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The effects of implementation delay on decision-making under uncertainty

Erhan Bayraktar*, Masahiko Egami

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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Abstract

In this paper, we accomplish two objectives: First, we provide a new mathematical characterization of the value function for impulse control problems with implementation delay and present a direct solution method that differs from its counterparts that use quasi-variational inequalities. Our method is direct, in the sense that we do not have to guess the form of the solution and we do not have to prove that the conjectured solution satisfies conditions of a verification lemma. Second, by employing this direct solution method, we solve two examples that involve decision delays: an exchange rate intervention problem and a problem of labor force optimization.

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1. Introduction

Implementation delays occur naturally in decision-making problems. Many corporations face regulatory delays, which need to be taken into account when the corporations make decisions under uncertainty. A decision made will be carried out only after certain amount of time elapses, for example, due to regulatory reasons. The decision involves optimally exercising a *real option*

* Corresponding author. Tel.: +1 734 9047024; fax: +1 734 7630937.
E-mail addresses: erhan@umich.edu (E. Bayraktar), egami@umich.edu (M. Egami).

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or optimally manipulating (with some associated cost) a state variable, which is the source of uncertainty. Several problems that fit into this framework can be found in the literature: The work of Bar-Ilan and Strange [6] constitutes the first study considering how delays affect rational investment behavior. Keppo and Peura [17] consider the decision making problem a bank has to solve when it is faced with a minimum capital requirement, a random income, and delayed (and costly) recapitalization. The bank's problem is to determine when to raise capital from its shareholders and the amount to be raised, given that this transaction requires a heavy preparatory work, which causes delay. Bar-Ilan and Strange [7] consider (irreversible) sequential (2 stage) investment decision problems given two sources of delay: one due to market analysis in the first stage and the other due to construction of a production facility in the second stage. In each stage the firm's problem is to decide whether to continue entering into the market (of that product) or to abandon it. See also Subramanian and Jarrow [24] who consider the problem of a trader (who is not a *price taker*) who wants to liquidate her position and encounters execution delays in an illiquid market. Alvarez and Keppo [3] study the impact of delivery lags on irreversible investment demand under revenue uncertainty. Øksendal et al. [20,15] consider the classical stochastic control of stochastic delay systems.

The problem of finding an optimal decision (in the presence of delays) can be characterized as a stochastic impulse control problem or an optimal stopping problem. In the papers cited above the impulse control problem or the optimal stopping problem were solved by using a system of quasi-variational inequalities. (See e.g. Bensoussan and Lions [8] and Øksendal and Sulem [21] for the relationship between control problems and quasi-variational inequalities.) In a different approach, Øksendal and Sulem [22] solve a version of delay problems, in which the controller decides on the magnitude of control at the time of decision-making before any delay (the decision is implemented after some delay). They convert the optimal impulse control problem with delayed reaction into a no-delay optimal stopping/impulse control problem. Note that choosing the control in this way introduces strong path dependence of the controlled process.

Here, we solve the impulse control problems with delays *directly* and the magnitude of the impulses are chosen at the time of action, not at the time of decision-making, by providing a new characterization of the *value function*. The controlled process is a non-Markov process in this case, too, since, depending on when a point in the state space is reached, it has different roles. But the controlled process in this case regenerates after a decision is implemented, and the value of the state process during the delay time depends on the past only through the value of the state process at the time of decision-making. We will only consider threshold and band policies in this paper, since we expect that the non-Markovian structure will make finding the optimal solution much more difficult if we allow more general strategies. For example, because of the lack of the Strong Markov property, we were unable to prove the concavity properties of the value function when the admissible strategies were a superset of band or threshold strategies.

Our results rely on the works of Dynkin [13,14] (see e.g. Theorem 16.4) and Dayanik and Karatzas [12], who give a general characterization of optimal stopping times of one dimensional diffusions, and on the work of Dayanik and Egami [11], who characterize the value function of stochastic impulse control problems. Our method is direct, in the sense that we do not have to guess the form of the solution and we do not have to prove that the conjectured solution satisfies conditions of a verification lemma as all the methods in the above literature do. Other works similar in vein to ours that provide different characterizations of the value function of impulse/singular control problems for one dimensional diffusions rather than solving variational inequalities are Alvarez [1,2], Alvarez and Virtanen [4], and Weerasinghe [25].

We give a geometric characterization of the value function, specifically, we find very general conditions on the reward function and the coefficients of the underlying diffusion under which the value function can be linearized (in the continuation region) after a suitable transformation. Then the problem of determining the value function is equivalent to determining the slope (if admissible strategies are threshold strategies), the slope and the intercept (if admissible strategies are band strategies) from first order conditions. To show the efficacy of our methodology we apply it to an optimization problem of a central bank that needs to carry out exchange rate intervention (this is the Krugman model of interest rates considered, among others, in Mundaca and Øksendal [18]) when there is delay in the implementation of its decisions. Also, using our methodology we will find optimal hiring and firing decisions of a firm that faces stochastic demand and has to conform to regulatory delays. Other works that deal with labor optimization problems are Bentolila and Bertola [9], and Shepp and Shiryaev [23] who model firing and hiring decisions as singular controls. It is also worth pointing out that an impulse control study when the underlying process is a superposition of a Brownian motion and a compound Poisson process (when the jumps are of phase type) is given by Bar-Ilan et al. [5] with management of foreign exchange reserves and labor optimization in mind.

The rest of the paper is organized as follows: In Section 2, we give a characterization of general threshold strategies with implementation delays and provide an easily implemented algorithm to find the value function and the optimal control. To illustrate our methodology, we will solve a delayed version of an example from Mundaca and Øksendal [18] (also see Øksendal [19]). A similar problem to the one we consider was solved in Øksendal and Sulem [22] in which the controller decides on the magnitude of control at the time of decision-making before any delay. In Section 3, we work with a band policy. In this section we work on the specific example of optimal hiring and firing decisions rather than providing a general characterization for the value function. We again provide an easily implemented algorithm to find the optimal control. Finally, we conclude in Section 4.

2. Optimal threshold strategies

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a standard Brownian motion $W = \{W_t; t \geq 0\}$ and consider the diffusion process X^0 with state pace $\mathcal{I} = (c, d) \subseteq \mathbb{R}$ and dynamics

$$dX_t^0 = \mu(X_t^0)dt + \sigma(X_t^0)dW_t \tag{2.1}$$

for some Borel functions $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to (0, \infty)$. (We assume that the functions μ and σ are sufficiently regular so that (2.1) makes sense.) Here we take c and d to be a natural boundaries. We use "0" as the superscript to indicate that X^0 is the uncontrolled process. We denote the infinitesimal generator of X^0 by $\mathcal{A}$ and consider the ODE $(\mathcal{A} - \alpha)v(x) = 0$. This equation has two fundamental solutions, $\psi(\cdot)$ and $\varphi(\cdot)$. We set $\psi(\cdot)$ to be the increasing and $\varphi(\cdot)$ to be the decreasing solution. $\psi(c+) = 0, \varphi(c+) = \infty$ and $\psi(d-) = \infty, \varphi(d-) = 0$ because both c and d are natural boundaries. First, we define an increasing function

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}. \tag{2.2}$$

Next, following Dynkin [14], p. 238, we define concavity of a function with respect F as follows: A real valued function u is called *F-concave* on (c, d) if, for every $c < l < r < d$ and $x \in [l, r]$,

$$u(x) \geq u(l)\frac{F(r) - F(x)}{F(r) - F(l)} + u(r)\frac{F(x) - F(l)}{F(r) - F(l)}.$$

Suppose that at any time $t \in \mathbb{R}_+$ and any state $x \in \mathbb{R}_+$, we can intervene and give the system an impulse $\xi \in \mathbb{R}$. Once the system gets intervened, the point moves from x to $y \in \mathbb{R}_+$ with associated reward and cost. An impulse control for the system is a double sequence,

$$\nu = (T_1, T_2, \dots T_i \dots ; \xi_1, \xi_2, \dots \xi_i \dots) \tag{2.3}$$

where $0 \le T_1 < T_2 < \cdots$ is an increasing sequence of $\mathbb{F}$-stopping times such that $T_{i+1} - T_i \ge \Delta$, and $\xi_1, \xi_2 \dots$ are $\mathcal{F}_{(T_i+\Delta)-}$ measurable random variables representing impulses exercised at the corresponding intervention times T_i with $\xi_i \in Z$ for all i where $Z \subset \mathbb{R}$ is a given set of admissible impulse values. The controlled process until the first intervention time is described as follows:

$$\begin{cases} \mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, & 0 \le t < T_1 + \Delta \\ X_{T_1+\Delta} = \Gamma(X_{(T_1+\Delta)-}, \xi_1) \end{cases} \tag{2.4}$$

with some mapping $\Gamma : (c, d) \times \mathbb{R} \to \mathbb{R}$. We consider the following performance measure associated with $\nu \in \mathcal{V}$ (= a collection of admissible strategies),

$$J^{\nu}(x) = \mathbb{E}^x \left[\int_0^{\infty} \mathrm{e}^{-\alpha s} f(X_s)\mathrm{d}s + \sum_{T_i < \infty} \mathrm{e}^{-\alpha(T_i+\Delta)} K(X_{(T_i+\Delta)-}, X_{T_i+\Delta}) \right]. \tag{2.5}$$

The objective (we shall call it the *delay problem*) is to find the optimal strategy ν^* (if it exists) and the value function:

$$v(x) \triangleq \sup_{\nu \in \mathcal{V}} J^{\nu}(x) = J^{\nu^*}(x). \tag{2.6}$$

Remark 2.1. The controlled process X is not a Markov process, since depending on whether a point is reached in the time interval $[T_i, T_i + \Delta)$ or not, that point has different roles. (The controlled process might jump or not at a given point depending on how it reaches to that point.) However, (1) the process regenerates at times $\{T_i + \Delta\}_{i \in \mathbb{N}}$, and (2) the value of the process at time $T \in (T_i, T_{i+\Delta})$, X_T, depends on the information up to T_i, $\mathcal{F}_{T_i}$, only through the value of the process at time T_i, X_{T_i}. Instead of finding the optimal strategy for a non-Markov process, we will use the hints of Markovian features to find the optimal *threshold strategy* (see Assumption 2.1).

The following is a standing assumption in Sections 2.1 and 2.2.

Assumption 2.1. We make the following assumptions in this section:

(a) We will assume that the set of admissible strategies is limited to *threshold strategies*. These strategies are determined by specifying two numbers $a \in (c, d)$ and $b \in (c, d)$ as follows: At the time the uncontrolled process hits level b, the controller decides to reduce the level of the process from $\xi_{T_i-} = b$ to $a < b$, through an intervention, and save the continuously incurred cost (which is high if the process is at a high level). But the implementation of this decision is subject to a delay of Δ units of time. Note that $\xi_{(T_i+\Delta)-}$ might be less than a. In that case the impulse applied increases the value of the process. Otherwise, if the value of the process is greater than a at time $(T_i + \Delta)-$ then the intervention reduces the level of the process to a.

(b) The running cost function $f : (c, d) \to \mathbb{R}$ is a continuous functions that satisfies

$$\mathbb{E}^x \left[\int_0^{\infty} \mathrm{e}^{-\alpha s} |f(X_s)|\mathrm{d}s \right] < \infty. \tag{2.7}$$

(c) For any point $x \in (c, d)$, we assume

$$K(x, x) < 0. \tag{2.8}$$

We make this assumption to account for the fixed cost of making an intervention.

