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Iterative method for solving the Neumann boundary value problem for biharmonic type equation☆

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Abstract

The solution of boundary value problems (BVP) for fourth order differential equations by their reduction to BVP for second order equations, with the aim to use the achievements for the latter ones attracts attention from many researchers. In this paper, using the technique developed by ourselves in recent works, we construct iterative method for the Neumann BVP for biharmonic type equation. The convergence rate of the method is proved and some numerical experiments are performed for testing it in dependence on the choice of an iterative parameter.

Keywords: Iterative method; Neumann problem; Biharmonic equation

1. Introduction

The solution of fourth order differential equations by their reduction to boundary value problems (BVP) for the second order equations, with the aim of using efficient algorithms for these, attracts attention from many researchers. Namely, for the biharmonic equation with the Dirichlet boundary condition, there is intensively developed the iterative method, which leads the problem to two problems for the Poisson equation at each iteration (see e.g. [4,8,9,11]). Recently, Abramov and Ulijanova [1] proposed an iterative method for the Dirichlet problem for the biharmonic type equation, but the convergence of the method is not proved. In our previous works [6,7], with the help of boundary or mixed boundary-domain operators appropriately introduced, we constructed iterative methods for biharmonic and biharmonic type equations associated with the Dirichlet boundary condition. It is proved that the methods are convergent with the rate of geometric progression. In this paper, we develop our technique in [4–7] for the Neumann BVP for the biharmonic type equation. Namely, we consider the following problem:

$$Lu \equiv \Delta^2 u - a\Delta u + bu = f \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = g_0, \qquad \frac{\partial \Delta u}{\partial \nu} = g_1 \quad \text{on } \Gamma, \tag{2}$$

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where Δ is the Laplace operator, Ω is a bounded domain in R^n $(n \geqslant 2)$, Γ is the sufficiently smooth boundary of Ω, ν is the outward normal to Γ and a, b are positive constants. Eq. (1) with other boundary conditions are met, for example, in [2,3]. An iterative method reducing the problem to a sequence of Neumann problems for Poisson equation will be proposed and investigated. Two different cases will be treated in dependence on the sign of $a^2 - 4b$.

2. Case $a^2 - 4b \geqslant 0$

In this case, we always can lead the original problem (1)–(2) to two Neumann problems for second order equation. To do this, let λ and μ be the roots of the quadratic equation

$$\xi^2 - a\xi + b = 0,$$

namely,

$$\lambda = \frac{a - \sqrt{a^2 - 4b}}{2}, \quad \mu = \frac{a + \sqrt{a^2 - 4b}}{2}. \tag{3}$$

Clearly, λ and $\mu > 0$. Put

$$L_1 = \Delta - \mu, \quad L_2 = \Delta - \lambda. \tag{4}$$

Then we have $L = L_1 L_2$ and the problem (1)–(2) is decomposed to the following problems:

$$\begin{aligned} &L_1 v \equiv \Delta v - \mu v = f(x) \quad \text{in } \Omega, \\ &\frac{\partial v}{\partial \nu} = g_1 - \lambda g_0 \quad \text{on } \Gamma, \end{aligned} \tag{5}$$

$$\begin{aligned} &L_2 u \equiv \Delta u - \lambda u = v(x) \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = g_0 \quad \text{on } \Gamma. \end{aligned} \tag{6}$$

These Neumann problems can be solved by known methods such as finite element method, boundary element method or finite difference method.

3. Case $a^2 - 4b < 0$

3.1. Construction of iterative method

Suppose λ and μ are positive numbers such that

$$\mu \geqslant \lambda, \quad \lambda + \mu = a, \quad b_1 = \lambda\mu < b. \tag{7}$$

Using the notations L_1 and L_2 given by (4) we set

$$v = L_2 u = \Delta u - \lambda u.$$

Then we get

$$L_1 v = \Delta v - \mu v = \Delta^2 u - a\Delta u + b_1 u.$$

Now, putting

$$\varphi = -\alpha u, \tag{8}$$

where

$$\alpha = b - b_1 \tag{9}$$

we can reduce the original problem (1), (2) to the problems

$$L_1 v = f + \varphi \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = g_1 - \lambda g_0 \quad \text{on } \Gamma, \tag{10}$$

$$L_2 u = v(x) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = g_0 \quad \text{on } \Gamma, \tag{11}$$

where φ as u is an unknown function but it is related with u by (8). Now consider the following iterative process for finding φ and simultaneously for finding u.

