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Quasi-uniform hyperspaces of compact subsets [☆]

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Abstract

Let $(X,\mathcal{U})$ be a quasi-uniform space, $\mathcal{K}(X)$ be the family of all nonempty compact subsets of $(X,\mathcal{U})$. In this paper, the notion of compact symmetry for $(X,\mathcal{U})$ is introduced, and relationships between the Bourbaki quasi-uniformity and the Vietoris topology on $\mathcal{K}(X)$ are examined. Furthermore we establish that for a compactly symmetric quasi-uniform space $(X,\mathcal{U})$ the Bourbaki quasi-uniformity $\mathcal{U}_*$ on $\mathcal{K}(X)$ is complete if and only if $\mathcal{U}$ is complete. This theorem generalizes the well-known Zenor–Morita theorem for uniformisable spaces to the quasi-uniform setting.

Keywords: Bourbaki quasi-uniformity; Vietoris topology; Small-set symmetric; Compactly symmetric; Complete

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1. Introduction

In general topology, one basic problem is to construct new spaces from old spaces. Then we study what kinds of properties possessed by the old spaces can be preserved by the new spaces. Hyperspaces are such objects constructed via existing spaces. The theory of hyperspaces begins with the well-known Hausdorff metric. Michael [14] has made

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major contributions to this field. Nowadays, it not only is a basic structure in general topology but also has many applications in other branches of Mathematics, such as mathematical economics, fractals, Banach space geometry, convex analysis and optimization (see [1,9,18]).

A *quasi-uniformity* on a (nonempty) set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ contains $\Delta(X)$, where $\Delta(X)$ is the diagonal of X, and (b) if $U \in \mathcal{U}$, then $V^2 \subseteq U$ for some $V \in \mathcal{U}$. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space* [7]. By $\tau(\mathcal{U})$, we denote the topology induced by $\mathcal{U}$ on X and if $\tau(\mathcal{U}) = \tau$ for some topology τ on X, then $\mathcal{U}$ is said to be *compatible* with τ. If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1}\colon U \in \mathcal{U}\}$ is also a quasi-uniformity on X and is called the *conjugate* of $\mathcal{U}$. Furthermore, a quasi-uniformity $\mathcal{U}$ is a uniformity provided $\mathcal{U} = \mathcal{U}^{-1}$.

Let $(X,\mathcal{U})$ be a quasi-uniform space, and let $\mathcal{P}_0(X)$ (respectively $\mathcal{K}(X)$) be the collection of all nonempty (respectively nonempty compact) subsets of $(X,\mathcal{U})$. On one hand, we can introduce a hyper-topology on $\mathcal{P}_0(X)$ in the following way. For a finite family $\mathcal{G}$ of nonempty subsets of $(X,\mathcal{U})$, define

$$\mathcal{G}^{\wedge} = \Big\{A \in \mathcal{P}_0(X)\colon\ A \subseteq \bigcup \mathcal{G} \text{ and } A \cap G \neq \emptyset \text{ for each } G \in \mathcal{G}\Big\}.$$

The *Vietoris topology* (or *finite topology*) on $\mathcal{P}_0(X)$, denoted by $2^{\tau(\mathcal{U})}$, is generated by taking $\{\mathcal{G}^{\wedge}\colon \mathcal{G}$ is a finite family of $\tau(\mathcal{U})\}$ as a base [14]. On the other hand, it is also possible to introduce a hyper-quasi-uniformity on $\mathcal{P}_0(X)$. For each $U \in \mathcal{U}$, set

$$H(U) = \big\{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X)\colon\ B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\big\}.$$

Then $\{H(U)\colon U \in \mathcal{U}\}$ is a base for a quasi-uniformity on $\mathcal{P}_0(X)$, which is called the *Bourbaki quasi-uniformity* induced by $\mathcal{U}$ and this quasi-uniformity is denoted by $\mathcal{U}_*$ [2,12]. If $\mathcal{U}$ is a uniformity on X, then $\mathcal{U}_*$ is nothing but the Hausdorff uniformity on $\mathcal{P}_0(X)$ induced by $\mathcal{U}$. Recently, Künzi and Ryser [12] investigated quasi-uniform properties of $(\mathcal{P}_0(X),\mathcal{U}_*)$. The main purpose of this paper is to study some properties of the quasi-uniform hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$, where we use $\mathcal{U}_*$ in the subspace $\mathcal{K}(X)$ instead of $\mathcal{U}_*|_{\mathcal{K}(X)}$.

