Strong characterizing sequences in simultaneous diophantine approximation

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Abstract

Answering a question of Liardet, we prove that if 1, x₁, x₂, ..., xₜ are real numbers linearly independent over the rationals, then there is an infinite subset A of the positive integers such that for real β, we have (|| || denotes the distance to the nearest integer)

\[ \sum_{n \in A} ||n\beta|| < \infty \]

if and only if β is a linear combination with integer coefficients of 1, x₁, x₂, ..., xₜ. The proof combines elementary ideas with a deep theorem of Freiman on set addition. Using Freiman’s theorem, we prove a lemma on the structure of Bohr sets, which may have independent interest. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

In [1], together with Jean-Marc Deshouillers, we proved the following theorem (|| || denotes the distance to the nearest integer).

Theorem. Assume that 1, x₁, x₂, ..., xₜ are real numbers linearly independent over the rationals. Then there is an infinite subset A of the positive integers such that for real β,
we have
\[ \lim_{n \in \mathbb{N}, n \to \infty} \|n\beta\| = 0 \]
if and only if \( \beta \in G \), where \( G \) is the group generated by 1, \( \alpha_1, \alpha_2, \ldots, \alpha_t \).

We call \( A \) a characterizing sequence of \( G \).

Actually, we proved there a stronger theorem: the same statement is true for any countable subgroup of the reals with 1 \( \in G \), but to extend the theorem for that case is a technical matter. For the sake of simplicity, in the present paper we consider only the special case. Liardet [2] asked the following problem: can one replace the condition
\[ \lim_{n \in \mathbb{N}, n \to \infty} \|n\beta\| = 0 \]
in the above theorem by
\[ \sum_{n \in A} \|n\beta\| < \infty? \]

Our answer is affirmative.

**Theorem.** Assume that 1, \( \alpha_1, \alpha_2, \ldots, \alpha_t \) are real numbers linearly independent over the rationals. Then there is an infinite subset \( A \) of the positive integers such that for real \( \beta \), we have
\[ \sum_{n \in A} \|n\beta\| < \infty, \]
if and only if \( \beta \in G \), where \( G \) is the group generated by 1, \( \alpha_1, \alpha_2, \ldots, \alpha_t \). Furthermore, for \( \beta \notin G \) we even have
\[ \lim_{n \in \mathbb{N}, n \to \infty} \inf \|n\beta\| > 0. \]

This is a strengthening of the quoted theorem of [1], so we may call such an \( A \) a strong characterizing sequence of \( G \).

Our proof combines the ideas of the proof in [1] with a deep theorem of Freiman on set addition. Using Freiman’s theorem, we prove a lemma on the structure of Bohr sets. Since this lemma (Lemma 1 below) may have independent interest, we state it here, in the Introduction.

Bohr sets are defined in the following way: if \( \alpha_1, \alpha_2, \ldots, \alpha_t \) are arbitrary (but fixed) real numbers (so independence is not assumed here), \( N \) is a positive integer
and \( \varepsilon > 0 \), let
\[
H_{N, \varepsilon} = \{ 1 \leq n \leq N: ||n \alpha_1|| \leq \varepsilon, ||n \alpha_2|| \leq \varepsilon, \ldots, ||n \alpha_t|| \leq \varepsilon \}.
\]
The implied constants in \( \ll \) depend only on \( t \) in the following lemma.

**Lemma 1.** Let \( \varepsilon > 0 \) be small enough (depending on \( t \)). Then
\[
H_{N, \varepsilon} \subseteq \left\{ \sum_{i=1}^{R} k_i n_i: 1 \leq k_i \leq K_i \text{ for } 1 \leq i \leq R \right\}
\]
with some \( R \geq 1 \) and suitable nonzero integers \( n_i \) and positive integers \( K_i \) satisfying \( R \ll 1 \),
\[
||n_i \alpha_j|| \ll \frac{\varepsilon}{K_i} \quad (1 \leq i \leq R, 1 \leq j \leq t)
\]
and
\[
|n_i| \ll \frac{N}{K_i} \quad (1 \leq i \leq R).
\]
Consequently, for any element \( n \) of the right-hand side of (1) we have
\[
|n| \ll N \quad \text{and} \quad ||n \alpha_j|| \ll \varepsilon \quad (1 \leq j \leq t).
\]

**Remark 1.** It would be interesting to analyze the dependence of \( R \) on the dimension \( t \) of the Bohr set.

**Remark 2.** Our work is related to the papers [3,4] (see [1] for more details in this connection).

2. **Lemmas on Bohr sets**

In this section \( \alpha_1, \alpha_2, \ldots, \alpha_t \) are arbitrary real numbers, and the implied constants in \( \ll \) depend only on \( t \).

