we notice that the composite
\( S^i \circ h'_j \) is nonzero if and only if the induced map in homology is. Consider

\[
\begin{align*}
S^n & \xrightarrow{h'_j} B \wedge K \overset{S^n \circ \text{Id}}{\longrightarrow} \Sigma^{2(p-1)j} \text{BP}(1) \wedge Z_l \\
& \xrightarrow{\rho} HZ/p \wedge Z_l \overset{\chi^p \circ \text{Id}}{\longrightarrow} \Sigma^{2(p-1)} HZ/p \wedge Z_l
\end{align*}
\]

For dimensional reasons, since \( Z_l \) is a space, if \( j > k \), \( \chi^p \circ \rho \circ h'_i \equiv * \). Since reduction from \( \text{BP}(1) \) to \( HZ/p \) is a monomorphism in homology, we find that \( S^i \circ h'_j \) is zero in homology. Hence \( S^i \circ h'_j = * \) when \( j > k \) so \( d_{0*} h'_j = 0 \).

3. Construction of \( \text{BP}(2)^k \)

In this section we prove Theorem \( \Lambda \) for \( n = 2 \). The proof uses the fact that it has already been proved for \( n = -1 \). To get started we need the following lemma which encapsulates the role adaptive complexes play in the construction.

**Lemma 3.1.** Let \( (T, h') \) be the adapted complex constructed in Section 2 and \( h: DT \to K_2^2 = \text{BP}(2) = X_0 \) the dual of \( h' \). Given parts (a), (b) and (c) of Theorem \( \Lambda \) for \( q \leq t \), i.e. cofiber sequences

\[
\begin{align*}
K^2_{q+1} & \xrightarrow{i^*_{q+1}} X_{q+1} \xrightarrow{p^*_q} X_q \xrightarrow{e_q} \Sigma K^2_{q+1}
\end{align*}
\]

with \( e_q i_q = d_q \), and given liftings \( h_q: DT \to X_q \) for \( q \leq t + 1 \). Then \( \ker i^*_q \cap \ker h^*_q = 0 \) for \( q \leq t \).

**Proof.** The proof is by induction on \( q \). There are two base cases. If \( q = 0 \) the result follows since \( i_0 = \text{Id} \). If \( q = 1 \), take \( v \in \ker i^*_1 \). Then \( v = p^*_1 w \) for some \( w \in H^* X_0 \). Now \( \ker p^*_1 = \text{Im} e^*_0 \) implies that either \( v = 0 \) in which case we are done or else \( w \notin \text{Im} e^*_0 = \ker d^*_0 = \ker h^*_0 \). The last equality is a consequence of Theorem 2.1. Thus \( 0 \neq h^*_0 w = h_1 p^*_1 w = h_1^* v \) where the second equality is a consequence of the fact that \( h_1 \) is a lifting of \( h_0 \). Hence \( \ker i^*_1 \cap \ker h^*_1 = 0 \).

Now assume inductively that \( \ker i^*_q \cap \ker h^*_q = 0 \) for \( q \leq s < t \). Take \( v \in \ker i^*_s \). Then \( v = p^*_s w \) for some \( w \in H^* X_s \). Now \( i^*_s w \in \ker d^*_s = \text{Im} d^*_s \) so choose \( x \in H^* K^2_{s+1} \) such that \( d^*_s x = i^*_s w \). Let \( w' = w - e^*_s x \). Then \( w' \in \ker i^*_s \) so by our induction hypothesis \( h^*_s w' \neq 0 \). But

\[
\begin{align*}
h^*_s w' &= h^*_s p^*_s (w - e^*_s x) = h^*_{s+1} v.
\end{align*}
\]

So \( \ker i^*_s + \ker h^*_s = 0 \). □

**Proof of Theorem \( \Lambda \) for \( n = 2 \). Let**

\[
\begin{align*}
\cdots & \xrightarrow{p_{q-1}} Y_{q-1} \xrightarrow{p_1} Y_1 \xrightarrow{e_1} \Sigma Y_1 \xrightarrow{h_1} HZ/p
\end{align*}
\]
be the tower whose homotopy inverse limit is $B(k)$, i.e. use $n = -1$ of Theorem A. We have the cofiber sequences

$$K_{q+1}^{-1} \xrightarrow{i_{q+1}} Y_{q+1} \xrightarrow{p_q} Y_q \xrightarrow{e_q} K_{q+1}^{-1}$$

and maps with the properties of Theorem A. One additional property of this case which we heavily depend on is given as Lemma 1.9 in [6], namely: for $Z$ a finite complex, $m \leq 2p(k+1) - 1$ and $\gamma : \Sigma^m DZ \to H\mathbb{Z}/p = Y_0$, any lifting of $\gamma$ to $Y_q$ lifts to $Y_{q+1}$. Let $(Z, h')$ be the complex adapted to $M_2(k)$ constructed in Section 2. Set $T = \Sigma^{2pk+3} DZ$ and let $h : T \to BP(2)$ be dual to $h'$.