2.1. Characterization of the value function

In this section, we will show that when we apply a suitable transformation to the value function corresponding to a particular threshold strategy (that is identified by a pair (a, b)), the transformed value function is linear on $(0, F(b))$. This characterization will become important in determining the optimal threshold strategy in the next section.

Let us define

$$g(x) \triangleq \mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s^0) ds \right]. \tag{2.9}$$

The following identity, which can be derived using the Strong Markov Property of X^0, will come in handy in a couple of computations below:

$$\mathbb{E}^x \left[\int_0^\tau e^{-\alpha s} f(X_s^0) ds \right] = g(x) - \mathbb{E}^x [e^{-\alpha\tau} g(X_\tau^0)], \tag{2.10}$$

for any stopping time τ under the assumption (2.7).
Now, let us simplify J^ν by splitting the terms in (2.5). We can write the first terms (the term with the integral) as

$$\begin{aligned}
\mathbb{E}^x \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] &= \mathbb{E}^x \left[\int_0^{T_1+\Delta} e^{-\alpha s} f(X_s^0) ds \right. \\
&\quad \left. + e^{-\alpha(T_1+\Delta)} \mathbb{E}^{X_{T_1+\Delta}} \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] \right] \\
&= g(x) - \mathbb{E}^x [e^{-\alpha(T_1+\Delta)} g(X_{T_1+\Delta}^0)] \\
&\quad + \mathbb{E}^x \left[e^{-\alpha(T_1+\Delta)} \mathbb{E}^{X_{T_1+\Delta}} \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] \right] \\
&= g(x) - \mathbb{E}^x [e^{-\alpha(T_1+\Delta)} g(X_{(T_1+\Delta)-})] \\
&\quad + \mathbb{E}^x \left[e^{-\alpha(T_1+\Delta)} \mathbb{E}^{X_{T_1+\Delta}} \left[\int_0^\infty e^{-\alpha s} f(X_s) ds \right] \right],
\end{aligned} \tag{2.11}$$

while the second term can be developed as

$$\begin{aligned}
&\mathbb{E}^x \left[\sum_{T_i < \infty} e^{-\alpha(T_i+\Delta)} K(X_{(T_1+\Delta)-}, X_{T_1+\Delta}) \right] \\
&= \mathbb{E}^x \left[e^{-\alpha(T_1+\Delta)} K(X_{(T_1+\Delta)-}, X_{T_1+\Delta}) + e^{-\alpha(T_1+\Delta)} \right. \\
&\qquad \left. \sum_{i=2}^\infty e^{-\alpha((T_i+\Delta)-(T_1+\Delta))} K(X_{(T_i+\Delta)-}, X_{T_i+\Delta}) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^x \Bigg[\mathrm{e}^{-\alpha(T_1+\Delta)} K(X_{(T_1+\Delta)-}, X_{T_1+\Delta}) + \mathrm{e}^{-\alpha(T_1+\Delta)} \\
&\quad \mathbb{E}^x \left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha((T_i+\Delta)\circ\theta(T_1+\Delta))} K(X_{(T_{i+1}+\Delta)-}, X_{T_{i+1}+\Delta}) \middle| \mathcal{F}_{T_1+\Delta} \right] \Bigg] \\
&= \mathbb{E}^x \Bigg[\mathrm{e}^{-\alpha(T_1+\Delta)} \Bigg\{ K(X_{(T_1+\Delta)-}, X_{T_1+\Delta}) + \mathbb{E}^{X_{T_1+\Delta}} \\
&\quad \left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} K(X_{(T_i+\Delta)-}, X_{T_i+\Delta}) \right] \Bigg\} \Bigg]
\end{aligned}
$$

where we used $T_{i+1} + \Delta = (T_1 + \Delta) + (T_i + \Delta) \circ \theta(T_1 + \Delta)$ with the shift operator $\theta(\cdot)$ in the second equality. Here, we relied on Remark 2.1. Combining the two terms, we can write (2.5) as

$$J^\nu(x) = \mathbb{E}^x[\mathrm{e}^{-\alpha(T_1+\Delta)}\{K(X_{(T_i+\Delta)-}, X_{T_i+\Delta}) - g(X_{(T_1+\Delta)-}) + J^\nu(X_{T_1+\Delta})\}] + g(x).$$

We define

$$u \triangleq J^\nu - g. \tag{2.12}$$

By adding and subtracting $g(X_{(T_1+\Delta)})$ to and from the first term we obtain

$$u(x) = \mathbb{E}^x[\mathrm{e}^{-\alpha(T_1+\Delta)} \bar{K}(X_{(T_1+\Delta)-}, X_{T_1+\Delta}) + u(X_{T_1+\Delta})] \tag{2.13}$$

in which

$$\bar{K}(x, y) \triangleq K(x, y) - g(x) + g(y). \tag{2.14}$$

since $T_1- = \tau_b$ with $\tau_b = \inf\{t \geq 0 : X_t^0 \geq b\}$ and the post intervention point is given by

$$X_{T_1+\Delta} = X_{\tau_b+\Delta} = X_{(\tau_b+\Delta)-} - \xi_1 \triangleq a. \tag{2.15}$$

From Remark 2.1

$$
\begin{aligned}
u(x) &= \mathbb{E}^x[\mathrm{e}^{-\alpha(\tau_b+\Delta)}\{\bar{K}(X_{\tau_b+\Delta}, a) + u(a)\}] \\
&= \mathbb{E}^x[\mathbb{E}^x[\mathrm{e}^{-\alpha(\tau_b+\Delta)}\{\bar{K}(X_{\tau_b+\Delta}, a) + u(a)\}|\mathcal{F}_{\tau_b}]] \\
&= \mathbb{E}^x[\mathrm{e}^{-\alpha\tau_b}\mathbb{E}^{X_{\tau_b}}[\mathrm{e}^{-\alpha\Delta}\{\bar{K}(X_\Delta, a) + u(a)\}]].
\end{aligned} \tag{2.16}
$$

Evaluating at $x = b$, we obtain $u(b) = \mathbb{E}^b[\mathrm{e}^{-\alpha\Delta}\{\bar{K}(X_\Delta, a) + u(a)\}]$. Therefore, (2.13) becomes

$$u(x) = \mathbb{E}^x[\mathrm{e}^{-\alpha\tau_b} u(X_{\tau_b})].$$

Hence we have finally

$$u(x) = \begin{cases} u_0(x) \triangleq \mathbb{E}^x[\mathrm{e}^{-\alpha\tau_b} u(b)], & x \in (c, b), \\ \mathbb{E}^x[\mathrm{e}^{-\alpha\Delta}(\bar{K}(X_\Delta, a) + u_0(a))], & x \in [b, d), \end{cases} \tag{2.17}$$

where the second equality is obtained when we plug $T_1 = 0$ in (2.13).

Using appropriate boundary conditions one can solve $(\mathcal{A}-\alpha)u = 0$ and obtain

$$\begin{aligned}
\mathbb{E}^x[\mathrm{e}^{-\alpha\tau_r}1_{\{\tau_r<\tau_l\}}] &= \frac{\psi(l)\varphi(x)-\psi(x)\varphi(l)}{\psi(l)\varphi(r)-\psi(r)\varphi(l)},\\
\mathbb{E}^x[\mathrm{e}^{-\alpha\tau_r}1_{\{\tau_l<\tau_r\}}] &= \frac{\psi(x)\varphi(r)-\psi(r)\varphi(x)}{\psi(l)\varphi(r)-\psi(r)\varphi(l)},
\end{aligned} \tag{2.18}$$

for $x \in [l, r]$ where $\tau_l \triangleq \inf\{t > 0; X_t^0 = l\}$ and $\tau_r \triangleq \inf\{t > 0; X_t^0 = r\}$ (see e.g. Dayanik and Karatzas [12]). By defining

$$W \triangleq (u/\varphi) \circ F^{-1}, \tag{2.19}$$

Eq. (2.17) becomes

$$W(F(x)) = W(F(c))\frac{F(b)-F(x)}{F(b)-F(c)} + W(F(b))\frac{F(x)-F(c)}{F(b)-F(c)}, \quad x \in (c, b], \tag{2.20}$$

We should note that $F(c) \triangleq F(c+) = \psi(c+)/\varphi(c+) = 0$ and

$$W(F(c)) = l_c \triangleq \limsup_{x\downarrow c} \frac{\bar{K}(x,a)^+}{\varphi(x)} \tag{2.21}$$

for any $a \in (c, d)$. For more detailed mathematical meaning of this value l_c, we refer the reader to Dayanik and Karatzas [12]. We have now established that $W(F(x))$ is a *linear function* in the transformed "continuation region".

2.2. An algorithm to compute the value function

Let us denote

$$r(x; a) \triangleq \mathbb{E}^x[\mathrm{e}^{-\alpha\Delta}\bar{K}(X_\Delta, a)] \tag{2.22}$$

and transform this function by

$$R(\cdot; a) \triangleq \frac{r(F^{-1}(\cdot), a)}{\varphi(F^{-1}(\cdot))}. \tag{2.23}$$

First stage: For a given pair $(a, b) \in (c, d) \times (c, d)$ we can determine (2.17) from the linear characterization (2.20). On $(0, F(b)]$ we will find $W(y) = \rho y + l_c$ (in which the slope is to be determined) from

$$\rho F(b) + l_c = R(F(b), a) + \mathrm{e}^{-\alpha\Delta}(\rho F(a) + l_c)\frac{\varphi(a)}{\varphi(b)}. \tag{2.24}$$

ρ can be determined as

$$\rho = \frac{R(F(b; a)) + l_c\left(\mathrm{e}^{-\alpha\Delta}\frac{\varphi(a)}{\varphi(b)} - 1\right)}{F(b) - \mathrm{e}^{-\alpha\Delta}\frac{\varphi(a)}{\varphi(b)}F(a)}. \tag{2.25}$$

Sometimes we will refer to ρ as $b \to \rho(b)$, when it becomes necessary to emphasize the dependence on b. The function u can be written as

$$u(x) = \begin{cases} u_0(x) \triangleq \rho\psi(x) + l_c\varphi(x) & x \le b \\ r(x,a) + \mathrm{e}^{-\alpha\Delta} u_0(a) & x > b. \end{cases} \tag{2.26}$$

Note that $(\mathcal{A} - \alpha)u(x) = 0$ for $x < b$. *Henceforth, to emphasize the dependence on the pair* (a, b) *we will write* $u^{a,b}(\cdot)$ *for the function* $u(\cdot)$.
Second stage: Our purpose in this section is to determine

$$u^a(x) \triangleq \sup_{b \in (c,d)} u^{a,b}(x), \quad x \in (c, d), \tag{2.27}$$

and to determine the constant b^*

$$u^a(x) = u^{a,b^*}(x), \quad x \in (c, d), \tag{2.28}$$

if it exists.

Let us fix a and treat ρ as a function of b parametrized by a.

Lemma 2.1. *Assume that the function* $R(\cdot; a)$ *defined in* (2.23) *is differentiable and that there exists a constant* $b^* \in (c, d)$ *satisfying* (2.28). *Then* b^* *satisfies the equation*

$$\rho F'(b) = \left.\frac{\partial}{\partial y} R(y; a)\right|_{y=F(b)} F'(b) - \mathrm{e}^{-\alpha\Delta}(\rho F(a) + l_c)\frac{\varphi(a)\varphi'(b)}{\varphi(b)^2}. \tag{2.29}$$

in which ρ *is given by* (2.25).