To prove Lemma 1 stated in the Introduction we need Lemma 2. If \( A \) and \( B \) are two subsets of the integers, then we write
\[
A + B = \{ a + b: a \in A, b \in B \}.
\]
Lemma 2. We have

\[ |H_{N,e} + H_{N,e}| \leq C|H_{N,e}|, \]

where \( C \) is a constant depending only on \( t \) (the dimension of the Bohr set).

Proof. It is clear that \( H_{N,e} + H_{N,e} \subseteq H_{2N,2e} \). We divide the interval \([1, 2N]\) into two parts, the interval \([-2e, 2e]\) into four parts, so the cube \([-2e, 2e]^t\) into \(4^t\) parts, and the lemma follows easily by the pigeon-hole principle. \( \square \)

Proof of Lemma 1. By Ruzsa’s version of Freiman’s theorem (see [5]; Freiman’s original work is [6]) and Lemma 2 we have

\[ H_{N,e} \subseteq \left\{ a + \sum_{i=1}^r l_i d_i; \ 1 \leq l_i \leq L_i \text{ for } 1 \leq i \leq r \right\} \]

with some \( r \geq 1 \) and suitable integers \( a \) and \( d_i \) and positive integers \( L_i \), where

\[ |H_{N,e}| \geq DL_1 L_2 \ldots L_r \]

with some \( 0 < D < 1 \). Here the numbers \( r \) and \( D \) depend only on \( C \) of Lemma 2 (so depend only on \( t \)).

Assume that \( L_i \geq \frac{3}{2D} \). Then it is clear that we can fix \( l_2, l_3, \ldots, l_r \) such that

\[ \left| \left\{ 1 \leq l_1 \leq L_1: a + \sum_{i=1}^r l_i d_i \in H_{N,e} \right\} \right| \geq DL_1 \geq 2. \]

Then there are two different numbers in this set, say \( l_1 \) and \( \lambda_1 \), with the property

\[ 0 < |l_1 - \lambda_1| < \frac{3}{2D}, \]

and since \( l_1 \) and \( \lambda_1 \) are elements of the above set, by the definition of \( H_{N,e} \) we have

\[ ||(l_1 - \lambda_1) d_j x_j|| \leq 2e \quad \text{for } 1 \leq j \leq t \]

and

\[ ||(l_1 - \lambda_1) d_1|| \leq N. \]

Applying this argument several times and taking least common multiple, we find a positive integer \( T \) such that

\[ T \leq 1, \quad ||Td_j x_j|| \leq e, \quad |Td_j| \leq N \quad (2) \]