In Section 2, relationships between the Bourbaki quasi-uniformity and the Vietoris topology on $\mathcal{K}(X)$ for a given quasi-uniform space $(X,\mathcal{U})$ are examined. In contrast to the uniform case, we observe that the Vietoris topology is always coarser than the topology of the Bourbaki quasi-uniformity on $\mathcal{K}(X)$, and they do not coincide in general. To reconcile the Bourbaki quasi-uniformity and the Vietoris topology on $\mathcal{K}(X)$, notions of compact symmetry and small-set symmetry for a quasi-uniform space are used.

Recall that a filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ is called a *Cauchy filter* if for each $U \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $F \times F \subseteq U$. A uniform space $(X,\mathcal{U})$ is called *complete* if every Cauchy filter on $(X,\mathcal{U})$ converges [6]. Moreover, a Tychonoff space is said to be *Dieudonné complete* if it has a compatible complete uniformity, equivalently, the finest compatible uniformity is complete. In 1970, using inverse limits, Zenor [19] proved that for a Tychonoff space (X,τ) the hyperspace $(\mathcal{K}(X), 2^{\tau})$ is Dieudonné complete if and only if (X,τ) is Dieudonné complete. Morita [15] improved Zenor's result by showing that for a uniform space $(X,\mathcal{U})$

(a) the Hausdorff uniformity $\mathcal{U}_*$ is compatible with the Vietoris topology $2^{\tau(\mathcal{U})}$, and

(b) $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.

Motivated by the above work of Zenor and Morita, we study the completeness of the quasi-uniform hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ of a given quasi-uniform space $(X,\mathcal{U})$ in Section 3. We prove that for a compactly symmetric quasi-uniform space $(X,\mathcal{U})$ the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete. This is a quasi-uniform analogy of the Zenor–Morita theorem for uniformisable spaces. We also give an example to show that the condition "compact symmetry" in this result cannot be removed or weakened to "local symmetry". Throughout the paper, $\mathbb{N}$ denotes the set of all natural numbers and $\omega = \mathbb{N} \cup \{0\}$. The topological spaces and quasi-uniform spaces are not assumed to satisfy any separation axioms, except those explicitly stated. For other undefined concepts about quasi-uniform spaces, see [7].

2. The Bourbaki quasi-uniformity on $\mathcal{K}(X)$

In this section, we discuss relationships between the Bourbaki quasi-uniformity and the Vietoris topology on $\mathcal{K}(X)$ for a given quasi-uniform space $(X,\mathcal{U})$. Some sufficient conditions under which the Bourbaki quasi-uniformity is compatible with the Vietoris topology are provided.

Proposition 2.1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then*

$$2^{\tau(\mathcal{U})} \subseteq \tau(\mathcal{U}_*) \quad \text{on } \mathcal{K}(X).$$

Proof. Let $\mathcal{G}$ be a finite family of open subsets of $(X,\mathcal{U})$ and let $A \in \mathcal{G}^\wedge$. Since A is compact, we can choose a $U_0 \in \mathcal{U}$ such that $U_0(A) \subseteq \bigcup \mathcal{G}$. Pick a point $x_G \in A \cap G$, and then choose a $U_G \in \mathcal{U}$ such that $U_G(x_G) \subseteq G$ for each $G \in \mathcal{G}$. Let $U = U_0 \cap (\bigcap_{G \in \mathcal{G}} U_G)$. We wish to show $H(U)(A) \subseteq \mathcal{G}^\wedge$. Let $B \in H(U)(A)$. Obviously, $B \subseteq U(A) \subseteq \bigcup \mathcal{G}$. On the other hand, for each $G \in \mathcal{G}$, $x_G \in U^{-1}(B)$ implies that $\emptyset \neq B \cap U(x_G) \subseteq B \cap G$. It follows that $B \in \mathcal{G}^\wedge$. Therefore $2^{\tau(\mathcal{U})} \subseteq \tau(\mathcal{U}_*)$ on $\mathcal{K}(X)$. □

Remark 2.2. Let (X,τ) be a topological space, and set $U_G = G \times G \cup (X - G) \times X$ for any open subset G of (X,τ). Then $\{U_G\colon G \in \tau\}$ is a subbase for the so-called *Pervin quasi-uniformity* $\mathcal{P}$ on (X,τ) [7]. It is shown in [2] and [13] that $\mathcal{P}_*$ is always compatible with 2^τ on $\mathcal{K}(X)$. However, the Bourbaki quasi-uniformity is not compatible with the Vietoris topology on $\mathcal{K}(X)$ generally, as Example 2.3 shows.

