Lipschitz and piecewise-$C^1$ regularity for scalar minimizers of affine simple integrals

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Abstract

Lipschitz, piecewise-$C^1$ and piecewise affine regularity is proved for AC minimizers of the “affine” integral $\int_a^b \{\rho(x)h(x') + \phi(x)\} \, dt$, under general hypotheses on $\rho : \mathbb{R} \to [1, +\infty)$, $\phi : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to [0, +\infty]$ with superlinear growth at infinity.

The hypotheses assumed to obtain Lipschitz continuity of minimizers are unusual: $\rho(\cdot)$ and $\phi(\cdot)$ are lsc and may be both locally unbounded (e.g., not in $L^1_{\text{loc}}$), provided their quotient $\phi/\rho(\cdot)$ is locally bounded. As to $h(\cdot)$, it is assumed lsc and may take $+\infty$ values freely.

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Consider the problem of minimizing the integral

$$\int_a^b \{\rho(x(t))h(x'(t))\} + \phi(x(t)) \, dt,$$

on the class $\mathcal{A}$ of all the absolutely continuous functions $x : [a, b] \to \mathbb{R}$ with fixed endpoints: $x(a) = A$, $x(b) = B$. Here the functions $\rho : \mathbb{R} \to [1, +\infty)$, $\phi : \mathbb{R} \to \mathbb{R}$ are assumed

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only lower semicontinuous bounded below; while the function \( h: \mathbb{R} \to [0, +\infty] \) is supposed convex lower semicontinuous and growing superlinearly at infinity:

\[
\frac{h(\xi)}{|\xi|} \to +\infty \quad \text{as} \quad |\xi| \to \infty.
\]  

(2)

As is well known, there exist minimizers of the integral (1) defined on \( \mathcal{A} \) (see, e.g., [3–5]). In a previous paper [7], I have shown that at least one such minimizer behaves very simply: it remains constant along a subinterval \([a', b']\) (with \(a' \leq b'\)), and either increases or decreases strictly along each one of the remaining two subintervals, \([a, a']\) and \([b', b]\), with derivatives “bounded away” from zero in the sense that

\[
y'(t) \notin \{0\} \cup \text{interior}\left(\partial h(\cdot)^{-1}(\partial h(0))\right) \quad \text{a.e.}
\]  

(3)

The aim of this paper is to prove extra regularity properties for minimizers, using as basic tool the above monotonicity property.

To begin with, Lipschitz continuity of minimizers is here obtained under hypotheses which allow \( \varphi(\cdot) \) and \( \rho(\cdot) \) to be both locally unbounded above, and \( y'(\cdot) \) to remain (during a positive measure time) on the boundary of the domain \( h^{-1}(\mathbb{R}) \). Such liberties were prohibited in previous papers (see, e.g., [1, Theorem 4.2]).

The second result is a refinement of this property: we obtain piecewise-\(C^1\) regularity of minimizers under slightly more restrictive hypotheses, which, however, are still so general that most Lagrangians coming from applications satisfy them (at least in case \( h(\cdot) \) takes only finite values). Indeed, besides (local) piecewise continuity of the functions \( \varphi(\cdot) \), \( \rho(\cdot) \), and openness of the domain \( h^{-1}(\mathbb{R}) \), we only assume (local) finiteness of the sets of nonsmooth vertices and 1-dim faces of \( \text{epi} h(\cdot) \), and of the boundary of each level set

\[
\{ s \in \mathbb{R}: \varphi(s) + q\rho(s) = \text{constant} \},
\]

for each number \( q < h(0) \).

Finally, existence of piecewise affine minimizers is proved in case \( h(\cdot) \) is itself (locally) piecewise affine.

These results improve the knowledge of regularity properties of minimizers contained in [2,6]. The convexity of \( h(\cdot) \) is here assumed only for simplicity: the results obtained extend easily to the case of \( h(\cdot) \) lsc but nonconvex, provided it is convex at zero, i.e. \( h^{**}(0) = h(0) \) (see [8]).

The first main result is the following theorem.