for \( 1 \leq j \leq t \) and for every \( 1 \leq i \leq r \) satisfying \( L_i \geq \frac{3}{2D} \). We want to improve the last two inequalities in (2).
To this end we assume again that \( L_1 \geq \frac{2}{\beta} \). If we fix suitably \( l_2, l_3, \ldots, l_r \), then we can find a residue class \( \tau \pmod{T} \) such that
\[
\left\{ 1 \leq l_1 \leq L_1 : l_1 \equiv \tau \pmod{T}, a + \sum_{i=1}^{r} l_id_i \in H_{N,\epsilon} \right\} \geq L_1.
\]
Hence there is an integer \( M_1 \geq L_1 \) and a number \( E > 0 \) depending only on \( t \) with the property that for every \( 1 \leq j \leq t \), there is a real \( x_j \) and there is an integer \( n \) such that with the notations
\[
S_{1,j} = \{ 1 \leq m \leq M_1 : ||x_j + m(Td_1x_j)|| \leq \epsilon \} \quad (3)
\]
and
\[
S_2 = \{ 1 \leq m \leq M_1 : |n + m(Td_1)| \leq N \}, \quad (4)
\]
we have
\[
|S_{1,j}| \geq EM_1, \quad |S_2| \geq EM_1. \quad (5)
\]
Recall from (2) that \( ||Td_1x_j|| \leq \epsilon \). Then it follows by (3) (dividing the interval \([1, M_1]\) into intervals of length smaller than \( \frac{1}{||Td_1x_j||} \)) that
\[
|S_{1,j}| \leq (1 + M_1||Td_1x_j||) \cdot \frac{\epsilon}{||Td_1x_j||}.
\]
If \( \epsilon \) is small enough (depending on \( t \)), then using (5) and \( M_1 \geq L_1 \) we get
\[
||Td_1x_j|| \leq \frac{\epsilon}{L_1}. \quad (6)
\]
On the other hand, by (2) and (4) we have
\[
|S_2| \leq \frac{N}{|Td_1|},
\]
and so (5) gives
\[
|d_1| \leq \frac{N}{L_1}. \quad (7)
\]
We see that (6) and (7) indeed improve (2).
Summing up: if \( \epsilon \) is small enough, we can divide \( \{1, 2, \ldots, r\} \) into a disjoint union
\[
\{1, 2, \ldots, r\} = I_1 \cup I_2
\]
such that

$$L_i < \frac{2}{D}$$

for \(i \in I_1\),

$$||Td_i x_j|| \leq \frac{\varepsilon}{L_i}$$

and

$$|d_i| \leq \frac{N}{L_i}$$

for \(i \in I_2\) and \(1 \leq j \leq t\). \hfill (8)

Now, it is clear that there is a set \(H_1\) of integers satisfying \(|H_1| \leq 1\) and \(H_{N, \varepsilon} \subseteq H_1 + H_2\), where

$$H_2 = \left\{ \sum_{i \in I_2} (Td_i) l_i : 1 \leq l_i \leq \left\lfloor \frac{L_i}{T} \right\rfloor \right\}.$$

Of course, we can assume that \(H_{N, \varepsilon} \cap (h + H_2) \neq \emptyset\) for every \(h \in H_1\), and so we know

$$||hx_j|| \leq \varepsilon \quad \text{for} \quad 1 \leq j \leq t \quad \text{and} \quad |h| \leq N$$

for \(h \in H_1\), if we know (9) for \(h \in H_2\) and \(h \in H_{N, \varepsilon}\). But for \(h \in H_2\) (9) follows from (8); for \(h \in H_{N, \varepsilon}\) (9) is true by definition. The lemma follows from the above observations (as \(n_i\) we can take \(Td_i(i \in I_2)\) and each element of \(H_1\)). \(\square\)

**Lemma 3.** If \(\omega\) is a real number, \(k \geq 1\) is an integer, and

$$||\omega||, ||2\omega||, ||4\omega||, \ldots, ||2^k \omega|| \leq \frac{\delta}{10},$$

then \(||\omega|| \leq \frac{\delta}{2^k}\).

**Proof.** We use induction on \(k\). The case \(k = 1\) is clear since

$$\frac{\delta}{2} < ||\omega|| \leq \frac{\delta}{10}$$

implies \(\delta < ||2\omega||\). If \(k > 1\), then by the \(k = 1\) case we have

$$||2^{j} \omega|| \leq \frac{\delta}{2} \quad \text{for} \quad 1 \leq j \leq k - 1$$

and then the assertion for \(k - 1\) implies the assertion for \(k\). \(\square\)

**Lemma 4.** If \(H_{N, \varepsilon}\) is a Bohr set, and \(\varepsilon > 0\) is small enough (depending on \(t\)), then there is a set \(S\) consisting of positive integers with the following three properties:

(i) \(\max_{n \in S} n \leq N\),
(ii) \(\sum_{n \in S} ||nx_j|| \leq \varepsilon\) for \(1 \leq j \leq t\),
(iii) \(\max_{n \in H_{N, \varepsilon}} ||n\beta|| \leq \max_{n \in S} ||n\beta||\) for every real \(\beta\).
Proof. We use the notations of Lemma 1. We define
\[ S = \{2^l|n_i|: 1 \leq 2^l \leq K_i, 1 \leq i \leq R\}. \]