As a natural way to reconcile the Bourbaki quasi-uniformity and the Vietoris topology on quasi-uniform hyperspaces, we will use some weak forms of symmetry in the original spaces. Recall that a quasi-uniform space $(X,\mathcal{U})$ is said to be *locally symmetric* [7] if for each $U \in \mathcal{U}$ and each point $x \in X$, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$. A well-known fact is that each locally symmetric quasi-uniform space is regular.

Example 2.3. Let $X = \omega$, and define $d: X \times X \to \mathbb{R}^+$ as follows

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/y & \text{if } x = 0,\ y > 0, \\ 1/x & \text{if } 0 < x < y, \\ 1 & \text{otherwise.} \end{cases}$$

Then (X,d) is a compact Hausdorff quasi-metric space. Let $\mathcal{U}_d$ be the quasi-uniformity induced by d. Note that 0 is the only nonisolated point in X and that the conjugate topology of $\mathcal{U}_d$ is discrete. By results of [7], it is straightforward to check that $(X,\mathcal{U}_d)$ is locally symmetric. However, the Bourbaki quasi-uniformity $(\mathcal{U}_d)_*$ is not compatible with the Vietoris topology $2^{\mathcal{T}_d}$ on $\mathcal{K}(X)$. In fact, we shall show later (see Example 3.8) that $(\mathcal{U}_d)_*$ is not complete.

In addition, $(\mathcal{K}(X),\mathcal{U}_*)$ is not locally symmetric. In fact, if $(\mathcal{K}(X),\mathcal{U}_*)$ is locally symmetric, then for any $n \in \mathbb{N}$ there exists a symmetric $\mathcal{W} \in \mathcal{U}_*$ such that $\mathcal{W}^2(X) \subseteq H(U_n)(X)$. Therefore, there is an $m \in \omega$ such that $H(U_m^{-1})(X) \subseteq \mathcal{W}(X) \subseteq H(U_n)(X)$. Let $A = \{0,1,2,\ldots,m\}$. Since $A \in H(U_m^{-1})(X)$ and $A \notin H(U_n)(X)$, we obtain a contradiction.

Since local symmetry is not well-behaved with respect to the space $(\mathcal{K}(X),\mathcal{U}_*)$ for a quasi-uniform space $(X,\mathcal{U})$ as shown in Example 2.3, we will turn our attention to some other weak forms of symmetry in the sequel.

Definition 2.4. A quasi-uniform space $(X,\mathcal{U})$ is called *compactly symmetric* if for each $U \in \mathcal{U}$ and each $C \in \mathcal{K}(X)$, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in C$.

Let $(X,\mathcal{U})$ be the quasi-uniform space defined in Example 2.3. We already knew that it is compact and locally symmetric. Since any compact and compactly symmetric quasi-uniform space must be a uniform space, $(X,\mathcal{U})$ is not compactly symmetric. In the following, we provide a compactly symmetric quasi-uniform space which is not a uniform space.

Example 2.5 [4]. Let $X = \mathbb{R} \setminus \{0\}$ and define $\mathcal{U}_d$ by the quasi-metric

$$d(x,y) = \begin{cases} \min\{y-x, 1\} & \text{if } x < 0 < y, \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Then (X,d) is a locally symmetric and quiet [5] quasimetric space such that $\tau(d)$ and $\tau(d^{-1})$ are the discrete topology on X. Thus $(X,\mathcal{U}_d)$ is compactly symmetric. However, $\mathcal{U}_d$ is not a uniformity.

Definition 2.6 [8]. Let $(X,\mathcal{U})$ be a quasi-uniform space and $U \in \mathcal{U}$. A subset B of X is called *U-small* provided that $B \subseteq U(A)$ whenever A is an open subset and $A \cap B \neq \emptyset$. The quasi-uniform space $(X,\mathcal{U})$ is called *small-set symmetric* if for each $U \in \mathcal{U}$ the open U-small subsets form a cover of X.

Note that it is shown in [10] that any compact small-set symmetric quasi-uniform space is a uniform space.

Lemma 2.7 [11]. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(X,\mathcal{U})$ is small-set symmetric if and only if $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$.*

Theorem 2.8. *Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(X,\mathcal{U})$ is either compactly symmetric or small-set symmetric, then $\mathcal{U}_*$ is compatible with $2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$.*

Proof. Following Proposition 2.1, it suffices to show that $\tau(\mathcal{U}_*) \subseteq 2^{\tau(\mathcal{U})}$. To do this, we prove that for each $U \in \mathcal{U}$ and each $A \in \mathcal{K}(X)$, the $\tau(\mathcal{U}_*)$-neighbourhood $H(U)(A)$ of A is also a $2^{\tau(\mathcal{U})}$-neighbourhood of A in the following two cases.