1. Given $\varphi^{(0)} \in L_2(\Omega)$, for example, $\varphi^{(0)} = 0$ in Ω.
2. Knowing $\varphi^{(k)}(x)$ in Ω $(k = 0, 1, \ldots)$ solve consecutively two problems

$$L_1 v^{(k)} = f + \varphi^{(k)} \quad \text{in } \Omega, \quad \frac{\partial v^{(k)}}{\partial \nu} = g_1 - \lambda g_0 \quad \text{on } \Gamma, \tag{12}$$

$$L_2 u^{(k)} = v^{(k)} \quad \text{in } \Omega, \quad \frac{\partial u^{(k)}}{\partial \nu} = g_0 \quad \text{on } \Gamma. \tag{13}$$

3. Compute the new approximation

$$\varphi^{(k+1)} = (1 - \tau_{k+1})\varphi^{(k)} - \alpha\tau_{k+1} u^{(k)}, \tag{14}$$

where τ_{k+1} is an iterative parameter to be chosen later.

3.2. Investigation of convergence

In order to investigate the convergence of the iterative process (12)–(14), firstly we rewrite (14) in the canonical form of two-layer iterative scheme [12]

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau_{k+1}} + (\varphi^{(k)} + \alpha u^{(k)}) = 0. \tag{15}$$

Now, we introduce the operator A defined by the formula

$$A\varphi = u, \tag{16}$$

where u is the function determined from the problems

$$L_1 v = \varphi \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{17}$$

$$L_2 u = v \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{18}$$

The properties of the operator A will be investigated in the sequel. Now, let us return to the problems (10), (11). We represent their solutions in the form

$$u = u_1 + u_2, \;\; v = v_1 + v_2, \tag{19}$$

where $u_1, u_2,\ v_1, v_2$ are the solutions of the problems

$$L_1 v_1 = \varphi \quad \text{in } \Omega, \quad \frac{\partial v_1}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{20}$$

$$L_2 u_1 = v_1 \quad \text{in } \Omega, \quad \frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{21}$$

$$L_1 v_2 = f \quad \text{in } \Omega, \quad \frac{\partial v_2}{\partial \nu} = g_1 - \lambda g_0 \quad \text{on } \Gamma, \tag{22}$$

$$L_2 u_2 = v_2 \quad \text{in } \Omega, \quad \frac{\partial u_2}{\partial \nu} = g_0 \quad \text{on } \Gamma. \tag{23}$$

According to the definition of A we have

$$A\varphi = u_1. \tag{24}$$

Since φ should satisfy the relation (8), taking into account the representation (19) we obtain the equation

$$(I + \alpha A)\varphi = -\alpha u_2, \tag{25}$$

where I is the identity operator.

Thus, we have reduced the original problem (1), (2) to the operator (25), whose right-hand side is completely defined by the data f, g_0 and g_1.

Proposition 3.1. *The iterative process* (12)–(14) *is a realization of the two-layer iterative scheme*

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau_{k+1}} + (I + \alpha A)\varphi^{(k)} = -\alpha u_2, \quad k = 0, 1, 2, \ldots \tag{26}$$

for the operator equation (25).

Proof. Indeed, if in (12), (13) we put

$$u^{(k)} = u_1^{(k)} + u_2, \quad v^{(k)} = v_1^{(k)} + v_2, \tag{27}$$

where u_2, v_2 are the solutions of Problems (22), (23), then we get

$$L_1 v_1^{(k)} = \varphi^{(k)} \quad \text{in } \Omega, \quad \frac{\partial v_1^{(k)}}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{28}$$

$$L_2 u_1^{(k)} = v_1^{(k)} \quad \text{in } \Omega, \quad \frac{\partial u_1^{(k)}}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{29}$$

From here it is easy to see that

$$A\varphi^{(k)} = u_1^{(k)}.$$

Therefore, taking into account the first relation of (27) and the above equality, from (15) we obtain (26). Thus, the proposition is proved. □

Proposition 1 enables us to lead the investigation of convergence of process (12)–(14) to the study of the scheme (26). For this reason we need some properties of the operator A.

Proposition 3.2. *If the numbers λ and μ satisfy the condition* (7) *then the operator A defined by* (16)–(18) *is linear*, *symmetric*, *positive and compact operator in the space $L_2(\Omega)$*.

