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INVERSE SCATTERING AT FIXED ENERGY FOR LAYERED MEDIA

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ABSTRACT. – In this article we show that exponentially decreasing perturbations of the sound speed in a layered medium can be recovered from the scattering amplitude at fixed energy. We consider the unperturbed equation $u_{tt} = c_0^2(x_n)\Delta u$ in $\mathbb{R} \times \mathbb{R}^n$, where $n \geq 3$. The unperturbed sound speed, $c_0(x_n)$, is assumed to be bounded, strictly positive, and constant outside a bounded interval on the real axis. The perturbed sound speed, $c(x)$, satisfies $|c(x) - c_0(x_n)| < C\exp(-\delta|x|)$ for some $\delta > 0$. Our work is related to the recent results of H. Isozaki (J. Diff. Eq. **138**) on the case where c_0 takes the constant values c_+ and c_- on the positive and negative half-lines, and R. Weder on the case $c_0 = c_+$ for $x_n > h$, $c_0 = c_h$ for $0 < x_n < h$, and $c_0 = c_-$ for $x_n < 0$ (IIMAS-UNAM Preprint 70, November, 1997). © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous montrons que l'amplitude de la diffusion à énergie fixée détermine les perturbations, exponentiellement décroissantes, de la vitesse du son dans un milieu stratifié. Nous considérons l'équation des ondes non perturbée $u_{tt} = c_0^2(x_n)\Delta u$ dans $\mathbb{R} \times \mathbb{R}^n$, $n \geq 3$, où $c_0(x_n)$ est strictement positive, bornée et constante en dehors d'un intervalle compact de l'axe réel. La vitesse du son perturbée, $c(x)$, vérifie $|c(x) - c_0(x_n)| < C\exp(-\delta|x|)$, $\delta > 0$. Ce travail est relié aux résultats récents de H. Isozaki (J. Diff. Eq. **138**) qui considère le cas où c_0 a deux valeurs constantes c_+ et c_- sur $\mathbb{R}_+$ et $\mathbb{R}_-$ ainsi qu'à ceux de R. Weder qui considère le cas où $c_0 = c_+$ pour $x_n > h$, $c_0 = c_h$ pour $0 < x_n < h$ et $c_0 = c_-$ pour $x_n < 0$ (IIMAS-UNAM Preprint 70, Novembre 1997). © Elsevier, Paris

0. Introduction

In this article we prove a uniqueness theorem in inverse scattering for the wave equation in a layered medium. We consider the wave equation in $R \times R^n, n \geq 3$, with a variable sound speed, $c(x)$,

$$\partial_t^2 u = c^2(x)\Delta u$$

as a perturbation of the wave equation with a sound speed, $c_0(x_n)$, which is a function of one variable,

$$\partial_t^2 u = c_0^2(x_n)\Delta u.$$

Thus the unperturbed wave equation could be used to model wave propagation in a medium composed of uniform layers with different physical properties. When one takes the scattering amplitude at fixed energy as the observed data, simple examples, e.g. infinitesimal perturbations of a homogeneous medium, show that it is not reasonable to expect to recover more than the Fourier transform of the perturbation restricted to a ball

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from this data. Hence one needs to assume that the perturbation will be determined by this restricted Fourier transform, and a natural way to do this is to assume exponential decay of the perturbation.

The precise formulation of the problem we study here is as follows. We consider perturbations of the operator $L_0 = -c_0^2(x_n)\Delta$ in R^n, where $c_0 \in L^\infty(R)$ satisfies $c_0(s) \geq c_{min} > 0$ for all s and

$$c_0(s) = \begin{cases} c_+, & \text{for } s > s_+ > 0, \\ c_- < c_+, & \text{for } s < s_- < 0. \end{cases}$$

The perturbed operators have the form $L = -c^2(x)\Delta$, where $c \in L^\infty(R^n)$ approaches c_0 at an exponential rate, i.e. $|c(x) - c_0(x_n)| < C\exp(-\delta|x|)$. We assume that c is bounded away from zero, and without loss of generality can take $c(x) \geq c_{min}$ for $x \in R^n$. With these hypotheses we have the following

THEOREM. – *The coefficient $c_0(x_n)$ and the scattering amplitude at energy $k^2 > 0$ determine $c(x)$.*

In this setting the scattering amplitude is more complicated than in two-body potential scattering because of the presence of critical angles of reflection and guided waves. In Section 1 of this paper we define the scattering amplitude, but the proof that this scattering amplitude can be determined from the asymptotic behavior of distorted plane waves is postponed until the Appendix. The organization of the rest of the paper is as follows. In Section 2 we outline the proof, based on [ER], of the theorem above. The proof involves a sequence of integral equations which must be solved to connect to scattering amplitude with the perturbation. In Section 3 we derive the estimates on integral operators needed for this argument, and in Section 4 we discuss the analytic continuation which connects the analogue of Faddeev's scattering amplitude with the Fourier transform of the perturbation.

That the method of [ER] could be applied in layered media was first recognized by Isozaki in [I]. He proved the result that we prove here in the case

$$c_0(s) = \begin{cases} c_+, & \text{for } s > 0, \\ c_- < c_+, & \text{for } s < 0. \end{cases}$$

The omission of the layer $s_- < s < s_+$ precludes the existence of guided waves, though the other features of the problem remain the same. This was rectified in [W] and a preliminary version of this article, [GR]. In [W] Weder introduced the layer $s_- < s < s_+$ and assumed that $c_0(s)$ took the constant value c_0 there, while [GR] used the hypotheses here, but with more regularity for $c_0(x_n)$ and $c(x)$. A difference in technique in [I] and [W] and the present work is in the method of extending the results of [ER] to nonsmooth coefficients. In [ER] the coefficients were assumed to be C^{n+5}. This choice arose because the coefficients of the first order derivatives (i.e. the magnetic potential) were among the unknown coefficients. In the problem here, while the unknown coefficients are initially those of the second order derivatives, one quickly reduces the problem to one where they are the coefficients of the zero order term, and hence analogous to potentials. For potentials R. Novikov showed in [N] that exponential decay without regularity was sufficient for uniqueness in the inverse problem. To obtain this result for layered media

Isozaki exploits the existence of an explicit operator with positive commutator with his L_0 to prove estimates of Mourre type, while Weder uses generalized eigenfunction expansions and a generalized limiting absorption principle from his previous work [W1] and [W2]. Here we work directly with the Green's function for the ordinary differential operator

$$-\frac{d^2}{ds^2} - \frac{k^2}{c_0^2(s)} - \lambda$$

and prove estimates by adapting an argument of Eskin based on duality in weighted L^2 spaces.

The asymptotics of the distorted plane waves for these problems have been studied in more generality and greater detail by T. Christiansen in [C]. For the result that the asymptotics determine the scattering amplitude one does not require uniform asymptotics at the critical angles or higher order terms, and this made it possible to use the simpler results that we present in the Appendix.

We wish to thank Gregory Eskin for helpful conversations and Ricardo Weder for sending us the preprint version of [W].

1. Definition of the scattering amplitude

The distorted plane waves u of energy k^2 are the solutions of $(L - k^2)u = 0$ of the form $\Phi + v$, where Φ is a generalized eigenfunction in the spectral representation of L_0 with $(L_0 - k^2)\Phi = 0$, and v is outgoing in the sense that $v = \lim_{\epsilon\to 0_+} v_\epsilon$ with

$$v_\epsilon = (L - k^2 - i\epsilon)^{-1}((L_0 - L)\Phi).$$

Since $(L_0 - L)\Phi = k^2(1 - c^2c_0^{-2})\Phi$, we also have

$$v_\epsilon = -\left(-\Delta - \frac{k^2 + i\epsilon}{c^2}\right)^{-1} q\Phi,$$

where $q = k^2(c_0^{-2} - c^{-2})$. Under our hypotheses the limit exists (see [dBP] or [BdMM]), and so we have

$$v = -\left(-\Delta - \frac{k^2}{c^2} - i0\right)^{-1} q\Phi = -\left(-\Delta - \frac{k^2}{c_0^2} - i0\right)^{-1} (q\Phi + qv) \tag{1}$$

or

$$v = -(L_0 - k^2 - i0)^{-1}(c_0^2(q\Phi + qv)).$$

We set $g = q\Phi + qv$ and

$$h(\xi, x_n) = \int_{R^{n-1}} e^{-i\xi\cdot x'} g(x', x_n)dx'. \tag{2}$$

We can define the scattering amplitude in terms of the asymptotics of v as $|x| \to \infty$. However, in this case, compared to potential scattering, these asymptotics are complicated. For ease of exposition we will define the scattering amplitude directly in terms of the expansion of g in generalized eigenfunctions of $c_0^{-2}(L_0 - k^2) = -\Delta - k^2 c_0^{-2}$, and discuss its relation to the asymptotics of v in Appendix I.

The generalized eigenfunctions of $-\Delta - k^2 c_0^{-2}$ have the form $\Phi(x) = \Phi(x, \zeta, \alpha, \lambda)$, $\zeta \in R^{n-1}$, $\alpha = 1, 2, 3, 4$, where

$$\Phi(x, \zeta, \alpha, \lambda) = (2\pi)^{\frac{1-n}{2}} e^{i\zeta \cdot x'} \phi_\alpha(x_n, \lambda),\ x = (x', x_n)$$

and $\{\phi_\alpha(x_n, \lambda)\}$ is an eigenfunction expansion for

$$G = -\frac{d}{ds^2} - \frac{k^2}{c_0^2(s)}$$

as an operator on $L^2(R)$. To describe the ϕ_α's we introduce the notation $k_\pm = k/c_\pm$; note that $k_- > k_+$. In all cases we have:

$$-\frac{d^2\phi_\alpha}{ds^2} - \frac{k^2}{c_0^2(s)}\phi_\alpha = \lambda\phi_\alpha.$$

For $\alpha = 1$,

$$\phi_1(s, \lambda) = a_+(\lambda)\exp(is(\lambda + k_+^2)^{1/2}),$$

when $s > s_+$. For $\alpha = 2$,

$$\phi_2(s, \lambda) = a_-(\lambda)\exp(-is(\lambda + k_-^2)^{1/2}),$$

when $s < s_-$. The functions ϕ_1 and ϕ_2 are defined with positive square roots for $\lambda > -k_+^2$, and we define them to be zero for $\lambda \leq -k_+^2$. For $\alpha = 3$,

$$\phi_3(s, \lambda) = a_0(\lambda)\exp(-s(-\lambda - k_+^2)^{1/2}),$$

when $s > s_+$. The function ϕ_3 is defined with a positive square root for $-k_-^2 < \lambda < -k_+^2$, and we define it to be zero outside this interval. We define $\phi_4(s, \lambda)$ to be a normalized λ-eigenfunction of G for λ in the point spectrum $\{\lambda_1, \ldots, \lambda_N\}$ of G, and zero elsewhere. Then, following [Wi], we may choose the coefficients $a_*(\lambda)$ so that the following Parseval formula holds (the actual coefficients are given in the Appendix):

$$f(s) = \sum_{\alpha=1}^{2} \int_{-k_+^2}^{\infty} \phi_\alpha(s, \lambda) \int_R \overline{\phi_\alpha(t, \lambda)} f(t) dt d\lambda + \int_{-k_-^2}^{-k_+^2} \phi_3(s, \lambda) \int_R \overline{\phi_3(t, \lambda)} f(t) dt d\lambda$$
$$+ \sum_{j=1}^{N} \phi_4(s, \lambda_j) \int_R \overline{\phi_4(t, \lambda_j)} f(t) dt$$

for $f \in C_0^\infty(R)$. We let Ψf denote the associated spectral representation of f, i.e.

$$\Psi f(\lambda) = (\Psi_1 f(\lambda), \ldots, \Psi_4 f(\lambda)) = \int_R f(t)(\overline{\phi_1(t, \lambda)}, \overline{\phi_2(t, \lambda)}, \overline{\phi_3(t, \lambda)}, \overline{\phi_4(t, \lambda)}) dt.$$

DEFINITION. – *The scattering amplitude is defined to be* $[\Psi h(\eta, \cdot)](-|\eta|^2)$, *where* h *is defined in* (2). *In the Appendix we will show that* $[\Psi h(\eta, \cdot)](-|\eta|^2)$ *can be recovered from the asymptotics of* v.