**Theorem 1.** Let \( \rho: \mathbb{R} \to [1, +\infty), \varphi: \mathbb{R} \to \mathbb{R} \) and \( h: \mathbb{R} \to [0, +\infty) \) be lower semicontinuous functions, bounded below, with \( h(\cdot) \) convex and having superlinear growth at infinity (as in (2)). If the quotient \( \varphi/\rho(\cdot) \) is bounded above on each bounded interval, then the integral (1), defined on the class \( \mathcal{A} \), has a minimizer which is Lipschitz continuous.

**Proof.** Assume, to fix ideas, \( A = x(a) \leq B = x(b) \). Let \([-A_0, A_0]\) be an interval containing, a priori, the values of the minimizers of the integral (1) defined on \( \mathcal{A} \); then there must exist some \( M > 0 \) for which

\[
\varphi(s) \geq -M, \quad -M \leq \frac{\varphi(s)}{\rho(s)} \leq M, \quad \text{for any} \quad s \in [-A_0, A_0].
\]
Let \( y(\cdot) \) be a minimizer of the convex integral (1) satisfying the monotonicity property explained above (see (3)). Then we need only to prove Lipschitz continuity along a subinterval where \( y(\cdot) \) is strictly monotone. For simplicity of notation and to fix ideas, let us assume \( y(\cdot) \) increasing strictly along the whole interval \([a, b]\). (If \( y(\cdot) \) decreased there, the arguments would be similar.) Notice that, in (3), \( \partial h(\cdot) \) is the subdifferential of the convex lower semicontinuous function \( h(\cdot) \) (see, e.g., \([5, 9]\)), while

\[
(\partial h(\cdot))^{-1}(\partial h(0)) := \{ \xi \in \mathbb{R} : \partial h(\xi) \cap \partial h(0) \neq \emptyset \}.
\]

Let us consider first the case in which

\[
y'(t) \in \text{interior} h^{-1}(\mathbb{R}) \quad \text{a.e. on } [a, b],
\]

Then, by the DuBois–Reymond differential inclusion (see \([1, \text{Theorem 4.1}]\)), there exists a constant \( c \) such that, setting \( q(s) := [c − φ(s)]/ρ(s) \), we have, for a.e. \( t \in [a, b] \),

\[
q(y(t)) \in Q(y'(t)), \quad \text{where } Q(ξ) := h(ξ) − ξ \partial h(ξ).
\]

Since, for \( t \in [a, b] \),

\[
|q(y(t))| ≤ |c| + \left| \frac{φ(y(t))}{ρ(y(t))} \right| ≤ |c| + M,
\]

the growth of \( h(\cdot) \) forces the essential boundedness of \( y'(\cdot) \), and \( y(\cdot) \) is Lipschitz continuous.

Let us consider now the other possible cases. Clearly, we may assume \( h^{-1}(\mathbb{R}) \) to be unbounded and closed, namely

either \( h^{-1}(\mathbb{R}) = (−\infty, ξ_0] \) or \( h^{-1}(\mathbb{R}) = [ξ_0, +∞) \),

with \( y'(t) = ξ_0 \) for \( t \) on a set \( P \) with positive measure.

In case \( 0 \in h^{-1}(\mathbb{R}) \), one easily checks that for a.e. \( t \) on \([a, b]\), either \( y'(t) \) is inside a bounded interval, or else it lies in the interior of \( h^{-1}(\mathbb{R}) \) (since \( y'(t) > 0 \) a.e., by (3)) and the above reasoning applies.

Therefore we may assume

either \( h^{-1}(\mathbb{R}) = [ξ_0, +∞) \subset (0, +∞) \) or \( h^{-1}(\mathbb{R}) = (−∞, ξ_0] \subset (−∞, 0) \).