Proof. The linearity of A is obvious. To establish the other properties of A let us consider the inner product $(A\varphi, \bar{\varphi})$ for two arbitrary functions φ and $\bar{\varphi}$ in $L_2(\Omega)$. Recall that the operator A is defined by (16)–(18). We denote by $\bar{u}$ and $\bar{v}$ the solutions of (17) and (18), where instead of φ there stands $\bar{\varphi}$. We have

$$\begin{aligned}(A\varphi, \bar{\varphi}) &= \int_\Omega u.\bar{\varphi}\,\mathrm{d}x = \int_\Omega u(\Delta\bar{v} - \mu\bar{v})\,\mathrm{d}x \\ &= \int_\Omega u\Delta\bar{v}\,\mathrm{d}x - \mu\int_\Omega u\bar{v}\,\mathrm{d}x = -\int_\Omega \nabla u\nabla\bar{v}\,\mathrm{d}x - \mu\int_\Omega u\bar{v}\,\mathrm{d}x.\end{aligned}$$

Noting that

$$\int_\Omega \bar{v}\Delta u\,\mathrm{d}x = -\int_\Omega \nabla\bar{v}\nabla u\,\mathrm{d}x$$

from the latter equality we get

$$(A\varphi, \bar{\varphi}) = \int_\Omega \bar{v}\Delta u\,\mathrm{d}x - \mu\int_\Omega u\bar{v}\,\mathrm{d}x.$$

In the above relation replacing $\Delta u = v + \lambda u$ we obtain

$$(A\varphi, \bar{\varphi}) = \int_\Omega v\bar{v}\,\mathrm{d}x + (\lambda - \mu)\int_\Omega u\bar{v}\,\mathrm{d}x.$$

Further, since $\bar{v} = \Delta\bar{u} - \lambda\bar{u}$ we have

$$\int_\Omega u\bar{v}\,\mathrm{d}x = \int_\Omega u(\Delta\bar{u} - \lambda\bar{u})\,\mathrm{d}x = -\int_\Omega u\nabla u\nabla\bar{u}\,\mathrm{d}x - \lambda\int_\Omega u\bar{u}\,\mathrm{d}x.$$

Hence,

$$(A\varphi, \bar{\varphi}) = \int_\Omega v\bar{v}\,\mathrm{d}x + (\mu - \lambda)\left(\int_\Omega \nabla u\nabla\bar{u}\,\mathrm{d}x + \lambda\int_\Omega u\bar{u}\,\mathrm{d}x\right).$$

Clearly,

$$(A\varphi, \bar{\varphi}) = (A\bar{\varphi}, \varphi)$$

and

$$(A\varphi, \varphi) = \int_\Omega v^2\,\mathrm{d}x + (\mu - \lambda)\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x + \lambda\int_\Omega u^2\,\mathrm{d}x\right) \geqslant 0$$

due to $\mu \geqslant \lambda$. Moreover, it is easy seen that $(A\varphi, \varphi) = 0$ if and only if $\varphi = 0$. So, we have shown that the operator A is symmetric and positive in $L_2(\Omega)$.

It remains to show the compactness of A. As is well known that if $\varphi \in L_2(\Omega)$ then the problem (17) has a unique solution $v \in H^2(\Omega)$, therefore, the problem (18) has a unique solution $v \in H^4(\Omega)$. Consequently, the operator A maps $L_2(\Omega)$ into $H^4(\Omega)$. In view of the compact imbedding $H^4(\Omega)$ into $L_2(\Omega)$ it follows that A is compact operator in $L_2(\Omega)$.

Thus, the proof of Proposition 3.2 is complete. □

Due to the above proposition the operator

$$B = I + \alpha A \tag{30}$$

is linear, symmetric, positive definite and bounded operator in the space $L_2(\Omega)$. More precisely, we have

$$\gamma_1 I < B \leqslant \gamma_2 I, \tag{31}$$

where

$$\gamma_1 = 1, \quad \gamma_2 = 1 + \alpha\|A\|. \tag{32}$$

In the future, we need the following estimate for the function u given by (17) and (18)

$$\|u\|_{H^4(\Omega)} \leqslant C\|\varphi\|_{L_2(\Omega)} \tag{33}$$

which follows from the general theory of BVP [10].

Before stating the result of convergence of the iterative process (12)–(14) we assume the following regularity of the data of the original problem (1), (2):

$$f \in L_2(\Omega), \quad g_0 \in H^{5/2}(\Gamma) \text{ and } g_1 \in H^{1/2}(\Gamma).$$

Then the problem (1), (2) has a unique solution $u \in H^4(\Omega)$. For the function u_2 determined by (22), (23) we have also $u_2 \in H^4(\Omega)$.