2. Formal solution of the problem

The methods in this section are taken largely from [ER] with the adaptations for layered media introduced in [I]. We will assume that the reader has some familiarity with this approach; [R] would be an adequate introduction. For vectors in R^n and C^n we will now use the notation ζ^2 for $\zeta_1^2 + \ldots + \zeta_n^2$.

From (1) we have the analog of (10) in [ER],

$$(3)\qquad h(\xi, x_n) + (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + \eta^2 - i0)^{-1} h(\eta, \cdot)](x_n) d\eta = -\hat{q}(\xi - \zeta, x_n)\phi_\alpha(x_n, \lambda),$$

where $\hat{q}(\xi - \zeta, x_n)$ is the Fourier transform of q in the first $(n-1)$-variables. Note that, if one is completely explicit $h = h(\xi, x_n; \zeta, \lambda, \alpha)$, but for simplicity of notation we will often suppress the (ζ, λ, α)-dependence, and write $h(\xi, x_n)$. Since $(L_0 - k^2)\phi_\alpha(x_n, \lambda)e^{ix' \cdot \zeta} = 0$ only for $\lambda = -\zeta^2$, the scattering amplitude is given more explicitly by

$$\{[\Psi_\beta h(\eta, \cdot; \zeta, -\zeta^2, \alpha)](-\eta^2) : \alpha, \beta = 1, 2, 3, 4,\ (\eta, \zeta) \in R^{n-1} \times R^{n-1}\}.$$

However, it will be convenient to continue to allow λ to take arbitrary real values.

In analogy with (16) in [ER] we will define the Faddeev extension, h^*, of the scattering data by means of the equation

$$(4)\qquad h^*(\xi, x_n, \sigma) + (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + \eta^2 + 0i \ \mathrm{sgn}(\eta \cdot \nu - \sigma))^{-1} h^*(\eta, \cdot, \sigma)](x_n) d\eta = -\hat{q}(\xi - \zeta, x_n)\phi_\alpha(x_n).$$

Here $\sigma \in R$ and $\nu \in S^{n-2}$, and from here on we will suppress the ν-dependence in h^*. To see the relation between h and h^* recall that

$$\lim_{\epsilon \to 0_+} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[((G + \eta^2 + i\epsilon)^{-1} - (G + \eta^2 - i\epsilon)^{-1}) f(\eta, \cdot)](x_n) d\eta$$

$$= \lim_{\epsilon \to 0_+} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n) \left[\Psi^* \left(\frac{-2i\epsilon}{(\lambda + \eta^2)^2 + \epsilon^2} \Psi f(\eta, \cdot) \right)\right](x_n) d\eta$$

$$= -\sum_{\alpha=1}^{3} 2\pi i \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)\phi_\alpha(x_n, -\eta^2)[\Psi_\alpha f(\eta, \cdot)](-\eta^2) d\eta$$

$$- \sum_{j=1}^{N} \frac{\pi i}{\sqrt{-\lambda_j}} \int_{\eta^2 = -\lambda_j} \hat{q}(\xi - \eta, x_n)\phi_4(x_n, -\eta^2)[\Psi_4 f(\eta, \cdot)](-\eta^2) dm_j,$$

where m_j is the volume measure on the sphere $\eta^2 = -\lambda_j$. Thus, substituting

$$(5)\qquad h(\xi, x_n; \delta, \lambda, \alpha) + (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + \eta^2 - 0i)^{-1} h(\eta, \cdot; \delta, \lambda, \alpha)](x_n) d\eta$$

for $-\hat{q}(\xi-\delta,x_n)\phi_\alpha(x_n,\lambda)$ in all places where it appears in (4), and then applying the inverse of the operator acting on h in (5) we arrive at the analog of (24) in [ER],
(6)

$$h^*(\xi,x_n;\zeta,\lambda,\alpha;\sigma)+2\pi i\sum_{\beta=1}^{3}\int_{\eta\cdot\nu>\sigma}h(\xi,x_n;\eta,-\eta^2,\beta)[\Psi_\beta h^*(\eta,\cdot;\zeta,\lambda,\alpha;\sigma)](-\eta^2)d\eta$$

$$+\sum_{j=1}^{N}\frac{\pi i}{\sqrt{-\lambda_j}}\int_{\{\eta^2=-\lambda_j,\eta\cdot\nu>\sigma\}}h(\xi,x_n;\eta,-\eta^2,4)[\Psi_4 h^*(\eta,\cdot;\zeta,\lambda,\alpha;\sigma)](-\eta^2)dm_j$$

$$=h(\xi,x_n;\zeta,\lambda,\alpha).$$

Hence, applying Ψ to both sides of (6) and evaluating it at $-\xi^2$, we see, assuming that the resulting system of integral equations for $[\Psi(h^*(\xi,\cdot))](-\xi^2)$ is uniquely solvable, that

$$\{[\Psi h(\xi,\cdot;\zeta,-\zeta^2,\alpha;\sigma)](-\xi^2):\alpha=1,2,3,4,\ (\xi,\zeta)\in R^{n-1}\times R^{n-1}\}$$

determines

$$\{[\Psi h^*(\xi,\cdot;\zeta,-\zeta^2,\alpha;\sigma)](-\xi^2):\alpha=1,2,3,4,\ (\xi,\zeta)\in R^{n-1}\times R^{n-1}\}.$$

We define $h_*(\xi,x_n;\zeta,\lambda,\alpha;\sigma)=h^*(\xi+\sigma\nu,x_n;\zeta+\sigma\nu,\lambda,\alpha;\sigma)$, and then (4) becomes
(7)

$$h_*(\xi,x_n;\sigma)+(2\pi)^{1-n}\int_{R^{n-1}}\hat{q}(\xi-\eta,x_n)[(G+(\eta+\sigma\nu)^2+0i\,\mathrm{sgn}(\eta\cdot\nu))^{-1}h_*(\eta,\cdot;\sigma)](x_n)d\eta$$

$$=-\hat{q}(\xi-\zeta,x_n)\phi_\alpha(x_n,\lambda).$$

We wish to extend $h_*(\sigma)$ to $h_*(z)$ for z in a set $\mathcal{D}_\epsilon$ of the form $\mathcal{D}_\epsilon=\{z:|\mathrm{Re}\{z\}|<\epsilon,\ \mathrm{Im}\{z\}>0\}$. For $z=i\tau$ this extension can be done directly. We define $h_*(\xi,x_n;i\tau)$ as the solution of

$$(8)\quad h_*(\xi,x_n;i\tau)+(2\pi)^{1-n}\int_{R^{n-1}}\hat{q}(\xi-\eta,x_n)[(G+(\eta+i\tau\nu)^2)^{-1}h_*(\eta,\cdot;i\tau)](x_n)d\eta$$

$$=-\hat{q}(\xi-\zeta,x_n)\phi_\alpha(x_n,\lambda),$$

or, more compactly,

$$h_*(i\tau)+A(i\tau)h_*(i\tau)=\tilde{q}.$$

We will show that on a suitable Banach space $\mathcal{A}$, containing $\tilde{q}$, $A(i\tau)$ has an extension to $A(z)$, a compact operator-valued analytic function on $\mathcal{D}_\epsilon$ for ϵ sufficiently small. Moreover, $A(z)$ extends continuously to $A(\sigma)$, the operator in (7), as z goes to the real axis. Since we will also show that the norm of $A(i\tau)$ goes to zero as τ goes to infinity, the inverse of $I+A(z)$ is meromorphic on $\mathcal{D}_\epsilon$ with a continuous extension to $-\epsilon<\sigma<\epsilon$ outside a closed set of measure zero. The functions in $\mathcal{A}$ will be holomorphic in a neighborhood of $\mathrm{Im}\{\xi\}=0$, and, since h_* will inherit the analyticity of $\hat{q}(\xi-\zeta,x_n)\phi_\alpha(x_n,\lambda)$ in ζ and λ, $h_*(\xi,x_n;\zeta,\lambda,\alpha;z)$ will by analytic in (ξ,ζ,λ,z) on

$$\mathcal{S}_\epsilon=\{|\mathrm{Im}\{\xi\}|<\epsilon\}\times\{|\mathrm{Im}\{\zeta\}|<\epsilon\}\times\{\mathrm{Re}\{\lambda\}>-k_+^2,\ |\mathrm{Im}\{\lambda\}|<\epsilon\}\times\{\mathcal{D}_\epsilon\cap F^c\}$$

for a discrete set F, when ϵ is sufficiently small.