Proof. From (2.26) it follows that the maximums of the functions $b \to u^{a,b}$ and $b \to \rho(b)$ are attained at the same point. Now taking the derivative of (2.24) and evaluating at $\rho_b = 0$ we obtain (2.29). □

To find the optimal b (given a) we solve the non-linear and implicit equation (2.29). Under certain assumptions on the function $(r/\varphi) \circ F^{-1}$, this equation has a unique solution as we show below.

Remark 2.2. On $y \ge F(b)$, the function W is given by

$$W(y) = \mathrm{e}^{-\alpha\Delta}(\rho F(a) + l_c)\frac{\varphi(a)}{\varphi(F^{-1}(y))} + R(y; a). \tag{2.30}$$

The right derivative of W at $F(b)$ is given by

$$W'(F(b)) = -\mathrm{e}^{-\alpha\Delta}(\rho F(a) + l_c)\frac{\varphi(a)}{\varphi(b)^2}\frac{\varphi'(b)}{F'(b)} + \left.\frac{\partial}{\partial y} R(y; a)\right|_{y=F(b)}. \tag{2.31}$$

Therefore, (2.29) implies that the left and the right derivative of W (recall that $W(y) = \rho y + l_c$ for $y < F(b)$) at $F(b)$ are equal (smooth fit).

Let us define

$$u_a(x) \triangleq \sup_{b \in (c,d)} \mathbb{E}^x[\mathrm{e}^{-\alpha\tau_b}\mathbb{E}^{X_{\tau_b}}[\mathrm{e}^{-\alpha\Delta}\{\bar{K}(X_\Delta, a) + u_a(a)\}]]. \tag{2.32}$$

The next lemma shows that (2.32) is well-defined. Below we show that under certain assumptions on $(r/\varphi) \circ F^{-1}$ this function is equal to u^a.

Lemma 2.2. *Assume that*

$$\sup_{x \in (c,d)} \mathbb{E}^x[\bar{K}(X_\Delta, a)] > 0 \tag{2.33}$$

for some $a \in (c, d)$. Let us introduce a family of value functions parameterized by $\gamma \in \mathbb{R}$ as

$$V_a^\gamma(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\mathrm{e}^{-\alpha(\tau+\Delta)} \left\{ \bar{K}(X^0_{\tau+\Delta}, a) + \gamma \right\} \right]$$
$$= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\mathrm{e}^{-\alpha\tau} \mathbb{E}^{X^0_\tau} \left[\mathrm{e}^{-\alpha\Delta} \left\{ \bar{K}(X^0_\Delta, a) + \gamma \right\} \right] \right], \tag{2.34}$$

here $\mathcal{S}$ is the set of all stopping times of the natural filtration of X^0. Then there exists a unique γ^* such that $V_a^{\gamma^*}(a) = \gamma^*$.

Proof. Let us denote

$$W_a^\gamma(F(x)) \triangleq \frac{V_a^\gamma(x)}{\varphi(x)}. \tag{2.35}$$

Consider the function $\gamma \to V_a^\gamma(a)$. Our aim is to show that there exists a fixed point to this function. Let us consider $V_a^0(a)$ first. Because (2.33) is satisfied we have that $V_a^0(a) > 0$. As γ increases, $V^\gamma(a)$ increases monotonically, by the right hand side of (2.34). Now, Lemma A.1 implies that for $\gamma_1 > \gamma_2 \geq 0$,

$$V_a^{\gamma_1}(x) - V_a^{\gamma_2}(x) \leq \gamma_1 - \gamma_2 \tag{2.36}$$

for any $x \in \mathbb{R}_+$. Note that $W_a^\gamma(F(a)) \geq R(F(a), a) + \frac{\mathrm{e}^{-\alpha\Delta}\gamma}{\varphi(a)}$ for all γ. However, since V has less than linear growth in γ as demonstrated by (2.36) we can see that there is a certain γ' large enough such that $W_a^\gamma(F(a)) = R(F(a), a) + \frac{\mathrm{e}^{-\alpha\Delta}\gamma}{\varphi(a)}$ for $\gamma \geq \gamma'$. This implies however

$$\varphi(a) W_a^{\gamma'}(F(a)) = \varphi(a) R(F(a), a) + \mathrm{e}^{-\alpha\Delta}\gamma'$$
$$\Leftrightarrow V_a^{\gamma'}(a) = r(a, a) + \mathrm{e}^{-\alpha\Delta}\gamma' < \gamma'$$

where the inequality is due to the assumption (2.8). For this γ', we have $V_a^{\gamma'}(a) < \gamma'$.

Since $\gamma \to V_a^\gamma$ is continuous, which follows from the fact that this function is convex, and increasing, $V_a^0 > 0$ and $V_a^{\gamma'}(a) < \gamma'$ implies that $\gamma \to V_a^\gamma$ crosses the line $\gamma \to \gamma$. $\square$

Lemma 2.3. *Assume that*

$$r(x, a) \text{ is lower semi-continuous.} \tag{2.37}$$

Let us define $R^\gamma(\cdot; a) \triangleq \frac{r^\gamma(F^{-1}(\cdot), a)}{\varphi(F^{-1}(\cdot))}$ where

$$r^\gamma(x, a) \triangleq \mathbb{E}^x[\mathrm{e}^{-\alpha\Delta}(\bar{K}(X_\Delta, a) + \gamma)]. \tag{2.38}$$

Then (2.35) is the smallest non-negative concave majorant of R^γ that passes through $(F(c+), l_c)$.

Proof. See for e.g. Dynkin [14] and Dayanik and Karatzas [12]. $\square$

Lemma 2.4. *Assume that (2.33) and (2.37) hold. Then u_a/φ is F-concave, i.e., α-excessive.*[1]

[1] A function f is called α-excessive function of X_0 if for any stopping time τ of the natural filtration of X^0 and $x \in (c, d)$, $f(x) \geq \mathbb{E}^x[\mathrm{e}^{-\alpha\tau} f(X^0_\tau)]$, see for e.g. Borodin and Salminen [10] and Dynkin [14] for more details.

Proof. This follows from Lemmas 2.2 and 2.3. For the equivalence of α-excessivity and F-concavity see e.g. Theorem 12.4 in Dynkin [14] and also Dayanik and Karatzas [12]. This fact can be observed from (A.8). $\square$

Lemma 2.5. *Assume that (2.33) and (2.37) hold. Then*

$$u^a(x) \le u_a(x), \quad x \in (c, d). \tag{2.39}$$

Proof. It follows from Lemma 2.4 that u_a is α-excessive. Also, observe from (2.32) that

$$u_a(x) \ge r(x; a) + \mathrm{e}^{-\alpha\Delta} u_a(a), \tag{2.40}$$

where $r(x, a)$ is as in (2.37). Let $\nu = \{T_1, T_2, \ldots, T_i, \ldots; \xi_1, \xi_2, \ldots, \xi_i, \ldots\}$ be an admissible control and let $T_0 = 0$. Without loss of generality we will assume that $r(b; a) > 0$, because otherwise the corresponding strategy will have a lower value function $J^\nu(x)$ associated to it. Since u_a is α-excessive,

$$\begin{aligned} &u_a(x) \ge \mathbb{E}^x[\mathrm{e}^{-\alpha T_1} u_a(X_{T_1})], \quad \text{and} \\ &\quad \mathbb{E}^x[\mathrm{e}^{-\alpha(T_i+\Delta)} u_a(X_{(T_i+\Delta)})] - \mathbb{E}^x[\mathrm{e}^{-\alpha T_{i+1}} u_a(X_{T_{i+1}})] \ge 0, \end{aligned} \tag{2.41}$$

for all $i = 1, \ldots, N-1$. Then

$$\begin{aligned} u_a(x) &\ge \mathbb{E}^x[\mathrm{e}^{-\alpha T_1} u_a(X_{T_1})] + \sum_{i=1}^{N-1} \mathbb{E}^x[\mathrm{e}^{-\alpha T_{i+1}} u_a(X_{T_{i+1}})] - \mathbb{E}^x[\mathrm{e}^{-\alpha(T_i+\Delta)} u_a(X_{(T_i+\Delta)})] \\ &= \mathbb{E}^x[\mathrm{e}^{-\alpha T_N} u_a(X_{T_N})] + \sum_{i=1}^{N-1} \mathbb{E}^x[\mathrm{e}^{-\alpha T_i} u_a(X_{T_i})] - \mathbb{E}^x[\mathrm{e}^{-\alpha(T_i+\Delta)} u_a(X_{(T_i+\Delta)})] \\ &\ge \sum_{i=1}^{N-1} \mathbb{E}^x[\mathrm{e}^{-\alpha T_i} r(X_{T_i}, a)], \end{aligned} \tag{2.42}$$

in which the inequality follows from (2.40) and the fact that u_a is non-negative. Now, using the monotone convergence theorem

$$\begin{aligned} u_a(x) &\ge \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha T_i} r(X_{T_i}, a)\right] = \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} \mathbb{E}^{X_{T_i}}\left[\bar{K}(X_\Delta, a)\right]\right] \\ &= \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} \mathbb{E}^{X_{T_i}}\left[K(X_\Delta, a) - g(X_\Delta) + g(a)\right]\right] \\ &= \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} K(X_{(T_i+\Delta)-}, X_{T_i+\Delta})\right] \\ &\quad + \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} (-g(X_{(T_i+\Delta)-}) + g(X_{(T_i+\Delta)}))\right] \\ &= \mathbb{E}^x\left[\sum_{i=1}^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)} K(X_{(T_i+\Delta)-}, X_{T_i+\Delta})\right] \\ &\quad + \mathbb{E}^x\left[\int_0^\infty \mathrm{e}^{-\alpha s} f(X_s) \mathrm{d}s\right] - g(x) = u^{a,b}(x). \end{aligned} \tag{2.43}$$

The third inequality follows from Remark 2.1. The fourth inequality can be derived from (2.11). The last equality follows from (2.12). Now taking the supremum over b, we obtain (2.39). □

Lemma 2.6. *Assume that (2.33) and (2.37) hold and that the function $x \to R(x; a)$ defined in (2.22) is concave and increasing on (a', d) for some $a' \in (a, d)$ and that*

$$\lim_{x \to F(d)} R(x; a) = \infty. \tag{2.44}$$

Then $u_a(x) = u^{a,b^*}(x)$ for a unique $b^* \in (c, d)$. Hence from Lemma 2.5 it follows that $u_a(x) = u^a(x) = u^{a,b^*}(x)$, $x \in (c, d)$.