We proceed by induction on $q$ with the hypothesis:

$H(t)$: For $q \leq t+1$, there are spectra $X_q$ and maps $e_q, i_{q+1}$ for $q \leq t$ so that properties (a)-(d) of Theorem A hold. Additionally, there are maps $\theta' : X_0 \to Y_q$ for $q \leq t+1$ so that for $q \leq t$

$$K_{q+1}^2 \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K_{q+1}^2$$

is a commutative diagram of cofibration sequences. Also, $h$ lifts to $X_{t+1}$.

Now $X_0$ is $BP(2)$ and $e_0$ is $d_0 : BP(2) \to \bigvee_{j \geq k} \Sigma^{2(p-1)j} BP(1)$. $X_1$ is defined to be the fiber of $e_0$ and setting $i_0$ to be the identity gives $e_0i_0 = d_0$. $\theta'_0 : BP(2) \to H\mathbb{Z}/p$ is the usual reduction map and $\theta'_0$ can be chosen to be any map satisfying (3.2), which completes the proof of $H(0)$.

$H(1)$ needs to be dealt with in a special manner. Define $e_1$ to fill in the following diagram of cofibrations, where the top square commutes because $d_1 \circ d_0 = 0$.

$$
\begin{array}{cccc}
\Sigma^{-1} BP(2) = \Sigma^{-1} K_0^2 & \to * \\
\downarrow d_0 & \downarrow \\
K_1^2 & \to \Sigma K_2^2 \\
\downarrow i_1 & \\
X_1 & \to \Sigma K_2^2 \\
\end{array}
$$

So by definition, $e_1i_1 = d_1$. To get $e_1\theta'_1 = \theta_2e_1$ we may have to adjust $e_1$ a bit. Setting $\Delta = e_1\theta'_1 - \theta_2e_1$ we find that

$$\Delta i_1 = e_1\theta'_1 - \theta_2d_1 = d_1\theta_1 - \theta_2d_1 = 0.$$

Thus the top square in the following diagram commutes defining $g$. 

If \( \rho : \text{BP}(2) \to H \) is the reduction map, then, since \( \rho^* : H^* H \to H^* \text{BP}(2) \) is surjective and \( K_{-1}^2 \) is a generalized Eilenberg-MacLane spectrum, \( g \) factors as
\[
\text{BP}(2) \xrightarrow{\rho = \theta_0} H = Y_0 \xrightarrow{s'} K_{-1}^2.
\]
Define \( e'_i = e_i - g' p_0 : Y_i \to K_{-1}^2 \). Notice that in the tower defining \( B(k) \) given in Theorem A, \( e_i \) can be replaced by \( e'_i \) since
\[
e'_i i_i = (e_i - g' p_0) i_i = e_i i_i = d_i.
\]
This replacement may require adjustment of the higher \( e_i \)'s but this can be done as in the original proof. So
\[
e'_i \theta_i - \theta_2 e_i = \Delta - g' p_0 \theta_i = \Delta - g' \theta'_0 p_0 = \Delta - \Delta = 0.
\]
To get the lifting of \( h \), notice that \( \theta'_i h_i \) lifts, so \( \theta_2 e_i h_i = e_i \theta'_i h_i = 0 \). \( T \) was chosen so that \( \text{BP}(0)^* T \to H \mathbb{Z}/p^* T \) is a monomorphism, hence \( e_i h_i = 0 \) and \( h_i \) lifts to \( h : T \to X_2 \), completing the verification of \( H(2) \).