Now consider an analytic curve $(\xi(s), \zeta(s), z(s))$, defined in $s_0 < \mathrm{Re}\{s\} < \infty$, $0 < \mathrm{Im}\{s\} < \epsilon'$ with a continuous extension to $\mathrm{Im}\{s\} = 0$, such that $(\xi(s), \zeta(s), -(\zeta(s) + z(s)\nu)^2, z(s))$ lies in $\mathcal{S}_\epsilon$. Since the scattering amplitude determines $[\Psi h_*(\xi, \cdot; \zeta, -(\zeta + \sigma\nu)^2, \alpha; \sigma)](-(\xi + \sigma\nu)^2)$, as long as $(\xi(s), \zeta(s), z(s))$ is real-valued for s in an interval on the real axis, the scattering amplitude will determine

$$[\Psi h_*(\xi(s), \cdot; \zeta(s), -(\zeta(s) + z(s)\nu)^2, \alpha; z(s))](-(\xi(s) + z(s)\nu)^2).$$

If $z(s) \to i\infty$ as $s \to \infty$, then the integral term in (8) goes to zero, and we may conclude that the scattering amplitude determines the asymptotic behavior of

$$[\Psi(\hat{q}(\xi(s) - \zeta(s), \cdot)\phi_\alpha(\cdot, -(\zeta(s) + z(s)\nu)^2, \alpha; z(s))](-(\xi(s) + z(s)\nu)^2)$$

as $s \to \infty$. The choice of $(\xi(s), \zeta(s), z(s))$ that we make here is precisely the one used in [I] and [ER]. Given $p \in R^n$ with $p_n \neq 0$ and $|p| < 2k_+$, we choose $\nu \in S^{n-2}$ with $\nu \cdot p' = 0$ and $\mu \in S^{n-1}$ with $\mu' \cdot \nu = 0$, $\mu \cdot p = 0$ and $\mu_n > 0$. Then we set:

$$\xi(s) = p'/2 + s\mu', \ \ \zeta(s) = -p'/2 + s\mu', \ \text{ and } z(s) = i\sqrt{s^2 + |p|^2/4 - k_+^2}.$$

With this choice of $z(s)$ for $s >> 0$

$$\sqrt{k_+^2 - (\xi(s) + z(s)\nu)^2} = \sqrt{(p_n/2 + s\mu_n)^2} = p_n/2 + s\mu_n$$

and

$$\sqrt{k_+^2 - (\zeta(s) + z(s)\nu)^2} = \sqrt{(p_n/2 - s\mu_n)^2} = -p_n/2 + s\mu_n.$$

From the definition of $\phi_1(x_n, \lambda)$, setting $\lambda(s) = -(\zeta(s) + z(s)\nu)^2$ and $\Lambda(s) = -(\xi(s) + z(s)\nu)^2$, we have:

$$\lim_{s\to\infty} (\overline{a_+(\Lambda(s))}a_+(\lambda(s)))^{-1}[\Psi_1(\hat{q}(\xi(s) - \zeta(s), \cdot)\phi_1(\cdot, \lambda(s))](\Lambda(s)) = \mathcal{F}q(p), \tag{9}$$

where $\mathcal{F}$ denotes the Fourier transform on R^n. In deriving (9) one only needs to use $|a_+(\lambda)^{-1}\phi_1(x_n, \lambda) - \exp(ix_n\sqrt{\lambda})| \to 0$ as $\lambda \to \infty$. Thus we have shown that the scattering amplitude determines the Fourier transform of the perturbation $q(x)$ on an open set. Since exponential decay of $q(x)$ implies that its Fourier tranform is analytic, this implies that the scattering amplitude determines q.

3. Integral equations

This section is devoted to the integral equations in **2.** The ingredient in all these equations which requires further study is the resolvent $(G - \lambda)^{-1}$. While we could represent $(G - \lambda)^{-1}$ in terms of generalized eigenfunctions, as we already did in the derivation of (6), it is simpler to use the standard Green's function construction here. When $\mathrm{Im}\{\lambda\} \neq 0$, the operator $(G - \lambda)^{-1}$ is an integral operator on $L^2(R)$ with the kernel

$$g(s, t.\lambda) = \frac{\phi_+(s, \lambda)\phi_-(t, \lambda)}{W(\lambda)} \text{ for } s > t \text{ and } g(s, t, \lambda) = \frac{\phi_-(s, \lambda)\phi_+(t, \lambda)}{W(\lambda)} \text{ for } s < t,$$

where $\phi_\pm$ are the solutions of $G\phi = \lambda\phi$ satisfying

$$\phi_+(s,\lambda) = e^{is(\lambda+k_+^2)^{1/2}}, \text{ for } s > s_+, \text{ and } \phi_-(s,\lambda) = e^{-is(\lambda+k_-^2)^{1/2}}, \text{ for } s < s_-,$$

with $\text{Im}\{(\lambda + k_\pm^2)^{1/2}\}$ chosen positive; $W(\lambda)$ is the Wronskian

$$W(\lambda) = \phi_+(s,\lambda)\phi'_-(s,\lambda) - \phi'_+(s,\lambda)\phi_-(s,\lambda).$$

Note that $W(\overline{\lambda}) = \overline{W(\lambda)}$. The properties of the Green's function that we will need are summarized in the following lemmas.

LEMMA 1. – *Let* $f(s,t,\lambda) = W(\lambda)g(s,t,\lambda)$, *then*

i) f *is a bounded function on* $R \times R \times \{\text{Im}\{\lambda\} \neq 0\}$,

ii) *for* $\alpha < 1/2$ *and any* $K < \infty$ *the Hölder norm in* λ *of the restriction of* f *to* $R\times R\times\{|\lambda| < K\}\cap\{\text{Im}\{\lambda\} > 0\}$ *and its restriction to* $R\times R\times\{|\lambda| < K\}\cap\{\text{Im}\{\lambda\} < 0\}$ *satisfy* $||f(s,t,\cdot)||_\alpha < ((1 + |s| + |t|)$, *and*

iii) $|W(\lambda)| > C|\lambda|^{1/2}$ *on* $\{|\lambda| > R_0\} \cap \{\text{Im}\{\lambda\} \neq 0\}$ *for* R_0 *sufficiently large.*

Proof. – Since we assume that $c_0(s)$ is constant outside the interval $s_- < s < s_+$, we can represent ϕ_- and ϕ_+ in terms of the following bases of solutions for $(G - \lambda)y = 0$: let $(y_1^+(s,\lambda), y_2^+(s,\lambda))$ and $(y_1^-(s,\lambda), y_2^-(s,\lambda))$ be the pairs of solutions to $(G - \lambda)y = 0$ satisfying

$$\begin{pmatrix} y_1 & y_1' \\ y_2 & y_2' \end{pmatrix} = I$$

at $s = s_+$ and $s = s_-$ respectively, then

$$\phi_+(s,\lambda) = e^{is_+(\lambda+k_+^2)^{1/2}} y_1^+(s,\lambda) + i(\lambda + k_+^2)^{1/2} e^{is_+(\lambda+k_+^2)^{1/2}} y_2^+(s,\lambda), \tag{10}$$

$$\phi_-(s,\lambda) = e^{-is_-(\lambda+k_-^2)^{1/2}} y_1^-(s,\lambda) - i(\lambda + k_-^2)^{1/2} e^{-is_-(\lambda+k_-^2)^{1/2}} y_2^-(s,\lambda).$$

The functions $y_j^\pm$, $j = 1, 2$, are entire $H^2([s_-, s_+])$-valued functions of λ satisfying the following estimates for $s \in [s_-, s_+]$:

$$|y_1^\pm(s,\lambda) - \cos\sqrt{\lambda}(s - s_\pm)| < C|\lambda|^{-1/2}\exp(|\text{Im}\sqrt{\lambda}||s - s_\pm|),$$

$$\left|y_2^\pm(s,\lambda) - \frac{\sin\sqrt{\lambda}(s - s_\pm)}{\sqrt{\lambda}}\right| < C|\lambda|^{-1}\exp(|\text{Im}\sqrt{\lambda}||s - s_\pm|),$$

$$|\partial_s y_1^\pm(s,\lambda) + \sqrt{\lambda}\sin\sqrt{\lambda}(s - s_\pm)| < C\exp(|\text{Im}\sqrt{\lambda}||s - s_\pm|),$$

$$|\partial_s y_2^\pm(s,\lambda) - \cos\sqrt{\lambda}(s - s_\pm)| < C|\lambda|^{-1/2}\exp(|\text{Im}\sqrt{\lambda}||s - s_\pm|).$$

These estimates are the results of Theorem 3 in Chapt. 1 of [PT] in this setting. With these estimates the lemma follows immediately, since we can write y_j^+ explicitly in terms of $\cos(\lambda+k_-^2)^{1/2}(s-s_-)$ and $(\lambda+k_-^2)^{-1/2}\sin(\lambda+k_-^2)^{1/2}(s-s_+)$ for $s < s_-$ and y_j^- explicitly in terms of $\cos(\lambda + k_+^2)^{1/2}(s - s_+)$ and $(\lambda + k_+^2)^{-1/2}\sin(\lambda + k_+^2)^{1/2}(s - s_+)$ for $s > s_+$.

LEMMA 2. – *The Wronskian* $W(\lambda)$ *does not vanish for* $\text{Im}\{\lambda\} \neq 0$. $W(\lambda)$ *has continuous extensions to* $\text{Im}\{\lambda\} = 0$ *from the upper and lower half-planes. These extensions* *agree for* $\lambda < -k_-^2$ *and hence* $W(\lambda)$ *is analytic on* $Re\{\lambda\} < -k_-^2$, *vanishing only at* $\lambda = \lambda_j$, $j = 1, \ldots, N$, *where it has simple zeros. The extensions of* $W(\lambda)$ *to* $\mathrm{Im}\{\lambda\} = 0$ *do not vanish for* $\lambda > -k_-^2$. *At* $\lambda = -k_-^2$ *one either has* $W(-k_-^2) \neq 0$ or $\lim_{\lambda \to -k_-^2} (\lambda + k_-^2)^{-1/2} W(\lambda) = i\gamma \neq 0$, $\gamma \in R$.

Proof. – The statements in the first two sentences follow from the observation that for $\lambda \in \{\mathrm{Im}\lambda \neq 0\} \cup \{\mathrm{Re}\lambda < -k_-^2\}$ $W(\lambda) = 0$ is equivalent to λ being an eigenvalue of G. The continuity of the extensions of $W(\lambda)$ to the real axis follows from (10). For the statements in the last two sentences we appeal to [CK]. In the notation of [CK] $\phi_+(s, \lambda) = f_+(s, (\lambda + k_-^2)^{1/2})$ and $\phi_-(s, \lambda) = f_-(s, (\lambda + k_-^2)^{1/2})$, where $f_\pm$ are the Jost functions for the potential $v(s) = -k^2/c_0^2(s) + k_-^2$. With these identifications Lemma 1.2 and Prop. 2.4 of [CK] contain the desired results. Note that (10) implies that $W(\lambda) = R(\lambda, \sqrt{\lambda + k_-^2})$, where $R(\lambda, z)$ is analytic near $(\lambda, z) = (-k_-^2, 0)$.