Proof. Since R is concave, R^γ in (2.38) is also concave on (a', d). The assumption in (2.44) implies that the smallest concave majorant W_a^γ in (2.35) is linear on $(F(c), F(b^\gamma))$ for a *unique* $b^\gamma \in (c, d)$ and is tangential to $R^\gamma(\cdot, a)$ at $F(b^\gamma)$ and coincides with $R^\gamma(\cdot, a)$ on $[F(b^\gamma), F(d))$. Together with Lemma 2.2 this implies that there exists a unique γ^* such that Eqs. (2.30) and (2.31) are satisfied when W is replaced by $W_a^{\gamma^*}$ and b is replaced by b^{γ^*}. Note that $W_a^{\gamma^*}$ corresponds to a strategy (a, b^{γ^*}). That is, if we start with $u^{a,b^{\gamma^*}}$ and transform it via (2.19) we get $W_a^{\gamma^*}$. On the other hand, using (2.35) and by substituting $\gamma = \gamma^*$ we have that $u_a(x) = \varphi(x) W_a^{\gamma^*}(F(x))$, $x \in (c, d)$. This lets us conclude that $u^{a,b^{\gamma^*}} = u_a(x)$, $x \in (c, d)$. We see that the unique b^* in the claim of the proposition is b^{γ^*}. □

Proposition 2.7. *Assume that the hypotheses of Lemma 2.6 are satisfied. Then there exists a unique solution to (2.29). If b^* is the unique solution of (2.29), then $u^a(x) = u^{a,b^*}(x)$.*

Proof. In the proof of Lemma 2.6, we have seen that there exists a unique b^* such that (2.30) and (2.31) are satisfied. Using Remark 2.2, we conclude that b^* is the unique solution of (2.29). □

Note that when the assumptions of Proposition 2.7 hold, the optimal threshold strategy is described by a single open interval in the state space of the controlled process. The conditions for the existence and uniqueness of the optimal interval are specified, essentially by the conditions on total reward function $\bar{K}(x, y)$ associated with one intervention from x to y (see (2.14) and (2.23)) and drift and volatility of the underlying diffusion as the function F that appears in (2.23) depends on them.

Third stage: Now, we let $a \in (c, d)$ vary and choose a^* that maximizes $\rho(a)$ and also find $b^* = b(a^*)$. Finally, we obtain the value function given in (2.6) by $v(x) = u(x) + g(x)$.

2.3. Example: Optimal exchange rate intervention when there is delay

To illustrate the procedure of solving impulse control problems with delay, we take an example from Mundaca and Øksendal [18] (also see Øksendal [19]) that considers the following foreign exchange rate intervention problem:

$$J_D^\nu(x) \triangleq \mathbb{E}^x \left[\int_0^\infty \mathrm{e}^{-\alpha s} X_s^2 \mathrm{d}s + \sum_i^\infty \mathrm{e}^{-\alpha(T_i + \Delta)} (c + \lambda |\xi_i|) \right] \tag{2.45}$$

where $X_t^0 = x + B_t$, in which B is a standard Brownian motion. Here, the superscript 0 is to indicate that the dynamics in consideration are of the uncontrolled state variable. In (2.45), $c > 0$ and $\lambda \geq 0$ are constants representing the cost of making an intervention. The problem without delays are solved by Øksendal [19] through quasi-variational inequalities and by Dayanik and Egami [11] using a direct characterization of the value function. In this problem, the Brownian motion represents the exchange rate of currency and the impulse control represents the interventions the central bank makes in order to keep the exchange rate in a given target window. At time T_i, such that $X_{T_i-} = b$, the central bank makes a commitment to reduce the exchange rate from b to $a < b$, which is implemented Δ units of time later. During the time interval $(T_i, T_i + \Delta]$ the central bank does not make any other interventions. Δ units later if the exchange rate is still greater than a, then the central bank reduces the exchange rate from $X_{(T_i+\Delta)-}$ to a and pays a cost of $c + \lambda(X_{(T_i+\Delta)-} - a)$. On the other hand, if Δ units of time later the exchange rate is less than a, the central bank chooses to increase the exchange rate to a at a cost of $c + \lambda(a - X_{(T_i+\Delta)-})$. This is a one-sided impulse control problem, in the sense that a control is triggered only if $X_t > b$ and there has not been any previous action in the interval $(t - \Delta, t)$.

The problem is to minimize the expected total discounted cost over all threshold strategies.

$$v_D(x) \triangleq \inf_{\nu} J_D^{\nu}(x). \tag{2.46}$$

A similar version of this problem is analyzed by Øksendal and Sulem [22], in which they take the controls $\xi_i \in \mathcal{F}_{T_i}$ for all i. (This introduces path dependence since the value of $X_{T_i+\Delta}$ is partially determined by $\mathcal{F}_{T_i}$.)

Instead of solving a minimization problem of (2.46), we will solve

$$v(x) = \sup_{\nu} \mathbb{E}^x \left[\int_0^{\infty} \mathrm{e}^{-\alpha s}(-X_s^2)\mathrm{d}s - \sum_i^{\infty} \mathrm{e}^{-\alpha(T_i+\Delta)}(c + \lambda|\xi_i|) \right]$$

and recover the value function by $v_D(x) = -v(x)$. (Here, the supremum is taken over all the threshold strategies.) The continuous cost rate is $f(x) = -x^2$ and the intervention cost is $K(x, y) = -c - \lambda|x - y|$ in our terminology. By solving the equation $(\mathcal{A} - \alpha)v(x) = \frac{1}{2}v''(x) - \alpha v(x) = 0$, we find that $\psi(x) = \mathrm{e}^{x\sqrt{2\alpha}}$ and $\varphi(x) = \mathrm{e}^{-x\sqrt{2\alpha}}$. Hence $F(x) = \mathrm{e}^{2x\sqrt{2\alpha}}$ and $F^{-1}(x) = \frac{\log x}{2\sqrt{2\alpha}}$. Using Fubini's theorem we can calculate $g(x)$ explicitly as:

$$g(x) = -\mathbb{E}^x \int_0^{\infty} \mathrm{e}^{-\alpha s}(x + B_s)^2 \mathrm{d}s = -\left(\frac{x^2}{\alpha} + \frac{1}{\alpha^2} \right).$$

We shall follow the procedure described in the last section: Let us fix $a > 0$ and consider

$$\begin{aligned} r(x, a) &= \mathbb{E}^x[\mathrm{e}^{-\alpha\Delta}\bar{K}(X_\Delta, a)] = \mathbb{E}^x[\mathrm{e}^{-\alpha\Delta}(-c - \lambda|X_\Delta - a| + g(a) - g(X_\Delta))] \\ &= \mathbb{E}^x\left[\mathrm{e}^{-\alpha\Delta}\left(-c - \lambda|x + B_\Delta - a| - \left(\frac{a^2}{\alpha} + \frac{1}{\alpha^2}\right) + \left(\frac{(x + B_\Delta)^2}{\alpha} + \frac{1}{\alpha^2}\right)\right)\right] \\ &= \mathrm{e}^{-\alpha\Delta}\left(-c - \lambda\left(2\Delta \exp\left(-\frac{(a - x)^2}{4\Delta^2} \right) + (a - x)\left(-1 + 2N\left(\frac{a - x}{\Delta}\right)\right)\right) \right. \\ &\quad \left. + \frac{x^2 - a^2 + \Delta}{\alpha} \right). \end{aligned} \tag{2.47}$$

The left boundary $-\infty$ is natural for a Brownian motion and, for any $a > 0$,

$$l_{-\infty} = \limsup_{x \downarrow -\infty} \frac{r(x, a)^+}{\varphi(x)} = 0.$$

It follows that $R(y)$ passes through $(F(-\infty), l_{-\infty}) = (0, 0)$. (See Dayanik and Karatzas [12] Proposition 5.12.)

Proposition 2.8. *For the function r in (2.47), there exists a unique solution to (2.29) for a fixed a.*

Proof. See Appendix A. □

Using the algorithm we described in Section 2.2 we find the optimal (a^*, b^*, ρ^*). Going back to the original space we get

$$V(x) = \sup_{a,b\in\mathbb{R}} u(x) = \varphi(x)W^*(F(x)) = \varphi(x)(\beta^*)F(x) = \rho^* \mathrm{e}^{x\sqrt{2\alpha}}.$$

on $x \in (-\infty, b^*]$. To get $v(x) = \sup_\nu J^\nu(x)$, we add back $g(x)$,

$$v(x) = V(x) + g(x) = \rho^* \mathrm{e}^{x\sqrt{2\alpha}} - \left(\frac{x^2}{\alpha} + \frac{1}{\alpha^2}\right).$$

Finally, flipping the sign we obtain the optimal cost function as

$$v_D(x) = \begin{cases} \hat{v}_o(x) \triangleq \left(\dfrac{x^2}{\alpha} + \dfrac{1}{\alpha^2}\right) - \rho^* \mathrm{e}^{x\sqrt{2\alpha}}, & 0 \le x \le b^*, \\ -\mathrm{e}^{-\alpha\Delta}\rho^* \mathrm{e}^{a^*\sqrt{2\alpha}} - r(x; a^*) + \dfrac{x^2}{\alpha} + \dfrac{1}{\alpha^2}, & b^* \le x. \end{cases} \tag{2.48}$$

Fig. 1 is obtained when the parameters are chosen to be $(c, \lambda, \alpha, \Delta) = (150, 50, 0.2, 1.0)$. We found the solution triplet to be $(a^*, b^*, \rho^*) = (5.066, 12.1756, 0.042423)$. The optimal cost function without delay, for the same parameters, has the solution triplet $(a_0, b_0, \rho_0) = (5.07723, 12.2611, 0.0492262)$. The continuation region shifts to the left with delay (it shrinks from $(-\infty, 12.2611)$ to $(-\infty, 12.1756)$), and the central bank acts more aggressively when it encounters delays (see Fig. 1(c)).

3. Firing costs and labor demand: Optimal band strategies

In this section, we will improve on the techniques of the previous section in order to study an impulse control corresponding to band policies when there are implementation delays. In particular, we will concentrate our attention on a specific example, which is of practical interest. We will find optimal hiring and firing decisions of a firm that faces stochastic demand and has to conform to regulatory delays when it is firing employees.

Recently, General Motors Corporation (GM) has decided to lay off 25,000 of its work force to cut back on its production and administrative costs. However "GM's UAW (United Auto Workers) contract essentially forces it to pay union employees during the life of the contract even if hourly workers are laid off and their plants are closed. But those protections only run through September 2007, when the current four-year pact with the union ends. GM spokesman Ed Snyder said the automaker has yet to reach any agreement with the UAW on the nature or the manner of the work force reduction".[2] This is a typical example of a firing cost and implementation delay a corporation faces when the workers are unionized. Another example of firing delay is caused by government regulations in Europe (see e.g. Bentolila and Bertola [9]).

[2] Source: June 7, 2005 CNN Money, "GM to cut 25,000 jobs" by Chris Isidore, http://money.cnn.com/2005/06/07/news/fortune500/gm_closings/.

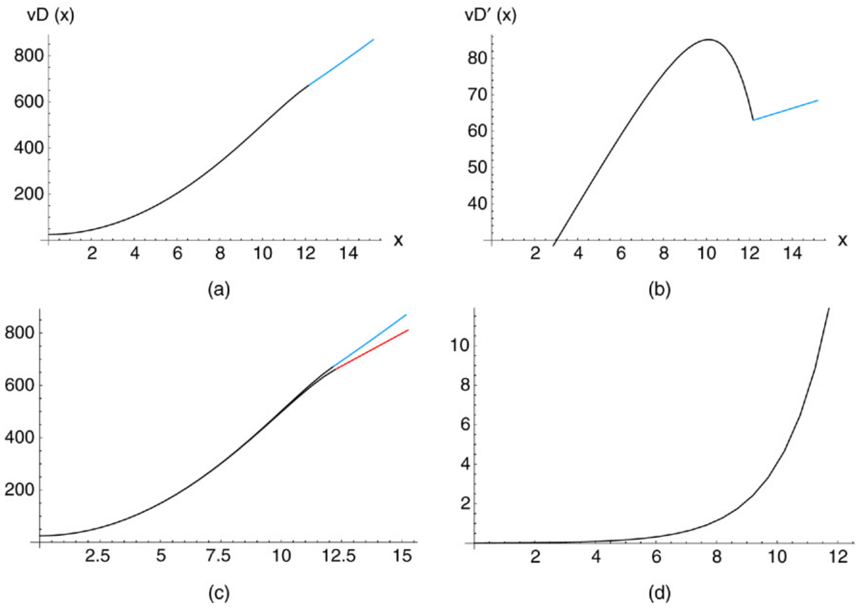

Fig. 1. (a) The optimal cost function $v_D(x)$. The dotted line and the solid line fit each other continuously at $b^* = 12.1756$. (b) The derivative of $v_D(x)$, showing that the smooth-fit principle holds at b^*. (c) Comparison of $v_D(x)$ with the cost function without delay $v_0(x)$. Note that v_D majorizes v_0. (d) Plot of the difference of $v_D(x) - v_0(x)$.