Now assume \( H(t-1) \) with \( t \geq 2 \). This proof of the induction step is essentially that of [6]. Since \( t+1 \geq 3 \), there is a map \( s_{t+1} : K_{t+1}^1 \to K_{t+1}^2 \) such that \( s_{t+1} \theta_{t+1} \) is the identity. Set
\[
e_i = s_{t+1} e_i \theta'_i : X_i \to 2K_{t+1}^2
\]
and let \( X_{t+1} \) be the fiber of \( e_i \). Then by our induction hypothesis,
\[
e_i i_i = s_{t+1} e_i \theta'_i i_i = s_{t+1} e_i i_i \theta_i = s_{t+1} e_i \theta_{t+1} d_i = s_{t+1} \theta_{t+1} d_i = d_i.
\]
To lift \( h \), to \( X_{t+1} \), notice that \( \theta'_i h_i \) is a lifting of \( \theta'_i h \). By Lemma 1.7 of [6] this must lift even further so we must have \( e_i \theta'_i h_i = 0 \) so \( 0 = s_{t+1} e_i \theta'_i h_i = s_{t+1} \theta_{t+1} e_i h_i = e_i h_i \) and \( h_i \) lifts. To get the diagram (3.2) we need to show
\[
X_i \xrightarrow{e_i} \Sigma K_{t+1}^2
\]
\[
\downarrow \theta'_i \quad \downarrow \theta_{t+1}
\]
\[
Y_i \xrightarrow{e_i} \Sigma K_{t+1}^1
\]
commutes and then simply define \( \theta'_{t+1} \) to satisfy (3.2). Since \( K_{t+1}^1 \) is an Eilenberg-MacLane spectrum, it suffices to show \( \theta_{t+1} e_i = e_i \theta'_i \) in cohomology. Lemma 3.1
indicates that it is sufficient to show
\[ i_t^* e_t^* \theta_t^* = i_t^* \theta_t^* e_t^* \] and
\[ h_t^* e_t^* \theta_t^* = h_t^* \theta_t^* e_t^*. \]
However
\[ \theta_{t+1} e_i h_t - e_i \theta_t^* i_t = \theta_{t+1} d_i - e_i \theta_t = \theta_{t+1} d_i - d_i \theta_t = 0 \]
and an earlier argument showed \( e_i h_t = 0 = e_i \theta_t^* h_t \). Finally, property (d) of Theorem A holds here because
\[ \theta_{t+1} h_t \rightarrow (K^2_{t+1})_k Z \rightarrow \theta_{t+1} h_t \]
is injective. This completes the proof of the inductive step and the theorem. \( \square \)

To finish this section, we need to complete the proof of Theorem B begun in the introduction. Recall that \( \text{BP}(2)^k \) is defined to be \( \text{holim}_q X_q \). We have already shown that if \( Z \) is a spectrum, \( w: \text{BP}(2)^k \rightarrow \text{BP}(2) \) the map from \( \text{holim}_q X_q \rightarrow X_0 \)
and \( m \leq 2p(k + 1) - 1 \) then \( \{ \text{Im } w_q: \text{BP}(2)^m \rightarrow \text{BP}(2)_m Z \} \leq \bigcup_{j > k} \ker S_j^i \) and we need to show that if \( HZ^p S^m \) is an \( F_p \) vector space then \( \text{Im } w_q = \bigcup_{j > k} \ker S_j^i \).

**Proof.** Smash all of the diagrams used to construct \( \text{BP}(2)^k \) with \( Z \). Take any \( f: S^m \rightarrow \text{BP}(2) \wedge Z \) with the property that \( f \in \cap_{j > k} \ker S_j^i \subseteq \text{BP}(2)_m Z \) and \( m \leq 2p(k + 1) - 1 \). The first condition on \( f \) guarantees that \( f \) lifts to \( f_1: S^m \rightarrow X_1 \wedge Z \). Let \( f_1 \) be the composite
\[ S^m \xrightarrow{f_1} X_1 \wedge Z \xrightarrow{h_1 \wedge \text{Id}} Y_1 \wedge Z. \]

Then by Lemma 1.9 of [6], \( f_1 \) lifts to \( f_2: S^m \rightarrow Y_2 \wedge Z \) so that \( (e_1 \wedge \text{Id}_Z) \circ f_1 = 0 \). Writing \( Z \) to denote the identity map on \( Z \), the commutativity of (3.2) tells us that
\[ 0 = (e_1 \wedge Z) \circ (\theta^i \wedge Z) \circ f_1 = (\theta_2 \wedge Z) \circ (e_1 \wedge Z) \circ f_1. \]

Since \( HZ^p S^m \) is an \( F_p \) vector space, this means that \( (e_1 \wedge Z) \circ f_1 = 0 \) and so \( f_1 \) lifts to \( f_2 \). A similar argument shows that \( f_2 \) and higher lifts can always be lifted. These arguments don't need \( HZ^p S^m \) an \( F_p \) vector space, since \( K_2 S \) is a wedge of \( HZ/p \)'s for \( s \geq 3 \), but do use the fact that \( K_2 \) is a wedge summand of \( K_2 \) for \( s \geq 3 \). Thus \( f \) lifts all the way up the tower to give \( f': S^m \rightarrow \text{BP}(2)^k \wedge Z \) with \( w \circ f' = f \).