Case (i). Suppose that $(X,\mathcal{U})$ is compactly symmetric. Choose a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in A$. Since A is compact, there exists a finite subset $A_1 \subseteq A$ such that $A \subseteq \bigcup_{a\in A_1} \operatorname{int}(V(a))$. Let

$$\mathcal{G} = \{\operatorname{int}(V(a)):\ a \in A_1\}.$$

Then $\mathcal{G}^\wedge$ is a $2^{\tau(\mathcal{U})}$-open set in $\mathcal{K}(X)$ containing A. For any $B \in \mathcal{G}^\wedge$, $B \subseteq V(A) \subseteq U(A)$. On the other hand, for each point $x \in A$, there exists a point $a \in A_1$ such that $x \in V(a)$. Choose a point $b \in B \cap V(a)$. It follows that $b \in V(a) \subseteq V^2(x) \subseteq U(x)$ which implies that $B \cap U(x) \neq \emptyset$ for each point $x \in A$. Consequently, we have $A \subseteq U^{-1}(B)$ and $\mathcal{G}^\wedge \subseteq H(U)(A)$ which implies that $H(U)(A)$ is a $2^{\tau(\mathcal{U})}$-neighbourhood of A.

Case (ii). Suppose that $(X,\mathcal{U})$ is small-set symmetric. Choose a $V \in \mathcal{U}$ such that $V^2 \subseteq U$. By Lemma 2.7, for $x \in X$ we have that $V^{-1}(x)$ is a $\tau(\mathcal{U})$-neighbourhood of x. Since A is compact, there is a finite subset $A_0 \subseteq A$ such that $A \subseteq \bigcup_{a\in A_0} \operatorname{int}(V^{-1}(a))$. Let

$$\mathcal{G} = \{\operatorname{int}(V(A))\} \cup \{\operatorname{int}(V(a)):\ a \in A_0\}.$$

If $B \in \mathcal{G}^\wedge$, then $B \subseteq V(A) \subseteq U(A)$. On the other hand, for each $B \in \mathcal{G}^\wedge$ and $a \in A_0$, $B \cap V(a) \neq \emptyset$ implies $a \in V^{-1}(B)$. It follows that $A \subseteq V^{-2}(B) \subseteq U^{-1}(B)$. Hence $A \in \mathcal{G}^\wedge \subseteq H(U)(A)$. Therefore, we have $\tau(\mathcal{U}_*) \subseteq 2^{\tau(\mathcal{U})}$ in either case. □

Corollary 2.9 [15]. *For a uniform space $(X,\mathcal{U})$, $\mathcal{U}_*$ is compatible with $2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$.*

The following result shows that for a quasi-uniform space $(X,\mathcal{U})$ both compact symmetry and small-set symmetry are preserved by the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$.

Theorem 2.10. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{K}(X),\mathcal{U}_*)$ is compactly (respectively small-set) symmetric if and only if $(X,\mathcal{U})$ is compactly (respectively small-set) symmetric.*

Proof. The necessity is trivial, since both small-set symmetry and compact symmetry are hereditary properties. To prove the sufficiency, we first assume that $(X,\mathcal{U})$ is compactly symmetric. To show that $(\mathcal{K}(X),\mathcal{U}_*)$ is compactly symmetric, let $\mathcal{A}$ be a nonempty compact subset of $(\mathcal{K}(X),\mathcal{U}_*)$ and $U \in \mathcal{U}$. By Theorem 2.8, $\tau(\mathcal{U}_*) = 2^{\tau(\mathcal{U})}$ on $\mathcal{K}(X)$, so

$\mathcal{A}$ is compact in $(\mathcal{K}(X), 2^{\tau(\mathcal{U})})$. It follows from [14, Theorem 2.5.2] that $\bigcup \mathcal{A}$ is compact in $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is compactly symmetric, there exists a symmetric $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ whenever $x \in \bigcup \mathcal{A}$. It follows that $(H(V))^2(A) \subseteq H(U)(A)$ for all $A \in \mathcal{A}$. Therefore $(\mathcal{K}(X), \mathcal{U}_*)$ is compactly symmetric.