The first two required properties of \( S \) are then trivial from Lemma 1. We prove the third one. We may assume that
\[ \max_{n \in S} ||n\beta|| < \frac{1}{10}. \]

Then by Lemma 3, we have
\[ ||n_i\beta|| \leq \frac{1}{K_i} \max_{n \in S} ||n\beta|| \]
for \( 1 \leq i \leq R \), and using Lemma 1, this proves the present lemma.

3. Proof of the Theorem

It is not needed for the general proof, but we think that it is interesting to give first a construction of a suitable set in the one-dimensional case: if \( t = 1, x = x_1, \)
\[ x = [a_0; a_1, a_2, \ldots] \]
is its continued fraction expansion, and \( p_m/q_m \) is the sequence of its convergents, then
\[ A = \{2^l q_m: 1 \leq 2^l \leq a_{m+1}, m = 1, 2, \ldots\} \]
is a set satisfying the conditions listed in the Theorem. This can be easily proved using Theorem 1* of [1] and our present Lemma 3, but instead of analyzing it further, we turn to the proof of the Theorem for any \( t \geq 1 \).

In the sequel, \( 1, x_1, x_2, \ldots, x_t \) are linearly independent over the rationals. The following lemma is a simple consequence of Lemma 2.2 in [1]. For the sake of completeness, we sketch its proof here.

Lemma 5. Let \( \varepsilon > 0, T \geq 1 \) and \( \delta > 0 \), and assume that \( \varepsilon T \leq \frac{1}{4} \). Then there is a positive integer \( N \) such that if
\[ \max_{n \in H_{N,\varepsilon}} ||n\beta|| \leq T\varepsilon \quad (\ast) \]
for a real \( \beta \), then
\[ ||\beta - (K_1 x_1 + \cdots + K_t x_t)|| < \delta \]
with some integers $K_1, \ldots, K_t$ satisfying

$$|K_1| + \cdots + |K_t| \leq T. \quad (**)$$

**Proof.** By a compactness argument, it is enough to prove the following:

**Statement.** Let $\varepsilon > 0, T \geq 1$ and assume that $\varepsilon T \leq \frac{1}{4}$. Then, if (*) is true for every positive integer $N$, then

$$\beta \equiv K_1 a_1 + \cdots + K_t a_t \pmod{1}$$

with some integers $K_1, \ldots, K_t$ satisfying (**).

To prove it, we note that by the conditions, the set

$$\{(n a_1, n a_2, \ldots, n a_t, n\beta): n \in \mathbb{Z}\}$$

is not dense in $(\mathbb{R}/\mathbb{Z})^{t+1}$, so, by Kronecker’s theorem, the numbers $a_1, a_2, \ldots, a_t, \beta$ and 1 cannot be linearly independent over the rationals. Hence, there are integers $K_1, K_2, \ldots, K_{t+1}$ and a positive integer $K$ such that

$$\beta \equiv K_1 \frac{a_1}{K} + \cdots + K_t \frac{a_t}{K} + \frac{K_{t+1}}{K} \pmod{1}.$$

We first prove that $K_1/K$ is an integer. If this is not the case, then there is an integer $1 \leq R < K$ such that $\|RK_1/K\| \geq 1/3$. For that $R$ and any $\delta > 0$, we can choose a large enough $r$ such that

$$||(R/K) - r a_1|| < \delta, \quad ||r a_2||, \ldots, ||r a_t|| < \delta,$$

and then, taking $n = rK$, this gives us (if $\delta$ is small enough) that $||n a_1||, \ldots, ||n a_t|| < \varepsilon$, but $||n\beta|| > 1/4$. This contradiction shows that $K$ divides $K_1$, and similarly, $K$ divides $K_2, \ldots, K_t$.