Bentolila and Bertola [9] address the issue of costly hiring and firing and its effects on unemployment rate in Europe using singular stochastic control. Here, we solve an impulse control problem since we are also taking fixed cost of labor adjustments into account. But our main purpose is to measure the effects on firing delay in the decisions of firms. As we shall see, it turns out that the controlled state variable is not Markov, therefore we will focus our attention completely on *band policies* (which we will define shortly) rather than trying to find the best impulse control policy. Our method of solving impulse control problem differs from its counterparts that use quasi-variational inequalities since we give a direct characterization of the value function as a linear function in the *continuation* region without having to guess the form of the solution and without having to prove that the conjectured solution satisfies the conditions of a verification lemma.

3.1. Problem setup

As in Bentolila and Bertola [9],[3] we will consider a firm with a linear production technology. In particular the quantity sold is $Q_t = AL_t$, $A \in \mathbb{R}_+$, in which L_t is the labor at time t. The selling price at time t, P_t, of the product is determined from

$$Q_t = Z_t P_t^{\frac{1}{\mu-1}}, \quad \mu \in (0, 1) \tag{3.1}$$

[3] The set up of Bentolila and Bertola [9] was brought to our attention by Keppo and Maull. In the INFORMS Annual Meeting in 2004, Keppo and Maull presented their partial results on the hiring and firing decisions of firms which they obtained by solving quasi-variational inequalities.

in which Z_t indexes the position of the direct demand curve whose dynamics follow

$$\mathrm{d}Z_t = Z_t b \mathrm{d}t + Z_t \sigma_t \mathrm{d}W_t \tag{3.2}$$

with a constant $b \in \mathbb{R}_+$. In Eq. (3.1) the quantity $1 - \mu$ is the firm's monopoly power. Let us denote the filtration generated by the demand process Z by $\mathcal{F} \triangleq (\mathcal{F}_t)_{t \geq 0}$. We will make the following assumption to guarantee that (3.2) has a unique strong solution. We assume that σ is bounded and adapted to the filtration of the Brownian motion W.

In our framework, if the firm produces excess products because of excess labor, the products produced are still all sold but at a cheaper price. The firm pays a wage, w, to its workers, therefore the net rate of profit that the firm makes at time t is given by

$$Q_t P_t - wL_t = Z_t^{1-\mu} (AL_t)^{\mu} - wL_t.$$

When the workers quit voluntarily, the firm bears no firing costs and we assume that the workers quit at rate δ, that is, without any intervention from the management the labor force follows the dynamics

$$\mathrm{d}L_t^0 = -\delta L_t^0 \mathrm{d}t. \tag{3.3}$$

Here, as in the previous section, the superscript 0 indicates that there are no controls applied. The firm makes commitments to change its labor force at times $\{S_i\}_{i \in \mathbb{N}}$ and $\{T_i\}_{i \in \mathbb{N}}$. At time S_i the firm makes a commitment to increase its labor force (which is immediately implemented), and at time T_i it makes a commitment to decrease its labor force, which is implemented Δ units of time later. During the time interval $(T_i, T_i + \Delta]$ the firm makes no commitments to change its labor force. Note that although at time T_i the firm decided to decrease its labor force, the labor force itself might move to very low levels following the dynamics (3.3), therefore at time $T_i + \Delta$ the firm may end up hiring to move to keep the production level up. However, if the labor force level is still very high at time $(T_i + \Delta)-$, then the firm ends up firing. Here, Δ represents the regulatory delays a firm faces when it is cutting off its work force.

The labor adjustments come at a cost: At time S_i the firm increases the labor by ζ_i $(\geq 0) \in \mathcal{F}_{S_i}$ (Here, for the sake of brevity we are taking the σ-algebras as a collection of mappings.) to $L_{S_i-} + \zeta_i$, then the associated cost is

$$c_1 \zeta_i + c_2 L_{S_i-}.$$

At time T_i, the firm makes a commitment to decrease the labor at time $T_i + \Delta$. If it ends up decreasing the labor force by $\eta_i (\geq 0) \in \mathcal{F}_{T_i+\Delta}$ to $L_{T_i+\Delta} = L_{(T_i+\Delta)-} - \eta_i$, then the associated cost is quantified as

$$c_3 \eta_i + c_4 L_{(T_i+\Delta)-},$$

which depends on the amount of labor force to be fired and the level of the total labor force as well. The latter component of costs is based on the following observations: When a corporation decides who to fire or which division to restructure, administrative costs will become larger in proportion to the size of the total labor force since the firm's operations are closely knitted among various divisions.

On the other hand, as we discussed above, if the labor force itself moves to very low levels itself during the Δ units of time, at time $T_i + \Delta$ the firm may end up hiring (in this case $\eta_i \leq 0$) to keep the production up at the cost of

$$c_1 |\eta_i| + c_2 L_{(T_i+\Delta)-}$$

for some positive constants c_1, c_2, c_3, c_4 and $\Delta \geq 0$. This cost becomes negligible as Δ becomes small because in that case the work force does not change much by itself. So the controls of the firm are of the form

$$\nu = (S_1, S_2, \ldots; \zeta_1, \zeta_2, \ldots; T_1, T_2, \ldots; \eta_1, \eta_2, \ldots),$$

where $0 \leq S_1 < S_2 < \cdots$ and $0 \leq T_1 < T_2 < \cdots$ are two increasing sequences of stopping times of the filtration $\mathcal{F}$. $T_{i+1} - T_i \geq \Delta$ and for any i there exists no j such that $T_i \leq S_j \leq T_{i+\Delta}$. The magnitudes of the impulses satisfy $\zeta_i (\geq 0) \in \mathcal{F}_{S_i}$ and $\eta_i (\in \mathbb{R}) \in \mathcal{F}_{T_i+\Delta}$ for all i. We call these type of controls *admissible* and we will denote the set of all admissible controls by $\mathcal{V}$. To each control $\nu \in \mathcal{A}$ we associate a profit function of the form

$$\begin{aligned} J^\nu(z,l) \triangleq \mathbb{E}\Bigg[& \int_0^\infty \mathrm{e}^{-rt} (Z_t^{1-\mu} (AL_t)^\mu - wL_t) \mathrm{d}t - \sum_i \mathrm{e}^{-rS_i} (c_1 \zeta_i + c_2 L_{S_i-}) \\ & - \sum_j \mathrm{e}^{-r(T_j+\Delta)} ((c_3 \eta_j + c_4 L_{(T_j+\Delta)-}) 1_{\{\eta_j > 0\}} \\ & + (c_1 \eta_j + c_2 L_{(T_j+\Delta)-}) 1_{\{\eta_j < 0\}}) \Bigg], \end{aligned} \tag{3.4}$$

which incorporates the profit and cost structure we have described so far. Here $r > b$ is a subjective rate of return that the firm uses to discount its future profits. In fact if $r < b$, then taking no action is optimal as we will point out below. Under the measure $\mathbb{P}$, we have that $L_0 = l$ and $Z_0 = z$ almost surely.

The objective of the company is then to maximize its profits by choosing the best possible strategy ν^* such that

$$v(z,l) \triangleq \sup_{\nu \in \mathcal{V}} J^\nu(z,l) = J^{\nu^*}(z,l), \tag{3.5}$$

if the optimal strategy ν^* exists. Hereafter, we will refer to v as the value function.

It looks as if the control problem defined in (3.5) involves two state variables, namely the demand Z and the labor force L. Recall that we have no control over the demand Z but we can control the labor force L by making hires and fires. But the only source of randomness is the demand process. In the sequel we will show that the optimal control problem (3.5) involves only one state variable. On denoting $\xi_t \triangleq L_t/Z_t$, $t \geq 0$ and the absolute changes in labor per unit of demand by $\beta_i \triangleq \zeta_i / Z_{S_i}$ and $\alpha_i \triangleq \eta_i / Z_{T_i+\Delta} \in \mathcal{F}_{T_i+\Delta}$, we can write the the profit function J^ν as

$$\begin{aligned} J^\nu(z,l) = \mathbb{E}\Bigg[& \int_0^\infty \mathrm{e}^{-rt} Z_t ((A\xi_t)^\mu - w\xi_t) \mathrm{d}t - \sum_i \mathrm{e}^{-rS_i} Z_{S_i-} (c_1 \beta_i + c_2 \xi_{S_i-}) \\ & \sum_j \mathrm{e}^{-r(T_j+\Delta)} ((c_3 Z_{T_j} \alpha_j + c_4 Z_{T_j+\Delta} \xi_{(T_j+\Delta)-}) 1_{\{\alpha_j > 0\}} \\ & + (c_1 Z_{T_j} \alpha_j + c_2 Z_{T_j+\Delta} \xi_{(T_j+\Delta)-}) 1_{\{\alpha_j < 0\}}) \Bigg]. \end{aligned} \tag{3.6}$$

Let us introduce a new probability measure $\mathbb{P}_0$ by

$$\left. \frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}\mathbb{P}} \right|_{\mathcal{F}_t} = \tilde{Z}_t, \quad \text{where } \tilde{Z}_t = \exp\left(\int_0^t \sigma_s \mathrm{d}W_s - \frac{1}{2} \int_0^t \sigma_s^2 \mathrm{d}s \right) \tag{3.7}$$

for every $0 \le t < \infty$. Using the representation of the profit function J^ν, we can write it as

$$J^\nu(z,l) = zI^\nu\left(\frac{z}{l}\right), \tag{3.8}$$

in which

$$\begin{aligned} I^\nu(\xi) \triangleq \mathbb{E}_0^\xi \Bigg[\int_0^\infty & \mathrm{e}^{(b-r)t} z\left((A\xi_t)^\mu - w\xi_t\right)\mathrm{d}t - \sum_i \mathrm{e}^{(b-r)S_i}(c_1\beta_i + c_2\xi_{S_i-}) \\ & - \sum_j \mathrm{e}^{(b-r)(T_j+\Delta)}((c_3\alpha_j + c_4\xi_{(T_j+\Delta)-})1_{\{\alpha_j>0\}} \\ & + (c_1\alpha_j + c_2\xi_{(T_j+\Delta)-})1_{\{\alpha_j<0\}}) \Bigg], \end{aligned} \tag{3.9}$$

where $\mathbb{E}^\xi$ is the expectation under $\mathbb{P}_0$ given that $\xi_0 = \xi$. Here, with a slight abuse of notation, on the right-hand-side of (3.8), we denoted

$$\nu = (S_1, S_2, \ldots; \beta_1, \beta_2, \ldots; T_1, T_2, \ldots; \alpha_1, \alpha_2, \ldots),$$

which is a control that is applied to the process ξ. The controls here are such that $\beta_i\,(\ge 0) \in \mathcal{F}_{S_i}$ and $\alpha_i\ (\in\mathbb{R}) \in \mathcal{F}_{T_i+\Delta}$. Again as before $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ are two increasing sequences of stopping times. We also assume that $T_{i+1} - T_i \ge \Delta \ge 0$ and that for any i there exists no j such that $T_i \le S_j \le T_{i+\Delta}$. With another slight abuse of notation we will denote the admissible set of controls we described here also by $\mathcal{V}$. As a result of the developments in the last part of this section we see that the process L_t/Z_t is the sufficient statistic of the problem in (3.5). In fact we can write the value function as

$$v(z,l) = zY\left(\frac{z}{l}\right), \quad \text{where } Y(\xi) \triangleq \sup_{\nu\in\mathcal{V}} I^\nu(\xi). \tag{3.10}$$

Under the measure $\mathbb{P}_0^\xi$ the dynamics of the process, ξ_t when there are no impulses applied follows

$$\xi_t^0 = \xi \exp\left(-(b+\delta)t - \int_0^t \sigma_s \mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s\right), \tag{3.11}$$

where B is a Wiener process under measure $\mathbb{P}_0$. Here, as before, the superscript 0 indicates that there are no controls/impulses applied.