Let us consider only the first case, the other one being less interesting. Define the number

\[
m := \begin{cases} 1 + \text{distance}[0, \partial h(ξ_0)] & \text{if } \partial h(ξ_0) \neq \emptyset, \\ 1 & \text{otherwise}; \end{cases}
\]

and, for \( n = 1, 2, \ldots \), define the function \( h_n : \mathbb{R} \to [0, +∞) \),

\[
h_n(ξ) := \begin{cases} h(ξ) & \text{for } ξ ≥ ξ_0, \\ h(ξ_0) + m\left(n(ξ_0 − ξ) + \frac{1}{ξ} − \frac{1}{ξ_0}\right) & \text{for } 0 < ξ ≤ ξ_0, \\ +∞ & \text{for } ξ ≤ 0. \end{cases}
\]

Consider the closed convex hull, or bipolar, \( h_0^{**} : \mathbb{R} \to [0, +∞) \) of \( h(\cdot) \), which is a convex lower semicontinuous function with \( (h_0^{**}(\cdot))^{-1}(\mathbb{R}) \) open. One easily checks that, because \( h(ξ) ≤ \lim inf_{h_n \to +∞} h_n(ξ) \), the sequence \( (h^{**}(ξ)) \) increases and converges to \( h(ξ) \) \( \forall ξ \), by
[5, Lemma 2.1, p. 241], and is equi-coercive in the sense of (2). Define, for each \( n \), the number

\[
\xi'_n := \min \{ \xi \geq \xi_0 : h^*_n(\xi) = h(\xi) \}.
\]

Clearly \( \xi'_n \) is well-defined and

\[
\xi_0 \leq \xi'_n \leq \xi, \quad \forall \xi \in h^{-1}(\min h(\mathbb{R})).
\]

Let \( y_n(\cdot) \) be a minimizer of the integral

\[
\int_a^b \left\{ \rho(x(t))h^*_n(x'(t)) + \varphi(x(t)) \right\} \, dt.
\]

Then clearly \( y_n(\cdot) \) increases strictly on \([a, b]\) and, again by [1, Theorem 4.1],

\[
q_n(y_n(t)) \in Q_n(y'_n(t)) \quad \text{a.e. on } [a, b],
\]

with \( q_n(s) := [c_n - \varphi(s)]/\rho(s) \) and \( Q_n(\xi) := h^*_n(\xi) - \xi \partial h^*_n(\xi) \).

Suppose, to begin with, the existence of a number \( n \) for which

\[
y'_n(t) \geq \xi'_n \quad \text{a.e. on } [a, b].
\]

Then \( y_n(\cdot) \) is also a minimizer of the integral (1), because for any \( x(\cdot) \in \mathcal{A} \) we have:

\[
\int_a^b \left\{ \rho(x(t))h^*_n(x'(t)) + \varphi(x(t)) \right\} \, dt \geq \int_a^b \left\{ \rho(x(t))h^*_n(x'(t)) + \varphi(x(t)) \right\} \, dt
\]

\[
\geq \int_a^b \left\{ \rho(y_n(t))h^*_n(y'_n(t)) + \varphi(y_n(t)) \right\} \, dt
\]

\[
\geq \int_a^b \left\{ \rho(y_n(t))h(y'_n(t)) + \varphi(y_n(t)) \right\} \, dt.
\]

Moreover, we may assume \( \xi'_n = \xi_0 \), because otherwise it would be \( y'_n(t) > \xi_0 \) a.e. on \([a, b]\), and this case has already been treated above. Defining the set

\[
K := \{ t \in [a, b] : y'_n(t) > \xi_0 \},
\]

we obtain

\[
q_n(y_n(t)) \in Q_n(y'_n(t)) \quad \text{a.e. on } K.
\]

But since \( |q_n(y_n(t))| \leq |c_n| + M \) on \( K \), \( y_n(\cdot) \) must be Lipschitz continuous.

The alternative for the case just treated is the following: for any \( n \), the set

\[
K^- := \{ t \in [a, b] : y'_n(t) < \xi_0 \}
\]

has positive measure. In such case, because \( h^*_n(\cdot) \geq 0 \), we must have

\[
q_n(y_n(t)) > 0 \quad \text{a.e. on } K^-.
\]
for any \( n \). Therefore \( c_n > -M \) for any \( n \), hence
\[
q_n(y_n(t)) = \frac{c_n - \varphi(y_n(t))}{\rho(y_n(t))} > -2M \quad \text{a.e. on } [a, b];
\]
and, defining
\[
\xi_1 := \max\{ \xi \geq 0: q \in Q(\xi) \text{ for some } q \geq -2M \},
\]
we obtain
\[
y_n'(t) \leq \xi_1 \quad \text{a.e. on } [a, b] \quad \forall n \in \mathbb{N}.
\]
This shows that letting \( n \to +\infty \), and following a standard procedure (see, e.g., [7, part (g) of the proof]), one obtains a new minimizer \( y_0(\cdot) \) of the convex integral (1) satisfying
\[
0 < \xi_0 \leq y_0'(t) \leq \xi_1 \quad \text{a.e. on } [a, b].
\]
thus completing the proof. \( \square \)

The second main result is presented in what follows.