4. Spacelike spectra and maps to Brown–Gitler spectra

**Proof of Corollary D.** Throughout this proof, all pairs \( (s, t) \) should be assumed to satisfy \( s - t \geq n - 2p(k + 1) + 2 \). The first thing to notice is that since \( Y \) is a finite spectrum, \( [Y, \text{BP}(2)^k] \cong [s, \text{DY} \wedge \text{BP}(2)^k] \), and we can replace \( E_{r}^{*,*}(Y, \text{BP}(2)^k) \) by \( E_{r}^{*,*}(\text{DY} \wedge \text{BP}(2)^k) \). Let \( f: \Sigma^s DZ \rightarrow Y \) be the map making \( Y \) spacelike. Then \( Df: \Sigma^s DZ \rightarrow \Sigma^{-s} Z \) has the property that \( Df' Z: H_{s} \Sigma^s DZ \rightarrow H_{s} \Sigma^{-s} Z \) is a monomorphism. \( Df \) induces a map \( i: E_{r}^{*,*}(\Sigma^s DZ \wedge \text{BP}(2)^k) \rightarrow E_{r}^{*,*}(\Sigma^{-s} Z \wedge \text{BP}(2)^k) \) which is monic for \( s \geq 3 \) since \( Df' Z \) is monic. \( Z \) being a space, Corollary C tells us that all differentials in \( E_{r}^{*,*}(\Sigma^{-s} Z \wedge \text{BP}(2)^k) \) are zero except possibly for \( d_1 \) and \( d_2 \).
The proof is by induction on $r$ in the assertion that for $s - t \geq n - 2(k+1)+2$,

$$E^s_r(DY \wedge BP(2)^k) \cong E^s_r(DY \wedge BP(2)^k), \text{for all } r \geq 3 \tag{A}$$

and

$$i: E^s_r(DY \wedge BP(2)^k) \to E^s_r((\Sigma^{-n} Z \wedge BP(2)^k) \text{ is monic for } s \geq 3. \tag{B}$$

The base case is $r = 3$ where (A) is immediate. As noted above, $i: E^3_r(DY \wedge BP(2)^k) \to E^3_r((\Sigma^{-n} Z \wedge BP(2)^k)$ is monic for $s \geq 3$. Now consider the first differential. In $E^s_r((\Sigma^{-n} Z \wedge BP(2)^k), d_1$ is zero on $E^s_r$ for $s \geq 2$ because in that range $K^{-l+1}$ is a wedge summand of $K^{-1+1}$, and maps to the original Brown-Gitler tower always lift. Said more explicitly, $x \in E^s_r((\Sigma^{-n} Z \wedge BP(2)^k)$ is a map $x: S^{-t} \to \Sigma^{-n} Z \wedge K^{-1}$.

$$\theta_{s+1}d_1x = \theta_{s+1}e_{s+1}x = e_{s+1} \theta_{s+1}i,x = 0$$

because $\theta_{s+1}i,x = S^{-t} \to \Sigma^{-n} Z \wedge Y_{s+1}$ lifts. However $s + 1 \geq 3$ implies that $\theta_{s+1}$ is an inclusion of a wedge summand so $d_1x = 0$. Thus

$$E^s_r((\Sigma^{-n} Z \wedge BP(2)^k) \cong E^s_r((\Sigma^{-n} Z \wedge BP(2)^k).$$

Therefore $d_1$ is zero on $E^s_r(DY \wedge BP(2)^k)$ for $s \geq 3$ since $0 = d_1ix = id_1x$ and $i$ is monic. A similar argument shows $d_2$ is zero on $E^s_r$ for $s \geq 1$ so

$$E^3_r((\Sigma^{-n} Z \wedge BP(2)^k) \cong E^3_r((\Sigma^{-n} Z \wedge BP(2)^k) \text{ for } s \geq 3 \text{ and}$$

$$E^3_r(DY \wedge BP(2)^k) \cong E^3_r(DY \wedge BP(2)^k) \text{ for } s \geq 5 \text{ and}$$

$$i: E^s_3(DY \wedge BP(2)^k) \to E^s_3((\Sigma^{-n} Z \wedge BP(2)^k) \text{ is monic for } s \geq 3.$$  

For the induction step, assume both are true for some $n \geq 3$. Since $d_nx$ has filtration $\geq 3$ for any $x \in E^s_{n+1}(DY \wedge BP(2)^k)$, $0 = d_nx = id_nx$. Since $i$ is monic in this range, $d_nx = 0$. Thus both $E^s_{n+1}$ terms are isomorphic to their respective $E^s_n$ terms and $i$ remains monic on $E^s_{n+1}$ for $s \geq 3$. This completes the induction step. \[ \square \]

One final comment is in order. [6] used these spectral sequences to construct pairings $BP(1)^{ij} \wedge BP(1)^{ij} \to BP(1)^{ij}$ and to prove the uniqueness of $BP(1)^{ij}$. I have not been able to prove the analogous results for the $BP(2)^k$ case because of the extra differential involved.

Acknowledgement

This paper stems from my Ph.D. thesis done under the supervision of John Jones at the University of Warwick. I want to thank him for suggesting the project to me and also the Natural Sciences and Engineering Research Council of Canada and the University of Western Ontario for providing me with support and a place to work while writing this paper. I also want to thank Paul Goerss for warning me of certain pitfalls and Mark Mahowald for helpful comments particularly regarding style.
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