3.2. Solution

The controlled process ξ is not a Markov process, because depending on whether the process reaches a point during the interval $(T_i, T_i + \Delta)$ or not, that point has different roles. That is, how the process reaches a particular point (path information) affects how the process will continue from that point. However, the process regenerates at times $\{T_i + \Delta\}_{i\in\mathbb{N}}$ and the value of the process at time $T \in (T_i, T_{i+\Delta})$, X_T, depends on the information up to T_i, $\mathcal{F}_{T_i}$, only through the value of the process at time T_i, X_{T_i}. Therefore, as we did in Section 2.1, assuming there is no history prior to time 0, i.e. $\mathcal{F}_0$ is a trivial sigma-algebra, we can develop

$$\begin{aligned} I^\nu(\xi) = \mathbb{E}_0^\xi \Big[& 1_{\{T_1<S_1\}} \mathrm{e}^{(b-r)(T_1+\Delta)} \left(C_1(\xi_{(T_1+\Delta)-}, \xi_{T_1+\Delta}) - g(\xi_{(T_1+\Delta)-}) + I^\nu(\xi_{T_1+\Delta})\right) \\ & + 1_{\{T_1>S_1\}} \mathrm{e}^{(b-r)S_1} \left(C_2(\xi_{S_1-}, \xi_{S_1}) - g(\xi_{S_1-}) + I^\nu(\xi_{S_1})\right) \Big], \end{aligned} \tag{3.12}$$

where

$$C_2(x,y) \triangleq -c_1(y-x)1_{\{y>x\}} - c_2 x, \quad \text{and,}$$

$$C_1(x,y) \triangleq -(c_3(x-y)+c_4 x)1_{\{x>y\}} + C_2(x,y)1_{\{y>x\}} \tag{3.13}$$

$$g(\xi) \triangleq \mathbb{E}_0\left[\int_0^\infty e^{(b-r)t}\left((A\xi_t^0)^\mu - w\xi_t^0\right)dt\right]. \tag{3.14}$$

On denoting $u(\xi) \triangleq I^{\tilde{\nu}}(\xi) - g(\xi)$, we can write

$$u(\xi) = \mathbb{E}_0^\xi[1_{\{T_1<S_1\}}e^{(b-r)(T_1+\Delta)}(\bar{C}_1(\xi_{(T_1+\Delta)-}, \xi_{T_1+\Delta}) + u(\xi_{T_1+\Delta}))] + \mathbb{E}_0^\xi[1_{\{T_1>S_1\}}e^{(b-r)S_1}(\bar{C}_2(\xi_{S_1-},\xi_{S_1}) + u(\xi_{S_1}))], \tag{3.15}$$

in which

$$\bar{C}_1(x,y) \triangleq C_1(x,y) - g(x) + g(y) \quad \text{and} \quad \bar{C}_2(x,y) \triangleq C_2(x,y) - g(x) + g(y). \tag{3.16}$$

In the rest of this section, we will analyze the following double sided threshold strategy (*band policy*) of the following form: (1) Whenever the marginal revenue product of labor hits level d, the firm makes a commitment to bring the marginal revenue product of labor to $c < d$. This may be achieved by firing employees if the marginal revenue product of labor is still greater than c after the delay. However, it is possible that after the delay the marginal revenue product of labor will be less than c. In this case, the firm makes hires. (2) Whenever the marginal revenue product of labor hits level p the firm increases it to $q > p$ (by hiring new employees). We will characterize the value function corresponding to an arbitrary band policy.

For the band policy we described above $S_1 = \tau_p$ and $T_1 = \tau_d$, and

$$\xi_{T_1+\Delta} = \xi_{(\tau_b+\Delta)-} - \alpha_1 = c \quad \text{and} \quad \xi_{S_1} = \xi_{S_1-} + \beta_1 = q.$$

Here, for any $x \in \mathbb{R}_+$, $\tau_x \triangleq \inf\{t \geq 0 : \xi_t^0 = x\}$. Let us introduce

$$u_0(\xi) \triangleq \mathbb{E}_0^\xi[e^{(b-r)\tau_d}1_{\{\tau_d<\tau_p\}}u(d)] + \mathbb{E}_0^\xi[e^{(b-r)\tau_p}1_{\{\tau_d>\tau_p\}}u(p)], \tag{3.17}$$

in which

$$u(d) = \mathbb{E}_0^d\left[e^{(b-r)\Delta}(\bar{C}_1(\xi_{\Delta-}^0, c) + u(c))\right] \quad \text{and} \quad u(p) = \bar{C}_2(p,q) + u(q). \tag{3.18}$$

From (3.15), (3.16) and (3.18) it can be seen that

$$u(\xi) = \begin{cases} \bar{C}_2(\xi,q) + u_0(q), & \xi \leq p; \\ u_0(\xi), & p \leq \xi \leq d; \\ r(\xi,c) + e^{(b-r)\Delta}u_0(c), & \xi \geq d. \end{cases} \tag{3.19}$$

in which

$$r(\xi,c) \triangleq \mathbb{E}_0^\xi[e^{(b-r)\Delta}\bar{C}_1(\xi_{\Delta-}^0, c)]. \tag{3.20}$$

Let us denote the fundamental solutions of $(\mathcal{A} + (b-r))f = 0$, by ψ (increasing) and φ (decreasing), and introduce $F \triangleq \psi/\varphi$. Using (2.18), on the interval (p,d) we can write u as

$$\frac{u(\xi)}{\varphi(\xi)} = \frac{u(d)}{\varphi(d)}\frac{(F(\xi)-F(p))}{(F(d)-F(p))} + \frac{u(p)}{\varphi(p)}\frac{(F(d)-F(\xi))}{(F(d)-F(p))}, \quad \xi \in (p,d). \tag{3.21}$$

Then, $W \triangleq \frac{u}{\varphi} \circ F^{-1}$, satisfies

$$W(y) = W(F(d)) \frac{y - F(p)}{F(d) - F(p)} + W(F(p)) \frac{(F(d) - y)}{(F(d) - F(p))}, \quad y \in [F(p), F(d)]. \tag{3.22}$$

Using the linear characterization (in the continuation region) of the band policies in (3.22), the following algorithm first determines the function u for an arbitrary band policy and goes on to find the best band policy.

First, let us define

$$R_1(x; c) \triangleq \frac{r(\cdot, c)}{\varphi(\cdot)} \circ F^{-1}(x) \quad \text{and} \quad R_2(x; q) \triangleq \frac{\bar{C}_2(\cdot, q)}{\varphi(\cdot)} \circ F^{-1}(x). \tag{3.23}$$

Algorithm. 1. For a given band policy which is characterized by the quadruplet (p, q, c, d) such that $p < q < c < d$, we can find the value function u in (3.19) using the linear characterization in (3.22). On $[F(p), F(d)]$ we will find $W(y) = \rho y + \tau$ (in which the slope ρ and the intercept τ are to be determined) from

$$\begin{aligned} &e^{(b-r)\Delta}(\rho F(c) + \tau) \frac{\varphi(c)}{\varphi(d)} + R_1(F(d); c) = \rho F(d) + \tau, \\ &(\rho F(q) + \tau) \frac{\varphi(q)}{\varphi(p)} + R_2(F(p); q) = \rho F(p) + \tau. \end{aligned} \tag{3.24}$$

ρ and τ are determined as

$$\begin{aligned} \rho &= \frac{\frac{R_2(F(p);q)}{1-\varphi(q)/\varphi(p)} \left(e^{(b-r)\Delta} \frac{\varphi(c)}{\varphi(d)} - 1 \right) + R_1(F(d); c)}{F(d) - e^{(b-r)\Delta} \frac{\varphi(c)}{\varphi(d)} F(c) + \frac{\varphi(q)/\varphi(p) F(q) - F(p)}{1-\varphi(q)/\varphi(p)} \left(1 - e^{(b-r)\Delta} \frac{\varphi(c)}{\varphi(d)} \right)}, \\ \tau &= \frac{\rho \left(\frac{\varphi(q)}{\varphi(p)} F(q) - F(p) \right) + R_2(F(p; q))}{1 - \frac{\varphi(q)}{\varphi(p)}}. \end{aligned} \tag{3.25}$$

Now u can be written as

$$u(\xi) = \begin{cases} u_0(q) + r_2(\xi, q), & x \le p, \\ u_0(\xi) \triangleq \rho \psi(\xi) + \tau \varphi(\xi), & p \le x \le d, \\ e^{(b-r)\Delta} u_0(c) + r_1(\xi, c), & x \ge d. \end{cases} \tag{3.26}$$

From this last expression, we observe that $(\mathcal{A} + (b - r))u(\xi) = 0$ for $\xi \in (p, d)$.

2. Note that ρ and τ are functions of (p, d) parametrized by (q, c). We will find an optimal pair (p, d) given (q, c) by equating the gradient of the function (ρ, τ) with respect to (p, d) to be zero. Now, differentiating the first equation in (3.24) with respect to d, and the second with respect to p, and evaluating them at $\tau_d = \rho_d = \tau_p = \rho_p = 0$ we obtain

$$\begin{aligned} &-(\rho F(q) + \tau) \frac{\varphi(q)}{\varphi(p)^2} \varphi'(p) - \rho F'(p) + \frac{\partial}{\partial y} R_2(y; q) \bigg|_{y=F(p)} F'(p) = 0 \\ &-e^{(b-r)\Delta}(\rho F(c) + \tau) \frac{\varphi(c)}{\varphi(d)^2} \varphi'(d) - \rho F'(d) + \frac{\partial}{\partial y} R_1(y; c) \bigg|_{y=F(d)} F'(d) = 0, \end{aligned} \tag{3.27}$$

in which ρ and τ are given by (3.25). To find the optimal (p, d) (given (c, q)) we solve the non-linear and implicit system of equations in (3.27).

Remark 3.1. On $[F(0), F(p)]$ the function W is given by

$$W(x) = \left((\rho F(q) + \tau)\frac{\varphi(q)}{\varphi(F^{-1}(x))}\right) + R_2(x; q), \tag{3.28}$$

and its left derivative at F(p), $W'(F(p)-)$, is given by

$$W'(F(p)-) = -(\rho F(q) + \tau)\frac{\varphi(q)}{\varphi(p)^2}\frac{\varphi'(p)}{F'(p)} + \frac{\partial}{\partial y}R_2(y; q)\bigg|_{y=F(p)}. \tag{3.29}$$

Therefore, the equation in (3.27) in fact implies that the left and the right derivative of W at $F(p)$ are equal (smooth fit). (Recall that $W(x) = \rho x + \tau y$ on $[F(p), F(d)]$.) Similarly, the second equation in (3.27) implies that the left and the right derivative of W at $F(d)$ are equal. This can be also expressed as: "R_2 shifted by an appropriate amount is tangential to the line $l(y) = \rho y + \tau$" at $F(p)$.