Theorem 3.3. *Let u be the solution of Problems* (1), (2) *and φ be the solution of Eq.* (25). *Then, if $\{\tau_{k+1}\}$ is the Chebyshev collection of parameters, constructed by the bounds γ_1 and γ_2 of the operator B, namely*

$$\tau_0 = \frac{2}{\gamma_1 + \gamma_2}, \quad \tau_k = \frac{\tau_0}{\rho_0 t_k + 1}, \quad t_k = \cos\frac{2k-1}{2M}\pi, \quad k = 1, \ldots, M \quad \rho_0 = \frac{1-\xi}{1+\xi}, \quad \xi = \frac{\gamma_1}{\gamma_2} \tag{34}$$

we have

$$\|u^{(M)} - u\|_{H^4(\Omega)} \leqslant C_1 q_M, \tag{35}$$

where

$$C_1 = C\|\varphi^{(0)} - \varphi\|_{L_2(\Omega)} \tag{36}$$

C being the constant in (33),

$$q_M = \frac{2\rho_1^M}{1+\rho_1^{2M}}, \quad \rho_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}. \tag{37}$$

In the case of the stationary iterative process, $\tau_k = \tau_0$ $(k = 1, 2, \ldots)$ we have

$$\|u^{(k)} - u\|_{H^4(\Omega)} \leqslant C_1 \rho_0^k, \quad k = 1, 2, \ldots. \tag{38}$$

Proof. According to the general theory of the two-layer iterative schemes (see [12]) for the operator equation (25) we have

$$\|\varphi^{(M)} - \varphi\|_{L_2(\Omega)} \leqslant q_M \|\varphi^{(0)} - \varphi\|_{L_2(\Omega)},$$

if the parameter $\{\tau_{k+1}\}$ is chosen by the formulae (34) and

$$\|\varphi^{(k)} - \varphi^*\|_{H^4(\Omega)} \leqslant \rho_0^k \|\varphi^{(0)} - \varphi^*\|_{L_2(\Omega)}, \quad k = 1, 2, \ldots$$

if $\tau_k = \tau_0$ $(k = 1, 2, \ldots)$. In view of these estimates the corresponding estimates (35) and (38) follow from (33) applied to the problems

$$L_1(v^{(k)} - v) = \varphi^{(k)} - \varphi \quad \text{in } \Omega, \quad \frac{\partial}{\partial \nu}(v^{(k)} - v) = 0 \quad \text{on } \Gamma,$$

$$L_2(u^{(k)} - u) = v^{(k)} - v \quad \text{in } \Omega, \quad \frac{\partial}{\partial \nu}(u^{(k)} - u) = 0 \quad \text{on } \Gamma,$$

which are obtained in the result of the subtraction (10) and (11) from (12) and (13), respectively. The theorem is proved. □

Remark. Theorem 3.3 theoretically ensures the error estimate (35) and the convergence of the iterative method (26) when the computing process is ideal, that is, all the computations are carried out with an infinity number of significant digits. But any computer makes calculations with a finite speed and a finite number of digits. In this case rounding errors may accumulate and the computing process may be unstable. In order to overcome this computational instability following Samarskii and Nikolaev [13,12] it is needed to use a stable collection of parameters $\{\tau_k^*\}$ calculated by the formula

$$\tau_k^* = \frac{\tau_0}{\rho_0 t_k^* + 1}, \quad t_k^* = -\cos\left(\frac{\pi}{2M}\theta_M^*(k)\right), \quad k = 1, \dots, M,$$

where θ_M^* is the sequence of the odd integers from 1 to $2M - 1$, determined by a rule given there.

An alternative way to treat the numerical instability of the Richardon's iterative method (26) is the use of the semi-iterative method based on stationary Richardon method as was recommended in [14] for solving a system of linear algebraic equations. But it requires extra computational work in computing sums of iterations.

In the case of the stationary process the value τ_0, as well known in [12,13], is optimal. But for calculating it we need to know $\|A\|$, which is difficult to be determined. Therefore, in the next section, first we shall determine $\|A\|$ experimentally from one particular case, and then use it to find nearly optimal value of the iterative parameter in some other cases.

4. Numerical results

We performed some limited experiments in MATLAB for testing the convergence of the iterative process (12)–(14) in dependence on iterative parameters, which are taken fixed at all iterations, i.e. $\tau_k \equiv \tau$. In the examples considered below the computational domain is a rectangle with the uniform grid including 65×65 nodes. The Neumann problems for the second order equations (12), (13) are discretized by difference schemes of second order approximation obtained by a variational method. The stopping criterion for the iterative process is $\|\varphi^{(k)}\|_\infty < 10^{-4}$.