To solve the integral equations (3), (4) and (8) in **3.** we will use the weighted L^2-spaces, $L_r^2(R^n)$, with norms

$$||f||_{2,r} = \left(\int_{R^n} |f(x)|^2 (1 + x^2)^r dx \right)^{1/2}.$$

In each case we will invert the partial Fourier transform in (2). Thus, (3) becomes

$$g(x) + q(x)[E_0 g](x) = q(x) e^{i\zeta \cdot x'} \phi_\alpha(x_n, \lambda), \tag{11}$$

where E_0 is the operator given by

$$E_0 f = \left(-\Delta - \frac{k^2}{c_0^2} - i0 \right)^{-1} f,$$

and for $f \in C_0^\infty(R^n)$ we have

$$[E_0 f](x) = (2\pi)^{1-n} \int_{R^{n-1}} \int_{R^{n-1}} e^{i\eta \cdot (x' - y')} [(G + \eta^2 - i0)^{-1} f(y', \cdot)](x_n) dy' d\eta.$$

We claim that (11) is a Fredholm equation in $L_r^2(R^n)$ for $r > r_0$. To prove this it will suffice to show that qE_0 is compact. We begin by showing that E_0 is a bounded operator from $L_r^2(R^n)$ to $L_{-r}^2(R^n)$ for $r > r_0$. Since q decays exponentially, any value of r_0 will suffice for us, and the value which we will use is far from optimal. We will bound the norm of E_0 as an operator from L_r^2 to L_{-r}^2 by duality. Thus it suffices to bound

$$\int_{R^n} \bar{f} E_0 g dx = (2\pi)^{1-n} \int_{R^{n-1} \times R} \overline{\hat{f}(\eta, x_n)} [(G + \eta^2 - i0)^{-1} \hat{g}(\eta, \cdot)](x_n) d\eta dx_n \tag{12}$$

by $C||f||_{2,r} ||g||_{2,r}$. Since G is a self-adjoint operator on $L^2(R^n)$ with spectrum

$$\{\lambda_1, \ldots, \lambda_N\} \cup [-k_-^2, \infty),$$

we have:

$$(2\pi)^{1-n} \int_R dx_n \int_{\{|\eta| > 1 + \sqrt{-\lambda_1}\}} \overline{\hat{f}(\eta, x_n)} [(G + \eta^2 - i0)^{-1} \hat{g}(\eta, \cdot)](x_n) d\eta \leq ||f||_{2,0} ||g||_{2,0}.$$

Thus we may assume that the integration in η in (12) is restricted to the ball $|\eta| < R_0 = 1 + \sqrt{-\lambda_1}$. By Lemma 2 $W(-\eta^2)$ vanishes to first order on the spheres $\{\eta^2 = -\lambda_j\}$, and we may have

$$\frac{1}{W(-\eta^2 + i0)} = \frac{r(\eta^2)}{(-\eta^2 + k_-^2 + i0)^{1/2}},$$

where $r(k_-^2) \neq 0$, but elsewhere $1/W$ is continuous. Using the notation of Lemma 1,

$$[(G + \eta^2 - i\epsilon)^{-1}\hat{g}](x_n) = \frac{1}{W(-\eta^2 + i\epsilon)} \int_R f(x_n, t, -\eta^2 + i\epsilon)\hat{g}(\eta, t)dt = \frac{h(\eta, x_n, \epsilon)}{W(-\eta^2 + i\epsilon)}.$$

From Lemma 1(ii) it follows that the Hölder norm $||h(\cdot, x_n, \epsilon)||_\alpha$, $\alpha < 1/2$, over $|\eta| < R_0$ is bounded by

$$C(1 + |x_n|) \int_R (1 + |t|) \sup_{\{|\eta| < R_0\}} (|\hat{g}(\eta, t)| + |\partial_\eta \hat{g}(\eta, t)|)dt,$$

uniformly for $\epsilon \in (0, 1]$. Since

$$|\hat{g}(\eta, t)| + |\partial_\eta \hat{g}(\eta, t)| < C\left(\int_{R^{n-1}} |g(x', t)|^2 (1 + |x'|^2)^r dx'\right)^{1/2}$$

when $r > 1 + n/2$, and the distribution

$$\mathcal{L}(\rho) = \lim_{\epsilon \to 0_+} \int_{-1}^{1} \frac{\rho(s)}{s + i\epsilon} ds$$

is bounded by the α-Hölder norm of ρ for any $\alpha > 0$, it follows that the quantity in (12) is bounded by $C||f||_{2,r}||g||_{2,r}$ for $r > 3 + n/2$. Thus by duality E_0 is bounded from L_r^2 to L_{-r}^2 in this range, and by exponential decay qE_0 is bounded from L_r^2 to $L_{r'}^2$ for all r'.

To complete the proof that qE_0 is compact on $L_r^2(R^n)$ for $r > r_0$ we note that the Sobolev estimate

$$||u||_{H^2(|x - x_0| < 1)} < C(||\Delta u||_{H^2(|x - x_0| < 2)} + ||u||_{H^0(|x - x_0| < 2)})$$

implies that $||E_0 f||_{H^2(|x - x_0| < 1)} < C||f||_{2,0}$ with C independent of x_0. Since qE_0 is bounded from L_r^2 to L_{r+1}^2, this shows that it is compact, and hence (3) is a Fredholm integral equation in $L_r^2(R^n)$ for $r > r_0$. If $(I + qE_0)h = 0$, for $h \in L_r^2(R^n)$, then $u = E_0 h$ is an outgoing solution to $(L - k^2)u = 0$. The limiting absorption theorem of [dBP] shows that $u = 0$ for any $k > 0$. Thus we have:

PROPOSITION 1. – *$I + qE_0$ is invertible on $L_r^2(R^n)$ for $r > 3 + n/2$, and equation* (3) *is solvable in the partial Fourier transform (as in* (2)*) of this space.*

The preceding arguments apply equally well to show that (4) is a Fredholm equation. Changing $(G + \eta^2 - i0)^{-1}$ to $(G + \eta^2 + 0i \operatorname{sgn}(\eta \cdot \nu - \sigma))^{-1}$ introduces a jump discontinuities across $\eta \cdot \nu = \sigma$ in $\phi_-(s, -\eta^2 - 0i \operatorname{sgn}(\eta \cdot \nu - \sigma))$ when $\eta^2 < k_-^2$, and in $\phi_+(s, -\eta^2 - 0i \operatorname{sgn}(\eta \cdot \nu - \sigma))$ when $\eta^2 < k_+^2$. Since the argument only requires that these functions be bounded when $\eta^2 < -\lambda_N - \delta$ for some $\delta > 0$, we have:

PROPOSITION 2. – *The equation* (4) *is Fredholm in the partial Fourier transform (as in* (2)*) of* $L^2_r(R^n)$ *for* $r > 3 + n/2$.

Next we need to show that the Faddeev equation, derived by evaluating (6) at $\lambda = -\zeta^2$, applying Ψ to both sides, and then evaluating Ψh and Ψh^* at $-\xi^2$, will be uniquely solvable when (4) is uniquely solvable. $[\Psi f](\lambda) = 0$ when $\lambda < \lambda_1$, and $[\Psi h(\xi, \cdot; \zeta, -\zeta^2; \alpha](-\xi^2)$ is continuous in all variables, except in the case that $W(-k_-^2) = 0$. In that case the normalizing coefficient $a_0(\lambda)$ behaves like $|\lambda + k_-^2|^{-1/4}$ near $\lambda = -k_-^2$. This follows from the explicit formulas for the normalizing coefficients in the Appendix. In all cases the Faddeev equation is Fredholm, if we pose it in the space of functions

$$S = C(\{\xi^2 \le k_+^2\})^2 \times C(\{k_+^2 \le \xi^2 \le k_-^2\}, a_0(-\xi^2)) \times \prod_{j=1}^{N} C(\{\xi^2 = -\lambda_j\}),$$

where $C(\mathcal{X}, a(x))$ denotes the space of functions continuous on the interior of $\mathcal{X}$ with the norm

$$||f|| = \sup_{\mathcal{X}} |f(x)|/a(x).$$

If $f = (f_1, f_2, f_3, f_{4,1}, \dots, f_{4,N}) \in S$ is a solution of the corresponding homogeneous equation, we have

$$(13) \qquad f_\alpha(\xi) + 2\pi i \sum_{\beta=1}^{3} \int_{\eta\cdot\nu>\sigma} [\Psi_\alpha h(\xi, \cdot; \eta, -\eta^2, \beta)](-\xi^2) f_\beta(\eta) d\eta$$

$$+ \sum_{j=1}^{N} \frac{\pi i}{\sqrt{-\lambda_j}} \int_{\{\eta^2=-\lambda_j, \eta\cdot\nu>\sigma\}} [\Psi_\alpha h(\xi, \cdot; \eta, -\eta^2, 4)](-\xi^2) f_{4,j}(\eta) dm_j = 0$$

for $\alpha = 1, 2, 3, (4,1), \dots, (4, N)$. If we evaluate Ψh at λ instead of $-\xi^2$, (13) gives an extension of f to $f(\xi, \lambda)$. Then, applying $\Psi^{(-1)}$ to the resulting equation gives a solution to the homogeneous version of (6), and we can reverse the derivation of (6) from (4) to get a solution – nontrivial since we have only applied invertible operators – to the homogeneous version of (4). Thus the Faddeev equation is uniquely solvable whenever (4) is uniquely solvable. This argument is analogous to the one given in [ER] in the paragraph following formula (26) in that article.

The critical step in the argument outlined in **2** is the proof that the norm of $A(i\tau)$ goes to zero as $\tau \to \infty$. This not only implies that (8) is solvable for τ large, but also by the analytic continuation argument it implies that (4) will be uniquely solvable for σ in an open interval on the real axis.

To bound the norm $A(i\tau)$ as an operator on $L^2_r(R^n)$ we will use the method used to prove Prop. 1. As in the proof of Prop.1, we first need to bound

$$I(f, g) = \int_{R^{n-1}\times R} \overline{\hat{f}(\eta, x_n)}[(G + \eta^2 - \tau^2 + 2i\tau\eta \cdot \nu)^{-1}\hat{g}(\eta, \cdot)](x_n) d\eta dx_n$$

in terms of the norms of f and g in L^2_r. We have:

$$|W(-(\xi + i\tau\nu)^2)| > C|(\xi + i\tau\nu)^2|^{1/2}$$

for $|(\xi+i\tau\nu)^2| > R_0$ from Lemma 1(iii), and from Lemma 2

$$|W(-(\xi+i\tau\nu)^2| > C|(\xi+i\tau\nu)^2+\lambda_j|$$

for $-(\xi+i\tau\nu)^2$ in a neighborhood of $\lambda_j,\ j=1,\dots,N$, and

$$|W(-(\xi+i\tau\nu)^2)| > C|(\xi+i\tau\nu)^2-k_-^2|^{1/2}$$

for $-(\xi+i\tau\nu)^2$ in a neighborhood of $-k_-^2$. Elsewhere W is nonzero and analytic on $C-[-k_-^2,\infty)$ with continuous limits on $[-k_-^2,\infty)$. Since

$$|(\xi+i\tau\nu)^2-b^2| = ((\xi^2-\tau^2+b^2)^2+(2\tau\xi\cdot\nu)^2)^{1/2}$$
$$\geq \frac{1}{2}(||\xi|^2-\tau^2+b^2|+2\tau|\xi\cdot\nu|) \geq \frac{\tau}{2}(||\xi|-\sqrt{\tau^2+b^2}|+|\xi\cdot\nu|),$$

it follows that $I(f,g)$ is bounded by a finite sum of terms of the form:

$$\frac{C}{\tau^{1/2}}\int_{R^{n-1}\times R}\frac{|\hat f(\eta,x_n)|(\int_R(1+s^2)|\hat g(\eta,s)|^2ds)^{1/2}}{(||\eta|-\sqrt{\tau^2+b^2}|+|\eta\cdot\nu|)^\alpha}d\eta dx_n, \tag{14}$$

where $b^2=0,k_-^2$, or $-\lambda_j,\ j=1,\dots,N$, and $\alpha=1/2$ or 1. The only complication in estimating (14) is that the sphere of codimension two where the denominator vanishes has radius $(\tau^2+b^2)^{1/2}$. Thus to bound the integral we need to have the numerator of the integrand bounded in the variables normal to the sphere and integrable in the variables tangent to the sphere. For $\tau>2$ we introduce coordinates $(t_1,t_2)=(\xi\cdot\nu,|\xi|-(\tau^2+b^2)^{1/2})$ near $|\xi|-(\tau^2+b^2)^{1/2}=\xi\cdot\nu=0$. Then the Jacobian of the transformation from ξ to $(t_1,t_2,\tau\omega')$, where ω' is the polar angular coordinate on the sphere $|\xi|-(\tau^2+b^2)^{1/2}=\xi\cdot\nu=0$, will be bounded above and below independently of τ. Thus, letting S_{t_1,t_2} be the sphere of codimension two obtained by fixing (t_1,t_1) and dA_{t_1,t_2} be the volume on it induced from Euclidean measure, (14) is bounded by