Now suppose that $(X,\mathcal{U})$ is small-set symmetric. Let $A \in \mathcal{K}(X)$ and let $U \in \mathcal{U}$. Consider the $\tau({\mathcal{U}_*}^{-1})$-neighbourhood of A,

$$(H(U))^{-1}(A) = \{B \in \mathcal{K}(X)\colon\ A \subseteq U(B),\ B \subseteq U^{-1}(A)\}$$

(note that $(H(U))^{-1} = H(U^{-1})$). Choose a $W \in \mathcal{U}$ such that $W^2 \subseteq U$. Since $(X,\mathcal{U})$ is small-set symmetric, for each $x \in X$ there is $V_x \in \mathcal{U}$ with $V_x^2(x) \subseteq W^{-1}(x)$. Thus $(V_x \cap W)^2(x) \subseteq W^{-1}(x)$ for all $x \in X$. By the compactness of A, there is a finite subset A' of A such that $A \subseteq \bigcup_{x \in A'}(V_x \cap W)(x)$. Let $V = \bigcap_{x \in A'}(V_x \cap W)$. We shall show that $H(V)(A) \subseteq H(U^{-1})(A)$. Let $B \in H(V)(A)$. Then $B \subseteq V(A)$ and $A \subseteq V^{-1}(B)$. Take any point $a \in A$. Since $A \subseteq V^{-1}(B)$, $(a,b) \in V$ for some point $b \in B$. Moreover, there is a point $x \in A'$ such that $a \in (V_x \cap W)(x)$. Hence $b \in (V_x \cap W)^2(x) \subseteq W^{-1}(x)$. It follows that $a \in W^2(b) \subseteq U(b)$. We have shown that $A \subseteq U(B)$. Now take any point $b \in B$. Since $B \subseteq V(A)$, $(a,b) \in V$ for some point $a \in A$. There is a point $x \in A'$ such that $a \in (V_x \cap W)(x)$. Hence $b \in (V_x \cap W)^2(x) \subseteq W^{-1}(x)$. Thus $a \in W^2(b) \subseteq U(b)$. We have shown that $B \subseteq U^{-1}(A)$. Therefore $B \in H(U^{-1})(A)$. We conclude that $(\mathcal{K}(X), \mathcal{U}_*)$ is small-set symmetric. □

In the following paragraphs we will discuss relationships between small-set symmetry and compact symmetry. Obviously, these two notions are equivalent for a compact quasi-uniform space. We first give a compactly symmetric quasi-uniform space which is not small-set symmetric. Then a relationship between these two concepts in one direction is established.

Example 2.11. A compactly symmetric quasi-uniform space which is not small-set symmetric. Let $X = \omega \cup \{p\}$ be equipped with its subspace topology induced by $\beta\omega$, where $p \in \beta\omega \setminus \omega$. Equip X with its Pervin quasi-uniformity $\mathcal{P}$. Since X is regular, $\mathcal{P}$ is locally symmetric. But $(X, \mathcal{P})$ is not small-set symmetric, since $\tau(\mathcal{P}^{-1})$ is discrete. If $K \subseteq X$ is compact, then K is finite. Otherwise, K is compact in $\beta\omega$. But any infinite closed subset of $\beta\omega$ has cardinality 2^c. This is a contradiction. Clearly each locally symmetric quasi-uniform space in which compact subsets are finite is compactly symmetric. Hence $(X, \mathcal{P})$ is compactly symmetric, but not small-set symmetric.

Proposition 2.12. *Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\tau(\mathcal{U})$ is a k-topology. If $(X,\mathcal{U})$ is compactly symmetric, then it is small-set symmetric.*

Proof. Let $G \in \tau(\mathcal{U}^{-1})$ be a nonempty subset. Consider an arbitrary $K \in \mathcal{K}(X)$. Then we have $G \cap K \in \tau(\mathcal{U}^{-1}|_K)$. Let $k \in G \cap K$ be a point. Choose a $V \in \mathcal{U}$ such that $V^{-1}(k) \cap K \subseteq G \cap K$. By compact symmetry, there exists a symmetric $W \in \mathcal{U}$ such that $W^2(k) \subseteq V(k)$ whenever $k \in K$. For each $k' \in W(k) \cap K$, we have

$k \in W(k') \subseteq V(k')$. Thus $k' \in V^{-1}(k) \cap K$. It follows that $W(k) \cap K \subseteq V^{-1}(k) \cap K$. Consequently, $G \cap K \in \tau(\mathcal{U}|_K)$. Since $(X, \tau(\mathcal{U}))$ is a k-space, G is open in $(X, \mathcal{U})$. Therefore, $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$. By Lemma 2.7, $(X, \mathcal{U})$ is small-set symmetric. □

We conclude this section by giving a connected, locally compact and small-set symmetric quasi-uniform space which is not compactly symmetric.