3. Next, we vary q and c to find the best band policy. Such a search can easily be carried out in Mathematica.

To obtain an explicit expression for g in (3.14) and r in (3.20) we make the following assumption. We will assume that $\sigma_t = \sigma > 0$ (a constant) in (3.2). Now, we can obtain g in (3.14) (see Appendix A) explicitly as

$$g(\xi) = \frac{A^{\mu}}{r - b + (b+\delta)\mu + \frac{1}{2}\sigma^2\mu - \frac{1}{2}\sigma^2\mu^2}\xi^{\mu} - \frac{w}{r+\delta}\xi \equiv k_1\xi^{\mu} + k_2\xi. \tag{3.30}$$

Note that if $r < b$, then $g(\xi) = \infty$, which implies that taking no action is optimal. The assumption in Proposition 3.1 that $\max(c_1 - c_2, c_3 + c_4) < |k_2|$ is for technical reasons, however it is not very restrictive. k_2 denotes the present value of the total wage that a firm pays per unit of marginal revenue product of labor and it should be greater than costs associated with one time hiring or firing of one unit of marginal revenue product of labor. Using (3.30) we can also calculate r in (3.20) explicitly as (see Appendix A)

$$\begin{aligned} r(\xi, c) = \mathrm{e}^{(b-r)\Delta}[&-(c_3 + c_4)\mathrm{e}^{-(b+\delta)\Delta}\xi N(d_1) + (c_1 - c_2)\mathrm{e}^{-(b+\delta)\Delta}\xi N(-d_1) \\ &+ c_3 c N(d_2) - c_1 c N(-d_2) - k_1 \exp(\epsilon)\xi^{\mu} - k_2\mathrm{e}^{-(b+\delta)\Delta}\xi + k_1 c^{\mu} + k_2 c] \end{aligned} \tag{3.31}$$

in which

$$\begin{aligned} d_1 &\triangleq \frac{1}{\sigma\sqrt{\Delta}}\log\left(\frac{\xi}{c}\right) + \left(\frac{1}{2}\sigma^2 - (b+\delta)\right)\frac{\sqrt{\Delta}}{\sigma}, \\ d_2 &\triangleq \frac{1}{\sigma\sqrt{\Delta}}\log\left(\frac{\xi}{c}\right) - \left(\frac{1}{2}\sigma^2 + (b+\delta)\right)\frac{\sqrt{\Delta}}{\sigma}, \\ \epsilon &\triangleq -\left(b + \delta + \frac{1}{2}\sigma^2(1-\mu)\right)\mu\Delta. \end{aligned} \tag{3.32}$$

Here the function $x \to N(x)$, $x \in \mathbb{R}$, denotes the cumulative distribution function of an $N(0, 1)$ (standard Gaussian) random variable. The infinitesimal generator $\mathcal{A}$ of the process ξ

Table 1
Measuring the effects of delay

	ρ	τ	p	q	c	d
$\Delta = 0$	0.0002003	38.1633	1.0664	2.125	7.240	35.728
$\Delta = 0.5$	0.0001725	38.1597	1.0661	2.100	7.120	36.640

is $\mathcal{A}u(x) \triangleq (\sigma^2/2)x^2u''(x) - (b+\delta)xu'(x)$, acting on smooth test functions $u(\cdot)$. Therefore the fundamental solutions of the equation $(\mathcal{A} + (b-r))u = 0$ are

$$\psi(x) \triangleq x^{\beta_1}, \qquad \varphi \triangleq x^{\beta_2}, \tag{3.33}$$

in which $\beta_1 > 1$ and $\beta_2 < 0$ are the roots of the following quadratic equation (in terms of β)

$$\frac{1}{2}\sigma^2\beta^2 - \left(\frac{1}{2}\sigma^2 + (b+\delta)\right)\beta + b - r = 0. \tag{3.34}$$

The next proposition justifies the second stage of our algorithm.

Proposition 3.1. *For a given* $(q, c) \in \mathbb{R}^2$*, such that* $(c_1q - (k_1q^{\mu} + k_2q)) < 0$ *there exits a unique solution* (p^*, d^*) *to the system of equations (3.27) if we further assume that* $\max(c_1 - c_2, c_3 + c_4) < |k_2|$*. Moreover,* $u^{p^*,q,c,d^*}(x) = \sup_{0<p<d} u^{p,q,c,s}(x)$, $x \geq 0$.

Proof. The proof is similar to that of Proposition 2.8. Also, see the remark below. □

Remark 3.2. The proof of Proposition 3.1 only relies on the following properties of the functions R_1 and R_2 defined in (3.23): (1) There exists a point $j \in (0, \infty)$ such that $y \to R_1(y; c)$ is concave and increasing on (j, ∞); (2) $\lim_{y\to\infty} R_1(y; c) = \infty$; (3) The function $y \to R_2(y, q)$ is increasing and concave on $(0, t)$ for some $t < F(q)$ and decreasing on (t, ∞); (4) Both $y \to R_1(y; c)$ and $y \to R_2(y, q)$ are differentiable.

Our results in this section can be generalized to the two-sided control of any one-dimensional diffusion and penalty functions satisfying the conditions in Remark 3.2. It is worth pointing out that Weeransinghe [25] has studied the two-sided bounded variation control within the framework of *singular stochastic control* of linear diffusions for a large class of cost functions by using the functional relationship between the value function of optimal stopping and that of singular stochastic control (see e.g. Karatzas and Shreve [16]).

3.3. Numerical example

In this section, we will give a numerical example for the labor problem with and without delay. We select the parameters as $b = 0.03, r = 0.06, \mu = 0.75, \sigma = 0.35, \delta = 0.1, A = 5, w = 2, \Delta = 0.5, c_1 = 0.05, c_2 = 0.1, c_3 = 2$ and $c_4 = 1$. The results we obtain are summarized in Table 1 and Fig. 2:

Both the slope ρ and the intercept τ are greater in the no-delay case and therefore, the value function corresponding to the no-delay problem $v^N(x)$ will dominate over that of the delay problem $v^D(x)$. On the right boundary, we have $(7.240, 35.728) \subset (7.120, 36.640)$ and on the left boundary the (c, d) pair has shifted to the left with delay. As a result, the continuation region (p, d) has expanded with delay: $\mathbf{C}^N \triangleq (1.0664, 35.728) \subset (1.0661, 36.640) \triangleq \mathbf{C}^D$. An explanation for this phenomenon can be made through the relative size of costs of firing and hiring, the size of delay parameter, the shape of g function, etc. In our example, the firing cost is relatively larger than hiring cost, the penalty of firing becomes smaller with delay (than without

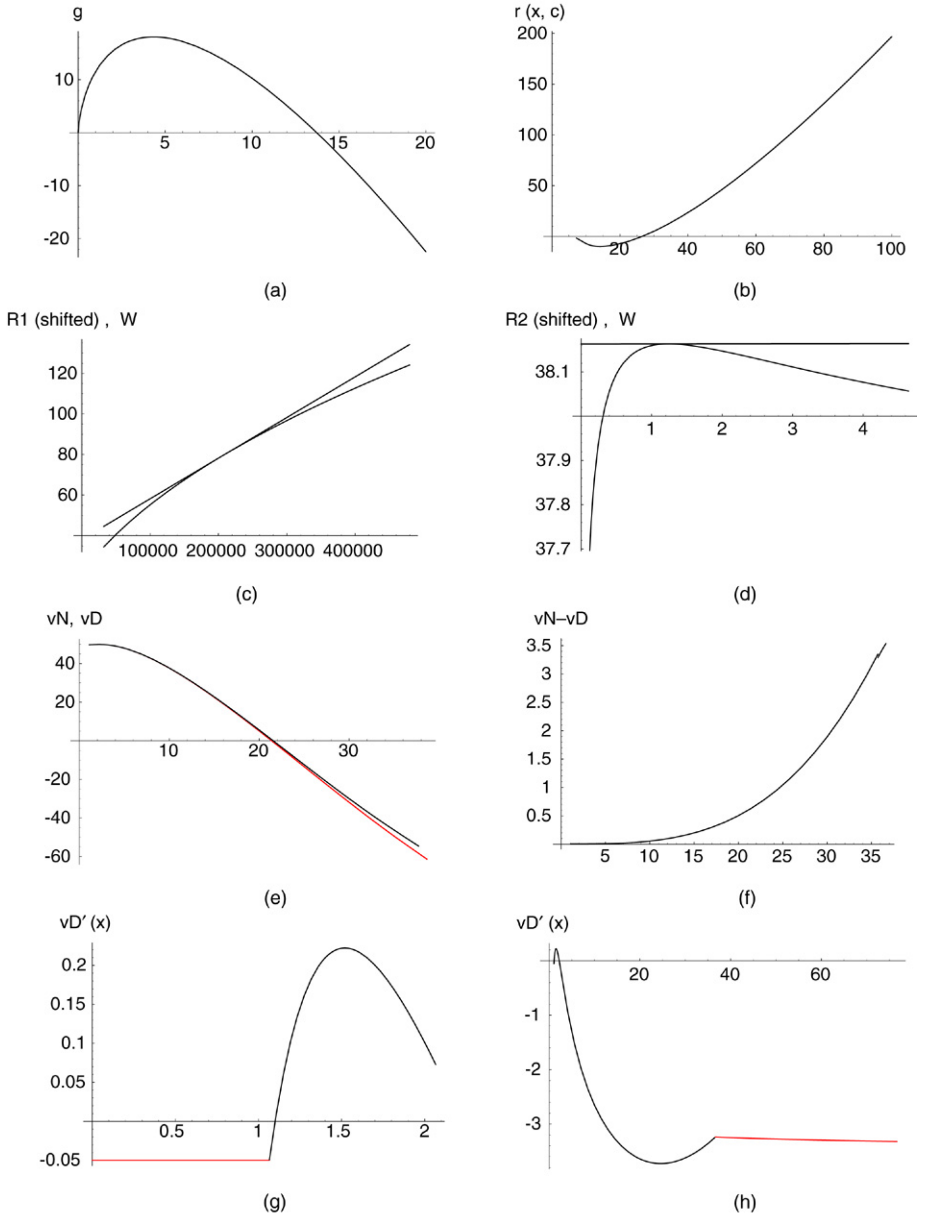

Fig. 2. (a) The graph of $g(x)$. (b) The graph of $r(x, c^*)$ for $\Delta > 0$. (c) The graph of line $\rho^* y + \tau^*$ we obtain via our algorithm and $R_1(y, c^*)$ after it is shifted vertically by $e^{(b-r)\Delta}(\rho F(c) + \tau)\frac{\varphi(c)}{\varphi(F^{-1}(y))}$. (d) The graph of the line $\rho^* y + \tau^*$ and $R_2(y, c^*)$ after it is shifted (see (3.28) for the amount of shift). (e) The two value functions, $v^N(x)$ ($\Delta = 0$) above and $v^D(x)$ ($\Delta > 0$) below. (f) Plot of difference, $v^N(x) - v^D(x)$. (f) Plot of difference, $v^N(x) - v^D(x)$. (g) (h) The derivatives match at $x = p$ and $x = d$ ($\Delta > 0$).

delay) which encourages the controller not to make hasty firing decisions, facing relatively large firing costs. Or since there is a chance that the process moves to the left during the delay period due to voluntary quits, this effect may help to reduce firing costs even though the decision making is postponed.