(15)

$$\frac{C_1}{\tau^{1/2}}\int_{t_1^2+t_2^2<1}dt_1dt_2\int_{S_{t_1,t_2}}\frac{(\int_R(1+s^2)|\hat f(\eta,s)|^2ds)^{1/2}(\int_R(1+s^2)|\hat g(\eta,s)|^2ds)^{1/2}}{(|t_1|+|t_2|)^\alpha}dA_{t_1,t_2}$$
$$+\frac{C_2}{\tau^{1/2}}\int_{t_1^2+t_2^2>1}\left(\int_R(1+s^2)|\hat f(\eta,s)|^2ds\right)^{1/2}\left(\int_R(1+s^2)|\hat g(\eta,s)|^2ds\right)^{1/2}d\eta.$$

The second integral in (15) is bounded by $\tau^{-1/2}||f||_{2,1}||g||_{2,1}$. To bound the first integral we observe that

$$\int_{S_{t_1,t_2}}\left(\int_R(1+s^2)|\hat f(\eta,s)|^2ds\right)^{1/2}\left(\int_R(1+s^2)|\hat g(\eta,s)|^2ds\right)^{1/2}dA_{t_1,t_2} \tag{16}$$
$$\leq\left(\int_{R\times S_{t_1,t_2}}(1+s^2)|\hat f(\eta,s)|^2dA_{t_1,t_2}ds\right)^{1/2}\left(\int_{R\times S_{t_1,t_2}}(1+s^2)|\hat g(\eta,s)|^2dA_{t_1,t_2}ds\right)^{1/2}.$$

The L^2-norm of a function h over the sphere $|\xi|=r_0,\ \xi\cdot\nu=s_0$ with $2<2s_0<r_0$ is bounded independently of r_0 and s_0 by the Sobolev norm $||h||_{H^2(R^{n-1})}$. Applying this in (16) and observing that $||\hat f(\cdot,s)||_{H^2(R^{n-1})} = ||f(\cdot,s)||_{L^2_2(R^{n-1})}$, we conclude that the first integral in (15) is bounded by $\tau^{-1/2}||f||_{2,3}||g||_{2,3}$. Thus we have $I(f,g) < C\tau^{-1/2}||f||_{2,r}||g||_{2,r}$ for $r\geq 3$. As in the proof of Prop. 1, combined with the exponential decrease of q, this gives:

PROPOSITION 3. – *The norm of $A(i\tau)$ as an operator on $L_r^2(R^n)$ is bounded by $C\tau^{-1/2}$ for τ large when $r \geq 3$.*

4. Analytic continuation

In this section we will show that $A(i\tau)$ has an analytic continuation to the domain $\mathcal{D}_\epsilon$, and complete the proof outlined in **2.** We will use the exponentially weighted analogs of the spaces $L_r^2(R^n)$

$$\mathcal{A}_\mu = \left\{ f : \int_{R^n} e^{2\mu|x|}|f(x)|^2 dx < \infty \right\},\ \mu > 0,$$

with the norms

$$||f||_\mu = \left(\int_{R^n} e^{2\mu|x|}|f(x)|^2 dx \right)^{1/2}.$$

Note that $f \in \mathcal{A}_\mu$ implies that the partial Fourier transform of f satisfies

$$\int_R |\hat{f}(\xi, x_n)|e^{\mu|x_n|/3} dx_n < C||f||_\mu \tag{17}$$

on $\{\xi \in C^{n-1} : |\mathrm{Im}\{\xi\}| \leq \mu/3\}$ and $\hat{f}$ is holomorphic in ξ on this set.

Recalling the definition of $A(i\tau)$ from (8), we have:

$$[A(i\tau)f](\xi, x_n) = (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + (\eta + i\tau\nu)^2)^{-1} f(\eta, \cdot)](x_n) d\eta.$$

We wish to define $A(\sigma + i\tau)$ for $|\sigma| < \epsilon$. To do this, in addition to the exponential decrease of q, we will need to use the following properties of G. The resolvent $(G - \lambda)^{-1}$ is analytic in λ as an operator on $L^2(R)$ for $\mathrm{Im}\{\lambda\} \neq 0$, and for $\mathrm{Re}\{\lambda\} < \lambda_1$. From (10) – using Lemma 1 (iii) and Lemma 2 to control $W(\lambda)$ – one sees that the kernel $g(t, s, \lambda)$ of the resolvent has analytic continuations across $(-k_+^2, \infty)$ from the upper and lower half-planes which satisfy

$$|g(s, t, \lambda)| < C|\lambda|^{-1/2} \exp((|\mathrm{Im}\sqrt{\lambda + k_+^2}| + |\mathrm{Im}\sqrt{\lambda + k_-^2}|)(|s| + |t|)). \tag{18}$$

We will apply these results with

$$\lambda = -(\eta + (\sigma + i\tau)\nu)^2 = -(\eta + \sigma\nu)^2 + \tau^2 - 2i\tau(\eta \cdot \nu + \sigma).$$

We write $A(i\tau) = A_1(i\tau) + A_2(i\tau)$, and discuss the analytic continations of A_1 and A_2 separately. The formula for A_1 and its analytic continuation to

$$\mathcal{D}_{\epsilon,T} = \{z = \sigma + i\tau : |\sigma| < \epsilon < 1,\ 0 < \tau < T\}$$

is

$$[A_1(z)\hat{f}](\xi, x_n) = (2\pi)^{1-n} \int_{\{|\eta| > R_T\}} \hat{q}(\xi - \eta, x_n)[(G + (\eta + z\nu)^2)^{-1} \hat{f}(\eta, \cdot)](x_n) d\eta,$$

where R_T is chosen large enough that $|\eta| > R_T - 1$ implies $-(\eta + \sigma\nu)^2 + \tau^2 < \lambda_1 - 1$ on $\mathcal{D}_{\epsilon,T}$. Since this implies $(G + (\eta + z\nu)^{-1}$ is analytic on $\mathcal{D}_{\epsilon,T}$ as an operator on $L^2(R)$ for $|\eta| > R_T$ with norm bounded by $C(1 + |\eta|^2)^{-1}$, it follows that the operator

$$[E_1(z)f](x) = \int_{\{|\eta|>R_T\}} e^{ix'\cdot\eta}[(G + (\eta + z\nu)^2)^{-1}\hat{f}(\eta, \cdot)](x_n)d\eta$$

is analytic on $\mathcal{D}_{\epsilon,T}$ as an operator on $L^2(R^n)$. Thus, since $|q(x)| < C\exp(-\delta|x|)$, $qE_1(z)$ is a compact operator-valued function on $\mathcal{A}_\mu$ for $\mu < \delta$, holomorphic on $\mathcal{D}_{\epsilon,T}$. Since $A_1(z)$ is the conjugation of $qE_1(z)$ by the partial Fourier transform, by simply identifying the space $\mathcal{A}_\mu$ with its partial Fourier transform we have:

PROPOSITION 4. – *On the domain $\mathcal{D}_{\epsilon,T}$ the function $A_1(z)$ is holomorphic with values in the compact operators on $\mathcal{A}_{\delta/3}$.*

The preceding argument applies equally well to part of A_2. If we set

$$[A_{2,1}(z)\hat{f}](\xi, x_n) = (2\pi)^{1-n} \int_{\{|\eta|<R_T, |\eta\cdot\nu|>2\epsilon\}} \hat{q}(\xi - \eta)[(G + (\lambda + z\nu)^2)^{-1}\hat{f}(\eta, \cdot)](x_n)d\eta,$$

then the analyticity of $(G - \lambda)^{-1}$ for $\text{Im}\{\lambda\} \neq 0$ allows us to re-use the preceding argument, and conclude that Prop. 4 holds for $A_{2,1}$.

The analytic continuation of the remainder of $A(i\tau)$ is a little more complicated, and makes essential use of the exponential decay of q. To continue $A_{2,2}(i\tau) = A(i\tau) - A_1(i\tau) - A_{2,1}(i\tau)$ we will deform the integration in the following way. We introduce cylindrical coordinates on R^{n-1} by defining:

$$\eta_\nu = \eta \cdot \nu, \ r = |\eta - \eta_\nu \nu|, \text{ and } \omega = r^{-1}(\eta - \eta_\nu \nu).$$

Writing the integral defining $A_{2,1}(i\tau)$ as an iterated integral over

$$\{\omega \in S^{n-2}\} \times \{|\eta_\nu| < 2\epsilon\} \times \{0 < r < (R_T^2 - \eta_\nu^2)^{1/2}\},$$

we replace the integration in r by integration over $\Gamma(\eta_\nu)$, where $\Gamma(-\eta_\nu) = \overline{\Gamma(\eta_\nu)}$ and for $\eta_\nu > 0$ the contour $\Gamma(\eta_\nu)$ is given by:

$$\{s : 0 \le s \le a\} \cup \{a + it : 0 \le t \le b\} \cup \{s + ib : a \le s \le c\}$$

$$\cup\{c + i(b - t) : 0 \le t \le b\} \cup \{s : c \le s \le (R_T^2 - \eta_\nu^2)^{1/2}\}.$$

We now need to choose a, b and c.