Example 2.13. A connected, locally compact and small-set symmetric quasi-uniform space which is not compactly symmetric. Let $\mathbb{R}$ be the set of all real numbers. For each $n \in \omega$, define

$$U_n = \left\{(x, y)\colon\ x \in \mathbb{R},\ x - \frac{1}{2^n} < y < x + \frac{1}{2^n}\right\} \cup \left\{(x, y)\colon\ y \in \mathbb{R},\ x > 2^n\right\}.$$

Let $\mathcal{U}$ be the quasi-uniformity on $\mathbb{R}$ generated by $\{U_n\colon\ n \in \omega\}$ as a base. It is easy to see that the topology $\tau(\mathcal{U})$ is the usual topology on $\mathbb{R}$ which is both connected and locally compact. Note that the conjugate topology $\tau(\mathcal{U}^{-1})$ is strictly coarser than $\tau(\mathcal{U})$. By Lemma 2.7, $(\mathbb{R}, \mathcal{U})$ is small-set symmetric. However, $(\mathbb{R}, \mathcal{U})$ is not compactly symmetric (even not point symmetric, see [7] for definition), since $\tau(\mathcal{U}) \subseteq \tau(\mathcal{U}^{-1})$ for any compactly symmetric quasi-uniformity $\mathcal{U}$ on $\mathbb{R}$.

3. The completeness of $(\mathcal{K}(X), \mathcal{U}_*)$

In this section, we use compact symmetry to extend the well-known Zenor–Morita theorem for uniformisable spaces to the quasi-uniform setting. Firstly we recall some definitions relating to the completeness of a quasi-uniform space.

Definition 3.1 [7]. A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called a *Cauchy filter* if for each $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(x) \in \mathcal{F}$. A quasi-uniform space $(X, \mathcal{U})$ is called *complete* (*convergence complete*) if every Cauchy filter on $(X, \mathcal{U})$ has a cluster point (converges).

Clearly, convergence completeness implies completeness, and the two notions coincide in the class of locally symmetric quasi-uniform spaces [7, Corollary 3.9], and therefore in the class of compactly symmetric quasi-uniform spaces. For convenience, we shall use certain convergent nets instead of filters. To this end, we recall the following definition.

Definition 3.2 [16]. A net $\{x_\alpha\colon\ \alpha \in D\}$ on $(X, \mathcal{U})$ is called an *$\mathcal{N}$-Cauchy net* if for each $U \in \mathcal{U}$ there exists a point $x \in X$ such that $\{x_\alpha\colon\ \alpha \in D\}$ is eventually in $U(x)$.

Lemma 3.3. *A quasi-uniform space $(X, \mathcal{U})$ is convergence complete if and only if every $\mathcal{N}$-Cauchy net in $(X, \mathcal{U})$ is convergent.*

Corollary 3.4. *A compactly symmetric quasi-uniform space $(X, \mathcal{U})$ is complete if and only if every $\mathcal{N}$-Cauchy net in $(X, \mathcal{U})$ is convergent.*

Theorem 3.5. *Let $(X,\mathcal{U})$ be a compactly symmetric quasi-uniform space. Then the hyperspace $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.*

Proof. Suppose that $(\mathcal{K}(X),\mathcal{U}_*)$ is complete. Let $\{x_\alpha\colon \alpha\in D\}$ be an $\mathcal{N}$-Cauchy net in $(X,\mathcal{U})$. Clearly, $\{\{x_\alpha\}\colon \alpha\in D\}$ is an $\mathcal{N}$-Cauchy net in $(\mathcal{K}(X),\mathcal{U}_*)$, so it has a cluster point $C\in\mathcal{K}(X)$. Fix any point $x_0\in C$. It easily follows that $\{x_\alpha\colon \alpha\in D\}$ clusters to x_0 in $(X,\mathcal{U})$. Hence $(X,\mathcal{U})$ is complete.

Conversely, suppose that $(X,\mathcal{U})$ is complete. Let $\{C_\alpha\colon \alpha\in D\}$ be an $\mathcal{N}$-Cauchy net in $(\mathcal{K}(X),\mathcal{U}_*)$. We show that $\{C_\alpha\colon \alpha\in D\}$ is convergent in $(\mathcal{K}(X),\mathcal{U}_*)$.

(1) For each $\alpha\in D$, define $F_\alpha=\bigcup\{C_\beta\colon \beta\in D,\ \beta\geqslant\alpha\}$. Let $\mathcal{F}=\mathrm{fil}\{F_\alpha\colon \alpha\in D\}$ and let $\mathcal{F}'$ be an ultrafilter containing $\mathcal{F}$.