Example 1. We take an exact solution $u = \sin x \sin y$ in the rectangle $[0, \pi] \times [0, \pi]$ and vary the coefficients a and b in the equation (1). Hence, the right-hand side of the equation is $f = (4 + 2a + b) \sin x \sin y$.

In all two examples we take $a = 1$, $\lambda = \mu = 0.5$ and change only b. First, we take $b = 1$ and make experiments for testing the convergence of the iterative method in dependence on the choice of the iterative parameter τ. The results of computation are presented in Table 1, where k is the number of iterations, err is the error of approximate solution u_{ap}, err $= \|u_{ap} - u\|_\infty$.

From Table 1 we see that it is possible to adopt experimentally $\tau_{\text{opt}} = 0.36$. Meanwhile, the formulae (32), (34) give

$$\tau_{\text{opt}} = \frac{2}{2 + \alpha\|A\|}, \tag{39}$$

where $\alpha = b - \lambda\mu$. From here we can calculate $\|A\| \approx 4.7407$. Note that for a fixed domain $\|A\|$ depends only on λ and μ. Therefore, this value of $\|A\|$ can be used for calculating good iterative parameter for the cases of other values

Table 1
Case $b = 1$

τ	k	err
0.20	30	0.0006
0.30	20	0.0006
0.35	16	0.0006
0.36	15	0.0006
0.37	16	0.0006
0.40	18	0.0005
0.45	39	0.0008
0.50	∞	

Table 2
Case $b = 0.5$

τ	k	err
0.40	13	0.0011
0.50	13	0.0011
0.55	9	0.0011
0.60	8	0.0011
0.6279	7	0.0011
0.65	7	0.0011
0.70	8	0.0009
0.80	15	0.0013

Table 3
Case $b = 1.5$

τ	k	err
0.20	31	0.0005
0.22	29	0.0005
0.25	25	0.0005
0.27	24	0.0005
0.2523	25	0.0005
0.30	41	0.0006
0.32	111	0.0001
0.35	∞	

Table 4
Case $b = 1$

τ	k	err
0.20	49	$1.2e-5$
0.30	32	$6.6e-6$
0.35	25	$4.9e-6$
0.36	26	$4.9e-4$
0.37	25	$5.2e-6$
0.40	24	$6.8e-5$
0.45	45	$2.1e-4$
0.50	∞	

Table 5
Case $b = 0.5$

τ	k	err
0.4	21	$1.4e-5$
0.5	16	$8.1e-6$
0.5	16	$8.1e-6$
0.55	14	$6.9e-6$
0.6279	12	$2.8e-6$
0.65	11	$6.9e-6$
0.7	10	$2.1e-4$
0.8	18	$2.1e-4$

Table 6
Case $b = 1.5$

τ	k	err
0.20	50	$8.1e-6$
0.22	46	$5.8e-6$
0.25	40	$5.2e-6$
0.2523	40	$4.5e-6$
0.27	38	$4.5e-6$
0.30	48	$2.1e-4$
0.32	127	$2.0e-4$
0.35	∞	

of b provided that a, λ, μ are unchanged. The results presented in Tables 2 and 3 support this assertion. In these tables as in Tables 4–6 below the values of τ in bold face are calculated by the formula (39) with $\|A\| = 4.7407$.

Example 2. We take an exact solution $u = (x^2-4)(y^2-1)$ in the rectangle $[-2, -2] \times [-1, 1]$. The right-hand side of the equation (1) is $f = 8 - 2a(x^2 + y^2 - 5) + b(x^2 - 4)(y^2 - 1)$.

The results of the convergence of the method are presented in Tables 4–6.

From Tables 1–6 it is clear that the values of iterative parameter τ computed by (39), where $\|A\|$ is found experimentally, are nearly optimal.

5. Concluding remark

In the paper, an iterative method was proposed for reducing the Neumann problem for biharmonic type equation to a sequence of Neumann problems for second order equations. The convergence rate of the method depends on the choice of the iterative parameter τ, whose optimal value is determined by the norm of the operator A. In its turn, this norm is fully defined for each domain and each value of the coefficient a in Eq. (1) if a choice of λ and μ is fixed, for example, $\lambda = \mu = a/2$. By experimental way it is possible to find this quantity. Theoretical estimate for $\|A\|$ is an interesting problem to be studied.

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