The choice of a and c is determined by the requirement that for r on the segments $[0, a]$ and $[c, R_T]$ and $|\eta_\nu| \le 2\epsilon$,

$$[(G + (\eta + z\nu)^2)^{-1}\hat{f}(\eta, \cdot)](x_n) = [(G + r^2 + (\eta_\nu + \sigma + i\tau)^2)^{-1}\hat{f}(\eta, \cdot)](x_n)$$

must be analytic in z on $\mathcal{D}_{\epsilon,T}$. Our choice for a is $a = k_+^2/2$. This insures that, taking ϵ sufficiently small ($\epsilon \le k_+/8$ will suffice), for $r \in [0, a]$, $|\eta_\nu| \le 2\epsilon$ and $z \in \mathcal{D}_{\epsilon,T}$ we can define $[(G + (\eta + (\sigma + i\tau)\nu)^2)^{-1}(\hat{f}(\eta, \cdot)](x_n)$ as the analytic continuation of

$([G+(\eta+i\tau))^2)^{-1}\hat{f}(\eta,\cdot)](x_n)$, across $[-k_+^2,\infty)$. The estimate (18) implies that we may do this for $f\in\mathcal{A}_{\delta/3}$ as long as

$$|\mathrm{Im}\sqrt{-(\eta+(\sigma+i\tau)\nu)^2+k_\pm^2}|\leq\delta/4. \tag{19}$$

Note that (19) will hold for ϵ sufficiently small for $r\in[0,a]$, $|\eta_\nu|\leq 2\epsilon$ and $z\in\mathcal{D}_{\epsilon,T}$, and this choice of ϵ can be made independent of T. For the segment $[c,R_T]$ we will have this analyticity as long as R_T is chosen large enough that $-r^2+\tau^2+16\epsilon^2<\lambda_1$ for $|\eta|>R_T-4\epsilon^2$. Our earlier choice of R_T suffices for this, if ϵ is sufficiently small. We choose the constant b so that the domain of integration stays within the domain of analyticity of $\hat{q}(\xi-\eta,x_n)$ in η for $|\mathrm{Im}\{\xi\}|\leq\delta/2$ and within the domain of analyticity of $\hat{f}(\eta,x_n)$ for $f\in\mathcal{A}_{\delta/3}$. In view of (17) it suffices to take $b<\delta/9$ for this.

Now we are ready to continue $A_{2,2}(i\tau)$ to $\mathcal{D}_{\epsilon,T}$ as an operator on $\mathcal{A}_{\delta/3}$. The procedure now is completely analogous to the one used in Section 2 of [ER]. We begin with the integral representation

$$[A_{2,2}(i\tau)\hat{f}](\xi,x_n)=(2\pi)^{1-n}\int_{\{|\eta|<R_T,|\eta\cdot\nu|<2\epsilon\}}\hat{q}(\xi-\eta)[(G+(\lambda+i\tau\nu)^2)^{-1}\hat{f}(\eta,\cdot)](x_n)d\eta,$$

and deform the integration in r to $\Gamma(\eta_\nu)$. This gives a new representation of $A_{2,2}(i\tau)\hat{f}$ for $\tau\in(0,T]$ by our choice of b. Next we define $A_{2,2}(\sigma+i\tau)\hat{f}$, using this representation with our convention for the analytic continuation of $(G+\lambda)^{-1}$ across $[-k_+^2,\infty)$. This gives an analytic continuation provided:

i) $\mathrm{Re}\{-r^2-(\eta_\nu+\sigma+i\tau)^2\}>-k_+^2$ and $|\mathrm{Im}\{(-r^2-(\eta_\nu+\sigma+i\tau)^2+k_\pm^2)^{1/2}\}|<\delta/4$, when $0\leq\tau\leq T$, $|\eta_\nu|\leq 2\epsilon$, $|\sigma|\leq\epsilon$, and $r\in[0,k_+^2/2]$ or $r=\mathrm{sgn}(\eta_\nu)(k_+^2/2+i(b-t)$, where $0\leq t\leq b$,

ii) $\mathrm{Im}\{r^2+(\eta_\nu+\sigma+i\tau)^2\}\neq 0$, when $\tau\geq 0$, $|\eta_\nu|\leq 2\epsilon$, $|\sigma|\leq\epsilon$, and $r=\mathrm{sgn}(\eta_\nu)(s+ib)$, where $k_+^2/2\leq s\leq c$,

iii) $\mathrm{Re}\{-r^2-(\eta_\nu+\sigma+i\tau)^2\}<\lambda_1$, when $0\leq\tau\leq T$, $|\eta_\nu|\leq 2\epsilon$, $|\sigma|\leq\epsilon$, and $r=\mathrm{sgn}(\eta_\nu)(c+i(b-t)$, where $0\leq t\leq b$ or $r\in[c,R_T]$.

Our choices of a, b and c guarantee that all of these properties hold except for i) and iii) on the short segments $r=k_+^2/2+i\mathrm{sgn}(\eta_\nu)t$, where $0\leq t\leq b$, and $r=c+i\mathrm{sgn}(\eta_\nu)(b-t)$, where $0\leq t\leq b$, respectively. To correct this we just choose b smaller and then take ϵ small enough that ii) holds again.

This completes the analytic continuation of $A_{2,2}$, and, since the proof from [ER] applies to show that $A_{2,2}(z)$ is compact, we conclude that Prop. 4 holds for $A_{2,2}(z)$, and hence for $A(z)$. If $I+A(i\tau)$ had a nontrivial null space in $\mathcal{A}_{\delta/3}$, it would have a nontrivial null space in $L_r^2(R^n)$, contradicting Prop. 3 for τ sufficiently large. Thus $(I+A(z))^{-1}$ is meromorphic on $\mathcal{D}_{\epsilon,T}$. It is also clear, as in [ER], from the contruction of the analytic continuation that $A(z)$ extends continuously to the operator $A(\sigma)$ in (7) as $\tau\to 0_+$. Thus we have justified the analytic continuation argument in **2.**

Appendix

In this Appendix we show that the scattering amplitude defined in **1.** can be recovered from the asymptotics of the scattered wave v. Because of the form of the leading terms in these asymptotic expansions, it will always suffice to compute asymptotics modulo functions in $L^2(R^n)$. Our starting point is the representation of the scattered wave v in terms of the "free" Green's function. This is given in (1), but in view of (3) and (11) we also have $v = E_0\tilde{h}$, where $\tilde{h}$ is the inverse Fourier transform in η of $h(\eta, x_n)$. Prop. 1 shows that $\tilde{h}$ is in $L^2_r(R^n)$, and from the equation

$$\tilde{h} + qE_0\tilde{h} = -q\Phi,$$

one sees that $\tilde{h}$ belongs to $\mathcal{A}_\mu$ for any $\mu < \delta$. Let $\chi \in C_0^\infty(R^{n-1})$ satisfy $\chi(\eta) = 1$ for $|\eta| < 1 + \sqrt{-\lambda_1}$, and also let χ denote the operator multiplying the Fourier transform in x' by χ. Since the norm of $(1 - \chi(\eta))(G + \eta^2)^{-1}$ as an operator from $L^2(R)$ to $L^2(R)$ is bounded uniformly in η, one sees from (11) that $E_0(I - \chi)$ is bounded from $L^2(R^n)$ to $L^2(R^n)$. Thus $E_0(I - \chi)\tilde{h} \in L^2(R^n)$. Likewise, let $\beta \in C^\infty(R)$ satisfy $\beta(\lambda) = 1$ for $\lambda < 1$ and $\beta(\lambda) = 0$ for $\lambda > 2$, and also let β denote the operator on $L^2(R)$ obtained by multiplying the spectral representation, Ψf by β. One sees directly that $E_0\chi(1 - \beta)$ is bounded from $L^2(R^n)$ to $L^2(R^n)$, and thus $E_0\chi(1 - \beta)\tilde{h} \in L^2(R^n)$.

For the remainder of v, i.e. $E_0\chi\beta\tilde{h}$, we have the representation:

$$(A.1) \qquad [E_0\chi\beta\tilde{h}](x) = (2\pi)^{1-n} \int_{R^{n-1}} e^{ix'\cdot\eta}\chi(\eta)[(G + \eta^2 - i0)^{-1}\beta h(\eta, \cdot)](x_n)d\eta.$$

We can express $\chi(\eta)(G + \eta^2 - i0)^{-1}\beta h(\eta, \cdot)$ in terms of the spectral representation Ψ:
$(A.2)$

$$(2\pi)^{n-1}[E_0\chi\beta\tilde{h}](x) = \sum_{\alpha=1}^{2} \int_{R^{n-1}} \int_{-k_+^2}^{\infty} e^{ix'\cdot\eta}\chi(\eta)\beta(\lambda)\frac{\phi_\alpha(x_n, \lambda)[\Psi_\alpha h(\eta, \cdot)](\lambda)}{\lambda + \eta^2 - i0}d\lambda d\eta +$$

$$\int_{R^{n-1}} \int_{-k_-^2}^{-k_+^2} e^{ix'\cdot\eta}\chi(\eta)\frac{\phi_3(x_n, \lambda)[\Psi_3 h(\eta, \cdot)](\lambda)}{\lambda + \eta^2 - i0}d\lambda d\eta +$$

$$\sum_{j=1}^{N} \phi_4(x_n, \lambda_j) \int_{R^{n-1}} e^{ix'\cdot\eta}\frac{\chi(\eta)[\Psi_4 h(\eta, \cdot)](\lambda_j)}{\lambda_j + \eta^2 - i0}d\eta.$$

We index the integrals in (A.2) by the components of Ψ that they contain and write

$$[E_0\chi\tilde{h}](x) = I_1(x) + I_2(x) + I_3(x) + I_{4,1}(x) + \ldots + I_{4,N}(x).$$

In what follows we will usually suppress the cutoffs χ and β to simplify the notation, and ask the reader to remember that all integrands can be assumed to have compact support in η and λ.

To compute the asymptotics of the integrals $I_j(x)$ as $|x| \to \infty$ we will use the following lemma.

LEMMA A.1. – *Assume that w and g are real-valued smooth functions with the properties:*

i) $\nabla g \neq 0$ *on the level set* $\{g = 0\}$,

ii) *the critical points of* w_0, *the restriction of* w *to* $\{g = 0\}$, *are nondegenerate, and*

iii) $\nabla g \cdot \nabla w \neq 0$ *at each of these critical points.*

Then, for $f \in C_0^\infty(R^m)$, *the function*

$$I(r) = (2\pi)^{-m} \int_{R^m} \frac{e^{irw(\xi)} f(\xi)}{g(\xi) - i0} d\xi$$

satisfies

$$I(r\theta) = \sum_{p \in S_+} C_p \frac{e^{iw(p)r}}{r^{(m-1)/2}} (f(p) + O(r^{-1}))$$

as $r \to \infty$. *The set* $S_+ = \{p \in \{g = 0\} : p$ *is a critical point for* w_0, *and* $\nabla w(p) \cdot \nabla g(p) > 0\}$. *The coefficient* C_p *is determined by the gradients of* w *and* g *at* p *and the Hessian of* w_0. *In the case that interests us most* $g(\xi) = \xi^2 - k^2$, $w(\xi) = \xi \cdot \theta$, $\theta \in S^{m-1}$, S_+ *is the point* $p = k\theta$ *and* $C_p = C_m(k) = (4\pi)^{-1} k^{(m-3)/2} (2\pi)^{(3-m)/2} exp(-i\pi(m-3)/4)$.