(2) $\mathcal{F}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$. Let $U\in\mathcal{U}$, and choose $V\in\mathcal{U}$ such that $V^2\subseteq U$. Since $\{C_\alpha\colon \alpha\in D\}$ is an $\mathcal{N}$-Cauchy net, there are $B\in\mathcal{K}(X)$ and $\alpha_V\in D$ such that $C_\alpha\subseteq V(B)$ and $B\subseteq V^{-1}(C_\alpha)$ for all $\alpha\geqslant\alpha_V$. By the compactness of B, there is a finite subset B' of B such that $B\subseteq\bigcup_{b\in B'}V(b)$. Consequently

$$F_{\alpha_V}=\bigcup_{\beta\geqslant\alpha_V}C_\alpha\subseteq V(B)\subseteq\bigcup_{b\in B'}V^2(b)\subseteq\bigcup_{b\in B'}U(b).$$

Since $\mathcal{F}'$ is an ultrafilter and $F_{\alpha_V}\in\mathcal{F}'$, $U(b)\in\mathcal{F}'$ for some $b\in B'$. Thus $\mathcal{F}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$.

(3) Set $C=\bigcap_{F\in\mathcal{F}}\overline{F}$. Then $C\in\mathcal{K}(X)$. From (2), we have $C\neq\emptyset$. Next we prove C is compact. Let $\mathcal{H}$ be a filter on C and let

$$\mathcal{G}=\mathrm{fil}\{W(H)\cap F\colon\ H\in\mathcal{H},\ W\in\mathcal{U},\ F\in\mathcal{F}\}.$$

Note that $\mathcal{G}$ is well-defined because $H\subseteq C$ for each $H\in\mathcal{H}$ and $C=\bigcap_{F\in\mathcal{F}}\overline{F}$. Let $\mathcal{G}'$ be an ultrafilter containing $\mathcal{G}$. We have shown that for each $U\in\mathcal{U}$ there are $B\in\mathcal{K}(X)$ and $\alpha_V\in D$ (where $V^2\subseteq U$) such that $F_{\alpha_V}\subseteq V(B)\subseteq\bigcup_{b\in B'}U(b)$ (where B' is some finite subset of B), so $F_{\alpha_V}\cap V(H)\subseteq V(B)\subseteq\bigcup_{b\in B'}U(b)$ for any $H\in\mathcal{H}$. Since $F_{\alpha_V}\cap V(H)\in\mathcal{G}'$, we have $U(b)\in\mathcal{G}'$ for some $b\in B'$. Thus $\mathcal{G}'$ is a Cauchy ultrafilter on $(X,\mathcal{U})$. So it converges to some point $y_0\in X$. Since $\mathcal{F}\subseteq\mathcal{G}$, $y_0\in C$. We see that $\mathcal{H}$ clusters to y_0. Let $U\in\mathcal{U}$ and $H\in\mathcal{H}$. Choose a symmetric $V\in\mathcal{U}$ such that $V^2(y_0)\subseteq U(y_0)$. Since $\mathcal{G}$ clusters to y_0, there is $z\in V(H)\cap V(y_0)$. By the symmetry of V, $x\in V^2(y_0)\subseteq U(y_0)$ for some $x\in H$. Therefore y_0 is a cluster point of $\mathcal{H}$. We conclude that C is compact.

(4) The net $\{C_\alpha\colon \alpha\in D\}$ is convergent to C in $(\mathcal{K}(X),\mathcal{U}_*)$. For each $U\in\mathcal{U}$, choose a symmetric $V\in\mathcal{U}$ such that $V^3(x)\subseteq U(x)$ whenever $x\in C$. Again, since $\{C_\alpha\colon \alpha\in D\}$ is $\mathcal{N}$-Cauchy, there are an $\alpha_0\in D$ and an $E\in\mathcal{K}(X)$ such that $(E,C_\alpha)\in H(V)$ for all $\alpha\geqslant\alpha_0$. Since $V(e)\cap C_\alpha\neq\emptyset$ for each $e\in E$ and each $\alpha\geqslant\alpha_0$, $\{F\cap V(e)\colon F\in\mathcal{F}\}$ is contained in a Cauchy filter on $(X,\mathcal{U})$. Thus, we have

$$\emptyset\neq\bigcap_{F\in\mathcal{F}}\overline{F\cap V(e)}\subseteq C\cap\overline{V(e)}\subseteq C\cap V^2(e).$$

It follows that $e\in V^2(C)$ for each $e\in E$. Hence $E\subseteq V^2(C)$ and $C_\alpha\subseteq V(E)\subseteq V^3(C)\subseteq U(C)$. Since $C\subseteq\overline{F_\alpha}\subseteq\overline{V(E)}\subseteq V^2(E)$, for each point $x\in C$ there must be a point $y \in E$ such that $x \in V^2(y)$. On the other hand, for each $\alpha \in D$ and $\alpha \geqslant \alpha_0$, there exists a point $x_\alpha \in C_\alpha$ with $y \in V(x_\alpha)$. Hence $x_\alpha \in V^3(x) \subseteq U(x)$. It follows that $x \in U^{-1}(x_\alpha)$ and $C \subseteq U^{-1}(C_\alpha)$ for all $\alpha \in D$ and $\alpha \geqslant \alpha_0$. Therefore $(C, C_\alpha) \in H(U)$.