4. Conclusion

In this paper we give a new characterization of the *value function* of one-sided and two-sided impulse control problems with implementation delays. We also provided easily implemented algorithms to find out the optimal control and the value function. Our methodology bypasses the need to guess the form of solution of quasi-variational inequalities and prove that this solution satisfies a verification lemma. Since our method directly finds the value function, we believe that this method can solve a larger set of problems than just with quasi-variational inequalities. Indeed, we applied our results to solving some specific examples. As an important application of a two-sided impulse control problem with decision delays we found out the optimal hiring and firing decisions of a firm facing regulatory delays and stochastic demand.

Here we considered a problem in which the decision maker needs to decide whether to take action and, after some delay, needs to decide the magnitude of her action. In the future, we will consider problems in which the decision maker takes action and waits for that action to be implemented. We will also consider a general characterization of the value function and the optimal controls when the decision delay is not a constant but depends on the magnitude of the action taken as in Subramanian and Jarrow [24] or it depends on the value of the state variable that is controlled as in Alvarez and Keppo [3].

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Appendix A

A.1. Derivations of (3.14) and (3.31)

Using (3.11) we can write (3.14) as

$$\begin{aligned} g(\xi) &= \mathbb{E}_0^{\xi}\left[\int_0^{\infty} A^{\mu}\xi^{\mu}\mathrm{e}^{(b-r)t}\exp\left(-(b+\delta)\mu t-\sigma\mu B_t-\frac{1}{2}\sigma^2\mu t\right)\mathrm{d}t\right] \\ &\quad - w\mathbb{E}_0^{\xi}\left[\int_0^{\infty}\xi\exp\left(-(b+\delta)t-\sigma B_t-\frac{1}{2}\sigma^2 t\right)\mathrm{d}t\right] \\ &= A^{\mu}\xi^{\mu}\int_0^{\infty}\exp\left[t\left(b-r-(b+\delta)\mu-\frac{1}{2}\sigma^2\mu+\frac{1}{2}\sigma^2\mu^2\right)\right]\mathrm{d}t \\ &\quad - w\xi\int_0^{\infty}\exp(-(b+\delta)t)\mathrm{d}t, \end{aligned} \tag{A.1}$$

from which we obtain (3.30) under the assumption that $r > b$. Here the second inequality follows from Fubini's theorem and using the Laplace transform of B_t.

In what follows we will present the derivation of (3.31). We can write (3.20) as

$$\begin{aligned} r(\xi,c) &= \mathrm{e}^{(b-r)\Delta}\mathbb{E}_0^{\xi}[(-c_3(\xi_\Delta-c)-c_4\xi_\Delta)1_{\{\xi_\Delta>c\}}+(-c_1(c-\xi_\Delta)-c_2\xi_\Delta)1_{\{\xi_\Delta<c\}} \\ &\quad -k_1\xi_\Delta^{\mu}-k_2\xi_\Delta+k_1c^{\mu}+k_2c]. \end{aligned} \tag{A.2}$$

Using (3.11) and the assumption that $\sigma_t = \sigma \in \mathbb{R}_+$, we compute

$$A \triangleq \mathbb{E}_0^{\xi}\left[1_{\{\xi_\Delta > c\}}\right] = N(d_2), \qquad B \triangleq \mathbb{E}_0^{\xi}\left[1_{\{\xi_\Delta < c\}}\right] = 1 - A = N(-d_2),$$
$$C(\theta) \triangleq \mathbb{E}^{\xi}\left[\xi_\Delta^{\theta}\right] = \xi^{\theta} \exp\left(-\left(b + \delta + \frac{1}{2}\sigma^2(1-\theta)\right)\theta\Delta\right), \tag{A.3}$$

where $\theta = 1$ or $\theta = \mu$. Here the third equality follows from the Laplace transform of B_t. We will also need to compute

$$D \triangleq \mathbb{E}_0^{\xi}\left[\xi_\Delta 1_{\{\xi_\Delta > c\}}\right]. \tag{A.4}$$

We will denote

$$\kappa = \exp\left(-\frac{1}{2}\sigma^2\Delta + \sigma\sqrt{\Delta}\eta\right),$$

in which $\eta = B_\Delta/\sqrt{\Delta}$, is an $N(0,1)$ random variable. Then $\xi_\Delta = \xi\exp(-(b+\delta)\Delta)\kappa$ and $A = \xi \mathrm{e}^{-(b+\delta)\Delta}\mathbb{E}_0^{\xi}\left[1_{\{\xi_\Delta > c\}}\kappa\right]$. Introducing a new probability measure Q by the radon–nikodym derivative $\mathrm{d}Q^{\xi}/\mathrm{d}P_0^{\xi} = \kappa$, we get

$$D = \mathrm{e}^{-(b+\delta)\Delta}\xi Q^{\xi}(\xi_\Delta > c).$$

Under the measure Q^{ξ}, $n \triangleq -\eta - \sigma\sqrt{\Delta}$ is $N(0,1)$ and we can write ξ_Δ in terms of n as

$$\xi_\Delta = \xi \exp\left(-\left(b + \delta - \frac{1}{2}\sigma^2\right)\Delta + \sigma\sqrt{\Delta}n\right). \tag{A.5}$$

Using (A.5), we can compute

$$D = \xi \mathrm{e}^{-(b+\delta)\Delta}N(d_1), \tag{A.6}$$

in which d_1 is given by (3.32). We can then immediately obtain,

$$E \triangleq \mathbb{E}_0^{\xi}\left[\xi_\Delta 1_{\{\xi_\Delta < c\}}\right] = \xi\mathrm{e}^{-(b+\delta)\Delta}(1 - Q^{\xi}(\xi_\Delta > c)) = \xi \mathrm{e}^{-(b+\delta)\Delta}N(-d_1). \tag{A.7}$$

Using (A.2)–(A.4) and (A.7) we obtain (3.20).

A.2. A technical lemma

Lemma A.1. *Define*

$$G(x,\gamma) \triangleq \sup_{\tau\in S}\mathbb{E}^x[\mathrm{e}^{-\alpha\tau}(h(X_\tau^0) + \gamma\mathrm{e}^{-\alpha\Delta})], \qquad x \in \mathbb{R},\ \gamma \in \mathbb{R},$$

for some Borel function h. Then for $\gamma_1 > \gamma_2$ we have that

$$G(x,\gamma_1) - G(x,\gamma_2) \le \gamma_1 - \gamma_2.$$

Proof. See the proof of Lemma 3.3 in Dayanik and Egami [11]. □

A.3. Proof of Proposition 2.8

The proof follows from the analysis of the function r. The following remark will be helpful in the analysis that follows.

Remark A.1. Let us denote $H(y) \triangleq (h/\varphi) \circ (F^{-1}(y))$, $y > 0$. If $h(\cdot)$ is twice-differentiable at $x \in \mathcal{I}$ and $y \triangleq F(x)$, then $H'(y) = m(x)$ and $H''(y) = m'(x)/F'(x)$ with

$$m(x) = \frac{1}{F'(x)} \left(\frac{h}{\varphi}\right)'(x), \quad \text{and} \quad H''(y)[(\mathcal{A} - \alpha)h(x)] \geq 0, \quad y = F(x), \tag{A.8}$$

with strict inequality if $H''(y) \neq 0$.

A.3.1. The analysis of the function r in (2.47)

Let us check the sign of $\left(\frac{r}{\varphi}\right)'(x) = \frac{r'\varphi - r\varphi'}{\varphi^2}(x)$ which is the same as the derivative of R as can be observed from the first equation in (A.8). The sign of $\left(\frac{r}{\varphi}\right)'(x)$ is the same as that of

$$\begin{aligned}
&\frac{\sqrt{2\alpha}}{\alpha}\left(x^2 - a^2 + \Delta - 2\alpha\lambda\Delta \exp\left(-\frac{(a-x)^2}{4\Delta^2}\right) - c\alpha\right) \\
&\quad + \lambda(a-x)\left(-\frac{1}{\Delta}\exp\left(-\frac{(a-x)^2}{4\Delta^2}\right) + \frac{1}{\Delta}\phi\left(\frac{a-x}{\Delta}\right) + \sqrt{2\alpha}\left(2N\left(\frac{a-x}{\Delta}\right) - 1\right)\right) \\
&\quad + \frac{2x}{\alpha} + \lambda\left(2N\left(\frac{a-x}{\Delta}\right) - 1\right).
\end{aligned} \tag{A.9}$$

Using the fact $2N\left(\frac{a-x}{\Delta}\right) < 1$ for $x > a$ and $-\frac{1}{\Delta}\exp\left(-\frac{(a-x)^2}{4\Delta^2}\right) + \frac{1}{\Delta}\phi\left(\frac{a-x}{\Delta}\right) < 0$ for $x > a$ sufficiently large, in this equation (for sufficiently large x) we identify the absolute value of the negative terms as $\frac{\sqrt{2\alpha}}{\alpha}\lambda\Delta\exp\left(-\frac{(a-x)^2}{4\Delta^2}\right) < \frac{\sqrt{2\alpha}}{\alpha}\lambda\Delta$, $c\alpha$ and $|\lambda\left(2N\left(\frac{a-x}{\Delta}\right) - 1\right)| < \lambda$. Since these negative terms are bounded, if we take sufficiently large value, say a', the sign of (A.9) is positive for $x \in (a', \infty)$. Moreover, we can directly calculate $\lim_{y\to+\infty} \frac{\partial}{\partial y} R(y; a) = 0$ to check the behavior of $R(y; a)$ for large y. We also know that $R(y; a) \triangleq (r(\cdot, a)/\varphi(\cdot)) \circ F^{-1}(y)$ is negative at $y = F(a)$. On the other hand, $1/\varphi(F^{-1}(y)) = \sqrt{y}$ is an increasing and concave function. It follows that $R(y; c) + \frac{\gamma}{\varphi(F^{-1}(y))}$ is an increasing function on $y \in (F(a'), \infty)$.

To investigate the concavity of $R(y; a)$, we set

$$\begin{aligned}
q(x, a) \triangleq{}& \frac{1}{2}x^2\frac{\lambda}{\Delta}\left(e^{-\frac{(a-x)^2}{\Delta}}\left(1 - \frac{2(a-x)^2}{4\Delta^2}\right) - 3\phi\left(\frac{a-x}{\Delta}\right) - \lambda(a-x)\phi'\left(\frac{a-x}{\Delta}\right)\right) \\
&+ \alpha x^2 - \alpha r(x, a)
\end{aligned}$$

so that $(\mathcal{A} - \alpha)r(x, a) = q(x, a)$ for every $x > 0$. We have $\lim_{x\to\infty} q(x) = -\infty$ if $\alpha < 4$. By the second equation in (A.8), the function $R(y; a)$ becomes concave eventually. Since $R(\cdot; a)$ is increasing and concave on (a'', ∞) for some $a'' > a'$ and $\lim_{y\to\infty} R(y; a) = \infty$ we can find a unique linear majorant to $R^\gamma(\cdot, a)$ in Lemma 2.3 (the linear majorant majorizes $R^\gamma(\cdot, a)$ in the continuation region and is equal to $R^\gamma(\cdot, a)$ in the stopping region). The rest of the proof is as in Proposition 2.7.

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