This lemma is well-known; it is contained in Theorem 8.3 of [AH], if one takes w as a coordinate function near $\{g = 0\}$.

This lemma applies directly with $m = n - 1$ to the integrals $I_{4,j}$, $j = 1, \ldots, N$, yielding ($r = |x'|, \theta = x'/|x'|$)

(A.3)

$$I_{4,j}(r\theta, x_n) = C_{n-1}((-\lambda_j)^{1/2}) \frac{e^{ir(-\lambda_j)^{1/2}}}{r^{(n-2)/2}} (\phi_4(x_n, \lambda_j)[\Psi_4 h((-\lambda_j)^{1/2}\theta, \cdot)](\lambda_j) + O(r^{-1})).$$

Since the λ_j and the eigenfunctions $\phi_j(x_n, \lambda_j)$ are assumed to be known, and the λ_j are distinct, one can determine $[\Psi_4 h((-\lambda_j)^{1/2}\theta, \cdot)](\lambda_j)$, $j = 1, \ldots, N$, $\theta \in S^{n-2}$ from the asymptotics of $I_{4,1}(r\theta, x_n) + \ldots + I_{4,N}(r\theta, x_n)$ as $r \to \infty$.

We cannot apply Lemma A.1 directly to $I_\alpha(x)$, $\alpha = 1, 2, 3$, because the functions $\phi(x_n, \lambda)[\Psi h(\eta, \cdot)](\lambda)$ have singularities at $\lambda = -k_\pm^2$. To make these singularities explicit we need to know the normalizing coefficients a_+, a_- and a_0. These may be computed as in [Wi], beginning with the identity

$$||P(E)f||^2 = \lim_{\epsilon \to 0_+} \frac{1}{2\pi i} \int_E (f, (G - (\lambda + i\epsilon)I)^{-1} f) - (f, G - (\lambda + i\epsilon)I)^{-1} f) d\lambda,$$

where $P(E)$ is the spectral family for G, and then using the explicit formula for the kernel of $(G - \lambda)^{-1}$. The results are:

$$a_+(\lambda) = \frac{(\lambda + k_-^2)^{1/4}}{\sqrt{\pi}|W(\lambda)|}, \ \lambda \in [-k_+^2, \infty),$$

(A.4)
$$a_-(\lambda) = \frac{(\lambda + k_+^2)^{1/4}}{\sqrt{\pi}|W(\lambda)|}, \ \lambda \in [-k_+^2, \infty),$$

$$a_0(\lambda) = \frac{(\lambda + k_-^2)1/4}{\sqrt{\pi}|W(\lambda)|}, \ \lambda \in [-k_-^2, -k_+^2].$$

Note that, since $W(\overline{\lambda}) = \overline{W(\lambda)}$, $|W(\lambda)|$ is well-defined for λ real. Now, since the analogue of (10) holds for $\phi_\alpha(s,\lambda)$, we see that for $\alpha = 1,2$, near $\lambda = -k_+^2$

$$(A.5) \qquad \phi_\alpha(x_n,\lambda)[\Psi_\alpha h(\eta,\cdot)](\lambda) = R_\alpha(x_n,\eta,\lambda,(\lambda+k_+^2)^{1/2}),$$

where $R_1(x_n,\eta,\lambda,\beta)$ and $R_2(x_n,\eta,\lambda,\beta)$ are restrictions of smooth functions to $[-k_+^2,\infty)$. This follows from Lemma 2, since only the squares of the coefficients $a_\pm$ enter in the formulas, and from the rapid decay of $h(\eta,x_n)$ in x_n. For $\alpha = 3$ the situation is a little more complicated, since now both $\lambda = -k_+^2$ and $\lambda = -k_-^2$ are in the support. Lemma 2 implies that near $\lambda = -k_-^2$ we either have

$$(a_0(\lambda))^2 = f(\lambda,(\lambda+k_-^2)^{1/2}),$$

where $f(\lambda,\tilde\beta)$ is a smooth function with $f(-k_-^2,0) = 0$ and $\partial f/\partial\tilde\beta(-k_-^2,0) \neq 0$, or $(a_0(\lambda))^2$ is the reciprocal of a function of this form. Thus we either have:

$$(A.6) \qquad \phi_3(x_n,\lambda)[\Psi_3 h(\eta,\cdot)](\lambda) = R(x_n,\eta,\lambda,(\lambda+k_-^2)^{1/2})$$

near $\lambda = -k_-^2$ with R smooth, or

$$(A.6') \qquad \phi_3(x_n,\lambda)[\Psi_3 h(\eta,\cdot)](\lambda) = (\lambda+k_-^2)^{-1/2} R(x_n,\eta,\lambda,(\lambda+k_-^2)^{1/2}).$$

However, $\phi_3(x_n,\lambda)[\Psi_3 h(\eta,\cdot)](\lambda)$ simply has the form given in (A.5) near $\lambda = k_+^2$. This is the information on the singularities that we will need.

To study $I_\alpha(x)$, $\alpha = 1,2$ and I_3 when $x_n > 0$, we first make the change of variables $\lambda = -k_+^2 + \beta^2$ in $I_1(x)$ and $I_2(x)$, and $\lambda = -k_+^2 - \tilde\beta^2$ in $I_3(x)$. Then for $\alpha = 1,2$,

$$I_\alpha(x) = \int_{R^{n-1}} \int_0^\infty e^{ix'\cdot\eta} \frac{\phi_\alpha(x_n,-k_+^2+\beta^2)[\Psi_\alpha h(\eta,\cdot)](-k_+^2+\beta^2)}{\eta^2+\beta^2-k_+^2-i0} 2\beta d\beta d\eta$$

and

$$I_3(x) = \int_{R^{n-1}} \int_0^{(k_-^2-k_+^2)^{1/2}} e^{ix'\cdot\eta} \frac{\phi_3(x_n,-k_+^2-\tilde\beta^2)[\Psi_3 h(\eta,\cdot)](-k_+^2-\tilde\beta^2)}{\eta^2-\tilde\beta^2-k_+^2-i0} 2\tilde\beta d\tilde\beta d\eta.$$

From (A.5) we see that the numerators of the integrands of $I_1(x)$ and $I_2(x)$ are smooth, and the numerator of $I_3(x)$ is smooth near $\tilde\beta = 0$.

We will only need the asymptotics of $I_\alpha(r\theta)$, $\alpha = 1,2,3$, as $r \to \infty$ for $\theta \in S^{n-1}$, $\theta_n \neq 0, -(1-(k_+/k_-)^2)^{1/2}$. The results will be uniform for θ in compact subsets of this set. Getting uniform asymptotics on all of S^{n-1} would be much more difficult, cf. [C]. For $\theta_n > 0$ the functions $\phi_1(r\theta_n,\beta^2-k_+^2)$ and $\phi_2(r\theta_n,\beta^2-k_+^2)$ become linear combinations of $\exp(ir\theta_n\beta)$ and $\exp(-ir\theta_n\beta)$ for r sufficiently large. The function $\phi_3(r\theta_n,-k_-^2+\tilde\beta^2)$ becomes a multiple of $\exp(-r\theta_n\tilde\beta)$. The exponential decrease of this function implies that the only significant contributions to $I_3(r\theta)$ when $\theta_n > 0$ come from a neighborhood of $\tilde\beta = 0$, and we will see that $I_3(r\theta)$ will not contribute to the leading asymptotics of $v(r\theta)$ for $\theta_n > 0$. For $\theta_n < 0$ and r sufficiently large the functions $\phi_1(r\theta_n,\beta^2-k_+^2)$ and $\phi_2(r\theta_n,\beta^2-k_+^2)$ are linear combinations of

$\exp(ir\theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2})$ and $\exp(-ir\theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2})$. Thus for r sufficiently large $I_1(r\theta)$ and $I_2(r\theta)$ are sums of integrals of the form:

$$(A.7) \qquad I(r\theta) = \int_{R^{n-1}} \int_0^\infty e^{irw(\theta,\eta,\beta)} \frac{f(-k_+^2 + \beta^2)[\Psi_\alpha h(\eta,\cdot)](-k_+^2 + \beta^2)}{\eta^2 + \beta^2 - k_+^2 - i0} 2\beta d\beta d\eta,$$

where $w(\theta, \eta, \beta) = \theta' \cdot \eta \pm \theta_n \beta$ when $\theta_n > 0$, and $w(\theta, \eta, \beta) = \theta' \cdot \eta \pm \theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2}$ when $\theta_n < 0$.

To study $I_3(x)$ when $x_n < 0$ we need to remove the singularities in the integrand. In order to do this we split $I_3(x)$ into two integrals using a cutoff $\rho(\lambda)$ which vanishes near $\lambda = -k_-^2$, and is chosen so that $1 - \rho(\lambda)$ vanishes near $\lambda = -k_+^2$. In the integral containing ρ we make the change of variables $\lambda = -k_+^2 - \tilde{\beta}^2$ and integral containing $1 - \rho$ we make the change of variables $\lambda = -k_-^2 + \hat{\beta}^2$. Then, for r sufficiently large and $\theta_n < 0$, $I_3(r\theta)$ is a sum of integrals either of the form

$$(A.8) \qquad I(r\theta) = \int_{R^{n-1}} \int_0^\infty e^{irw(\theta,\eta,\tilde{\beta})} \frac{f(-k_+^2 - \tilde{\beta}^2)[\Psi_\alpha h(\eta,\cdot)](-k_+^2 - \tilde{\beta}^2)}{\eta^2 - \tilde{\beta}^2 - k_+^2 - i0} 2\tilde{\beta} d\tilde{\beta} d\eta$$

with $w(\theta, \eta, \tilde{\beta}) = \theta' \cdot \eta \pm \theta_n(-\beta^2 + k_-^2 - k_+^2)^{1/2}$, or of the form

$$(A.9) \qquad I(r\theta) = \int_{R^{n-1}} \int_0^\infty e^{irw(\theta,\eta,\hat{\beta})} \frac{f(-k_-^2 + \hat{\beta}^2)[\Psi_\alpha h(\eta,\cdot)](-k_-^2 + \hat{\beta}^2)}{\eta^2 + \hat{\beta}^2 - k_-^2 - i0} 2\hat{\beta} d\hat{\beta} d\eta$$

with $w(\theta, \eta, \beta) = \theta' \cdot \eta \pm \theta_n \hat{\beta}$. Note that the factor of $\hat{\beta}$ in (A.9) cancels the singularity from $(A.6')$.