Combining the previous arguments, $(\mathcal{K}(X), \mathcal{U}_*)$ is complete. □

Corollary 3.6 [15]. *Let $(X,\mathcal{U})$ be a uniform space. Then $(\mathcal{K}(X),\mathcal{U}_*)$ is complete if and only if $(X,\mathcal{U})$ is complete.*

Corollary 3.7 [19]. *Let (X,τ) be a Tychonoff space. Then $(\mathcal{K}(X), 2^\tau)$ is Dieudonné-complete if and only if (X,τ) is Dieudonné-complete.*

Example 3.8. The condition "compact symmetry" in Theorem 3.5 cannot be weakened to "local symmetry". Let (X,d) be the quasi-metric space which is defined in Example 2.3. We know that $(X,\mathcal{U}_d)$ is locally symmetric, but neither small-set symmetric nor compactly symmetric. For each $n \in \omega$, set $K_n = \{1,2,\ldots,n\}$ and $A_n = \{0,n+1\} \cup K_n$. Then $K_m \in H(U_{1/n})(A_n)$ whenever $m > n$, where $U_{1/n} = \{(x,y)\colon d(x,y) < 1/n\}$. It follows that $\{K_n\colon n \in \omega\}$ is an $\mathcal{N}$-Cauchy sequence in $(\mathcal{K}(X), (\mathcal{U}_d)_*)$. However, $\{K_n\colon n \in \mathbb{N}\}$ has no cluster point. Therefore $(\mathcal{K}(X), (\mathcal{U}_d)_*)$ is not complete.

Let $(X,\mathcal{U})$ be a quasi-uniform space. A filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called *right K-Cauchy* if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U^{-1}(x) \in \mathcal{F}$ for all $x \in F$. The space $(X,\mathcal{U})$ is called *right K-complete* if every right K-Cauchy filter on $(X,\mathcal{U})$ is convergent [17]. Recall that a filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called *stable* if for each $U \in \mathcal{U}$, $\bigcap_{F\in\mathcal{F}} U(F)$ belongs to $\mathcal{U}$. In [12], Künzi and Ryser successfully generalized the Isbell–Burdick theorem as follows: For a quasi-uniform space $(X,\mathcal{U})$ the Bourbaki quasi-uniformity $\mathcal{U}_*$ on $\mathcal{P}_0(X)$ is right K-complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. In light of this result, it appears reasonable to conjecture that $(\mathcal{K}(X),\mathcal{U}_*)$ is right K-complete if and only if $(X,\mathcal{U})$ is right K-complete. Unfortunately, the following example shows that this conjecture is false.

Example 3.9. Let $X = \mathbb{R}$ (where $\mathbb{R}$ denotes the reals), and define

$$U_n = \{(x,y)\colon x \in X,\ y < -2^n\} \cup \left\{(x,y)\colon x \in X,\ x - \frac{1}{2^n} < y < x + \frac{1}{2^n}\right\}$$

for each $n \in \omega$. Let $\mathcal{U}$ be the quasi-uniformity on X generated by $\{U_n\colon n \in \omega\}$. Let $\{x_\alpha\colon \alpha \in D\}$ be a right K-Cauchy net in $(X,\mathcal{U})$. Then there exists an $\alpha_0 \in D$ such that $\overline{\{x_\alpha\colon \alpha \geqslant \alpha_0\}}$ is bounded from above. Since any subset of X which is closed and bounded from above in $(X,\mathcal{U})$ is compact, $\{x_\alpha\colon \alpha \in D\}$ is convergent in $(X,\mathcal{U})$. Therefore $(X,\mathcal{U})$ is right K-complete. For each $n \in \omega$, set $K_n = (-\infty, n]$. Then it is easy to check that $\{K_n\colon n \in \omega\}$ is a right K-Cauchy sequence in $(\mathcal{K}(X),\mathcal{U}_*)$. But $\{K_n\colon n \in \omega\}$ is not convergent in $(\mathcal{K}(X),\mathcal{U}_*)$. In fact, suppose that $K_n \to C \in \mathcal{K}(X)$. Then $K_n \subseteq U_1(C)$ for n large enough. Since C is bounded from above, we obtain a contradiction.

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