We now prove that $K_{t+1}/K$ is also an integer. If not, then for a $1 \leq R < K$ we have $||RK_{t+1}/K|| \geq 1/3$. For any $\delta > 0$ we can choose a large enough $r$ such that with $n = R + rK$ we have $||n a_1||, \ldots, ||n a_t|| < \delta$. Then, similarly as above, for small enough $\delta$ we will have $||n a_1||, \ldots, ||n a_t|| < \varepsilon$, but $||n\beta|| > 1/4$. Hence $K$ divides $K_{t+1}$. So we can assume that $K = 1$, i.e.,

$$\beta \equiv K_1 a_1 + \cdots + K_t a_t \pmod{1}$$

and it is easy to see that our condition can be satisfied only if (**) is true. Lemma 5 is proved. \[\square\]

We now prove the theorem. Let $\delta_k$ be a strictly decreasing sequence (to be determined later) tending to 0. Then, by Lemma 5, we can choose a sequence $N_k$ of
positive integers such that \( H_{N_k,2^{-k-2}} \neq \emptyset \), and if

\[
\max_{n \in H_{N_k,2^{-k-2}}} ||n\beta|| \leq \frac{1}{2^k}
\tag{10}
\]

for a real \( \beta \), then

\[
||\beta - (K_1\alpha_1 + \cdots + K_i\alpha_i)|| < \delta_k
\tag{11}
\]

with some integers \( K_1, \ldots, K_i \) satisfying

\[
|K_1| + \cdots + |K_i| \leq 2^k.
\tag{12}
\]

By Lemma 4, for large enough \( k \), say for \( k \geq K_0 \) we can choose a set \( S_k \) for \( H_{N_k,2^{-k-2}} \) satisfying the properties listed in that lemma. Observe that by (ii) of Lemma 4, we have

\[
\lim_{k \to \infty} \left( \min_{n \in S_k} n \right) = \infty.
\tag{13}
\]

Define

\[
A = \bigcup_{k \geq K_0} S_k.
\tag{14}
\]

Assume that for a real \( \beta \) we have

\[
\lim_{n \in A, n \to \infty} ||n\beta|| = 0.
\tag{15}
\]

Then, by (13) and (14), we must have

\[
\lim_{k \to \infty} \left( \max_{n \in S_k} ||n\beta|| \right) = 0,
\]

and so by (iii) of Lemma 4, (10) is valid for large enough \( k \), if \( \beta \) satisfies (15). This implies (see (11) and (12)) that for such \( \beta \) and for every large enough \( k \), one has

\[
||\beta - (K_{1,k}\alpha_1 + \cdots + K_{i,k}\alpha_i)|| < \delta_k
\tag{16}
\]

for suitable integers satisfying

\[
|K_{1,k}| + \cdots + |K_{i,k}| \leq 2^k.
\tag{17}
\]

Using (16) for \( k \) and \( k + 1 \), and using also that \( \delta_k \) is decreasing, we find that

\[
||(K_{1,k} - K_{1,k+1})\alpha_1 + \cdots + (K_{i,k} - K_{i,k+1})\alpha_i|| < 2\delta_k.
\tag{18}
\]
If we define

\[
\delta_k = \frac{1}{2} \min_{0 < |K_1| + \cdots + |K_t| \leq 2^{k+2}} ||K_1 x_1 + \cdots + K_t x_t||,
\]

then we obtain from (18) (using (17) for \(k\) and \(k+1\)) that

\[
K_{j,k} = K_{j,k+1} \quad \text{for } 1 \leq j \leq t.
\]

This is true for every large enough \(k\), so there are integers \(K_j\) for every \(j\) such that \(K_{j,k} = K_j\) for large \(k\). Since \(\delta_k \to 0\), this easily implies \(\beta \in G\) by (16). Hence we proved that if (15) is true for \(\beta\), then \(\beta \in G\).

On the other hand, for every \(1 \leq j \leq t\), by the definition of the sets \(S_k\), by (ii) of Lemma 4 and by (14) we obtain

\[
\sum_{n \in A} ||nx_j|| \leq \sum_{k \geq K_0} \sum_{n \in S_k} ||nx_j|| \leq \sum_{k \geq K_0} 2^{-k-2} \leq 1.
\]

This proves the theorem. \(\square\)

References