Now all the integrands that we have to consider (except for $I_3(r\theta)$ when $\theta_n > 0$) have the form required for Lemma A.1. However, we still need to show that the endpoints at $\beta = 0$, $\tilde{\beta} = 0$ and $\hat{\beta} = 0$ do not contribute to the leading term in the asymptotics. In (A.7) and (A.9) the level sets of the denominators of the integrands are spheres. Since we are assuming that $\theta_n \neq 0, -(1 - (k_+/k_-)^2)^{1/2}$, the critical point of w on the sphere where the denominator vanishes is not at $\beta = 0$ in (A.7) or $\hat{\beta} = 0$ in (A.9). Introducing spherical coordinates (s, ω) in (A.7) and (A.9) with $\beta = \hat{\beta} = s\omega_n$, one sees that $\partial w/\partial\omega_n \neq 0$ on $(s, \omega) = (k_+, \omega', 0)$ in (A.7) and on $(s, \omega) = (k_-, \omega', 0)$ in (A.9). We can cut the integrands off to small neighborhoods of $s = k_+$ in (A.7) and $s = k_-$ in (A.9), and the discarded portions will be Fourier transforms of functions in $L^2(R^n)$, and hence will not contribute to the leading asymptotics. Next we write the remaining integrals as sums of integrals with integrands supported in the interior of $\{\omega_n > 0\}$ so that Lemma A.1 applies to them, and integrals with integrands with supports so close to $s = k_\pm$, $\omega_n = 0$ that $\partial w/\partial\omega_n \neq 0$ on them. In these integrals we can write $\exp(irw) = (ir\partial w/\partial\omega_n)^{-1}\partial/\partial\omega_n \exp(irw)$ and integrate by parts in ω_n. Each integration by parts gives an additional factor of r^{-1} and a boundary term which is an integral over $\omega_n = 0$. Lemma A.1 with $m = n - 1$ can be applied to the boundary terms – we can repeat the integration by parts as often as needed – because integration by parts in ω_n does not change the denominators. Thus the contributions from the the boundaries $\beta = 0$ and $\hat{\beta} = 0$ will be $O(r^{-1+(2-n)/2}) = O(r^{-n/2})$, and will not contribute to the leading term in the asymptotics.

The argument of the preceding paragraph applies to the integrals (A.8) as well. The only change is that the level sets of the denominators are not spheres. However, again since we assume that $\theta_n \neq -(1-(k_+/k_-)^2)^{1/2}$, we can choose a coordinate $\tilde{\omega}_n$ near $\eta^2 = k_+^2$, $\beta = 0$ transverse to $\tilde{\beta} = 0$ by parts such that $\eta^2 - \beta^2$ is independent of $\tilde{\omega}_n$. Then we can repeat the argument of the preceding paragraph using integration by parts in $\tilde{\omega}$. This shows that the contribution from the boundary $\tilde{\beta} = 0$ will not contribute to the leading term in the asymptotics. The same reasoning shows that the contribution of $I_3(r\theta)$ to the asymptotics when $\theta_n > 0$ is negligible as well.

Applying Lemma A.1 to (A.7), and absorbing the terms from the boundary $\beta = 0$ in the $O(r^{-n/2})$ correction, we have from (A.7) with $\theta_n > 0$ modulo terms in $L^2(R^n)$:
$(A.10)$
$$I(r\theta) = C_n(k_+)\frac{e^{ik_+r}}{r^{(n-1)/2}}(f(-k_+^2(1-\theta_n^2))[\Psi_\alpha h(k_+\theta', \cdot)](-k_+^2(1-\theta_n^2)) + O(r^{-1/2})),$$

when $w(\theta,\eta,\beta) = \theta' \cdot \eta + \theta_n\beta$, and $I(r\theta) = O(r^{-n/2})$, when $w(\theta,\eta,\beta) = \theta' \cdot \eta - \theta_n\beta$, because there are no critical points in the interior in this case. When $\theta_n < 0$, the contribution from the integral in (A.7) modulo terms in $L^2(R^n)$ is:
$(A.11)$
$$I(r\theta) = C_n(k_-)\frac{e^{ik_-r}}{r^{(n-1)/2}}(f(-k_-^2(1-\theta_n^2))[\Psi_\alpha(k_-\theta', \cdot)](-k_-^2(1-\theta_n^2)) + O(r^{-1/2})),$$

when $w(\theta,\eta,\beta) = \theta' \cdot \eta - \theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2}$, and $I(r\theta) = O(r^{-n/2})$, when $w(\theta,\eta,\beta) = \theta' \cdot \eta + \theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2}$. Likewise applying Lemma A.1 to (A.9) and absorbing the contributions from the boundary $\hat{\beta} = 0$ in $O(r^{n/2})$, we have modulo terms in $L^2(R^n)$:
$(A.12)$
$$I(r\theta) = C_n(k_-)\frac{e^{irk_-}}{r^{(n-1)/2}}(f(-k_-^2(1-\theta_n^2))[\Psi_3 h(k_-\theta', \cdot)](-k_-^2(1-\theta_n^2)) + O(r^{-1/2})),$$

when $w(\theta,\eta,\beta) = \theta' \cdot \eta - \theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2}$, and $I(r\theta) = O(r^{-n/2})$ when $w(\theta,\eta,\beta) = \theta' \cdot \eta + \theta_n(\beta^2 + k_-^2 - k_+^2)^{1/2}$. Finally, since $\theta_n \neq -(1-(k_+/k_-)^2)^{1/2}$, we can assume that the support of $1-\rho$ was chosen sufficiently small that there are no critical points of w_0 in the support of the integrand in (A.8), and thus the contribution from (A.8) is $O(r^{-n/2})$.

Using (A.10-12) we can now give the leading asymptotics of $[E_0\chi\beta\tilde{h}](r\theta)$ for $\theta_n \neq 0, -(1-(k_+/k_-)^2)^{1/2}$. Note that the terms $I_{4,j}(r\theta)$ are exponentially decreasing in this case, and do not contribute. To specify the functions "$f(-k_\pm^2(1-\theta_n^2)^{1/2})$" which appear, we introduce the following notation. For $s < s_-$

$$\phi_1(s,\lambda) = a_1(\lambda)\exp(is(\lambda + k_-^2)^{1/2}) + b_1(\lambda)\exp(-is(\lambda + k_-^2)^{1/2}),$$

for $s > s_+$

$$\phi_2(s,\lambda) = a_2(\lambda)\exp(is(\lambda + k_+^2)^{1/2}) + b_2(\lambda)\exp(-is(\lambda + k_+^2)^{1/2})$$

and for $s < s_-$

$$\phi_3(s,\lambda) = a_3(\lambda)\exp(is(\lambda + k_-^2)^{1/2}) + \overline{a_3(\lambda)}(\lambda)\exp(-is(\lambda + k_-^2)^{1/2}).$$

Then for $\theta_n > 0$ we have:
$(A.13)$

$$[E_0\chi\beta\tilde{h}](r\theta) = C_n(k_+)\frac{e^{ik_+r}}{r^{(n-1)/2}}(a_+(-k_+^2(1-\theta_n^2))[\Psi_1 h(k_+\theta',\cdot)](-k_+^2(1-\theta_n^2))+ a_2(-k_+^2(1-\theta_n^2))[\Psi_2 h(k_+\theta',\cdot)](-k_+^2(1-\theta_n^2)) + O(r^{-1/2})),$$

and for $\theta_n < -(1-(k_+/k_-)^2)^{1/2}$
$(A.14)$

$$[E_0\chi\beta\tilde{h}](r\theta) = C_n(k_-)\frac{e^{ik_-r}}{r^{(n-1)/2}}(b_1(-k_-^2(1-\theta_n^2))[\Psi_1 h(k_-\theta',\cdot)](-k_-^2(1-\theta_n^2))+ a_-(-k_-^2(1-\theta_n^2))[\Psi_2 h(k_-\theta',\cdot)](-k_-^2(1-\theta_n^2)) + O(r^{-1/2})).$$

For the remaining interval, $-(1-(k_+/k_-)^2)^{1/2} < \theta_n < 0$, we have
$(A.15)$

$$[E_0\chi\beta\tilde{h}](r\theta) = C_n(k_-)\frac{e^{ik_-r}}{r^{(n-1)/2}}(\overline{a_3(-k_-^2(1-\theta_n^2))}[\Psi_3 h(k_-\theta',\cdot)](-k_-^2(1-\theta_n^2)) + O(r^{-1/2})).$$

Since $a_3(\lambda) \neq 0$ for $-k_-^2 < \lambda < -k_+^2$, we conclude from (A.15) that $\Psi_3 h(\xi,\cdot)](-\xi^2)$ is determined by the asymptotics of v. To reach this conclusion for $[\Psi_\alpha h(\xi,\cdot)](-\xi^2)$, $\alpha = 1, 2$, using (A.13) and (A.14), it now suffices to know that the determinant of the matrix

$$\begin{pmatrix} a_+(\lambda) & a_2(\lambda) \\ b_1(\lambda) & a_-(\lambda) \end{pmatrix}$$

does not vanish for $\lambda > -k_+^2$. Computing the Wronskian of $\phi_1(s,\lambda)$ and $\overline{\phi_2(s,\lambda)}$ for $s < s_-$ and for $s > s_+$, and equating the results (as in Lemma 1.1 (iii) of [CK]), we have for $\lambda > -k_+^2$

$$-a_+(\lambda)\overline{a_2(\lambda)}(\lambda + k_+^2)^{1/2} = a_-(\lambda)b_1(\lambda)(\lambda + k_-^2)^{1/2}.$$

Thus the determinant is strictly positive for $\lambda > -k_+^2$.

All that we still need to do to complete this argument is show that $I_1(x) + I_2(x) + I_3(x)$ decays sufficiently rapidly as $|x'| \to \infty$ that $\Psi_4 h(\xi,\cdot)](-\xi^2)$ will be determined by the asymptotics given in (A.3). For this it suffices to show that $I_1(x) + I_2(x) + I_3(x)$ is a sum of terms which are square-integrable over the slab $\{0 < x_n < 1\}$ and terms which are $o(|x'|^{(2-n)/2})$. Using smooth cutoffs in λ, we can write the integrals in (A.2) as sums of integrals over $\lambda < -\epsilon$ and $-2\epsilon < \lambda < 2$ with ϵ chosen small enough that $\phi_\alpha(x_n,\lambda)[\Psi_\alpha h(\eta,\cdot)](\lambda)$, $\alpha = 1, 2$, is a smooth function of λ on $\lambda > -2\epsilon$. Applying Lemma A.1 with $m = n - 1$ to the integration in η in the integrals over $\lambda \leq -\epsilon < 0$, for $\alpha = 1, 2, 3$, we get a sum of terms of the form

$$r^{(2-n)/2}\int_{-k_\pm^2}^{-\epsilon} e^{i(-\lambda)^{1/2}r}C_{n-1}((-\lambda)^{1/2}\phi_\alpha(x_n,\lambda)[\Psi_\alpha h((-\lambda)^{1/2})\theta',\cdot)](\lambda)d\lambda,$$

plus lower order terms. These are $o(r^{(2-n)/2})$ by the Riemann-Lebesgue lemma. Finally, in the integrals over $\lambda > -2\epsilon$ we may integrate by parts in λ reduce the singularity from $(\eta^2 + \lambda - i0)^{-1}$ to $\log(\eta^2 + \lambda + i0)$. Thus these integrals also give terms which are square-integrable over $0 < x_n < 1$, and the proof is complete.

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