**Theorem 2.** Let \( \rho(\cdot), \varphi(\cdot), h(\cdot) \) be as in Theorem 1 and assume, in addition:

(i) the domain \( h^{-1}(\mathbb{R}) \) is open (e.g., \( h(\mathbb{R}) \subset \mathbb{R} \)), and \( \text{epi } h(\cdot) \) has finitely many nonsmooth vertices and 1-dim faces on each bounded interval;

(ii) \( \varphi(\cdot), \rho(\cdot) \) are (locally) piecewise continuous and the boundary of each level set of \( \varphi(\cdot) + q\rho(\cdot) \),
\[
\partial\{ s \in I: \varphi(s) + q\rho(s) = c \},
\]
is finite, for each \( q < h(0), c \in \mathbb{R} \), and each bounded interval \( I \).

Then the integral (1), defined on the class \( \mathcal{A} \), has a minimizer which is piecewise-C¹.

**Corollary 3.** Let \( \rho(\cdot), \varphi(\cdot), h(\cdot) \) be as in Theorem 2 and assume, in addition:

(i') \( h(\cdot) \) is (locally) piecewise affine.

Then the integral (1), defined on the class \( \mathcal{A} \), has a minimizer which is piecewise affine.

**Remark 1.** Let us precise better what is meant, in Theorem 2, by “finitely many nonsmooth vertices and 1-dim faces on each bounded interval.” Consider, for each \( q < h(0) \), the set
\[
T_q = Q^{-1}(q) := \{ \xi \in \mathbb{R}: q \in Q(\xi) \},
\]
with \( Q(\cdot) \) as in (4), i.e. the set of points \( \xi \) of \( h^{-1}(\mathbb{R}) \) over which the supporting lines to the epigraph of \( h(\cdot) \) meet the vertical axis at the point \( (0, q) \). Clearly there exist at most
countably many values of \( q < h(0) \), which we denote by \( q_1, q_2, \ldots \), for which the set \( T_q \) has more than two points (hence has nonempty interior). Define the union 
\[
T := \bigcup_{r=1}^{+\infty} T_{q_r}
\]
and denote by \( Q_T \) the set \( \{q_1, q_2, \ldots\} \). Define the countable set \( D \) of non-differentiability points \( \xi_d \neq 0 \) of \( h(\cdot) \), and the set 
\[
Q_D := \bigcup_{\xi_d \in D} \partial Q(\xi_d),
\]
where \( \partial Q(\xi_d) \) denotes the boundary of the set \( Q(\xi_d) \), with \( Q(\cdot) \) as in (4).

In Theorem 2, what one really needs to assume is:

- the set \( D \),
- the boundary \( \partial T \) of the set \( T \),
- the boundary \( \partial \{s \in \mathbb{R}: \varphi(s) + q\rho(s) = c\} \), for each \( q \in Q_D \cup Q_T \),

should be all finite on each bounded interval.

**Proof.** (a) By [7] and Theorem 1, there exists a Lipschitz minimizer \( y(\cdot) \) of (1) which is constant along a subinterval \([a', b']\) and strictly monotone along each one of the two other subintervals, \([a, a']\) and \([b', b]\). We will change \( y(\cdot) \) along \([a, a']\) and along \([b', b]\), in order to obtain a new minimizer \( x(\cdot) \) satisfying the desired properties. To simplify the notation, let us assume \( y(\cdot) \) to, say, increase strictly, with \( x'(t) > 0 \), along \([a, b]\); if \( y(\cdot) \) decreased strictly along \([a, b]\), the arguments would be similar.

By the DuBois–Reymond differential inclusion (4), there exists a constant \( c \) such that, defining \( q(s) := \frac{c - \varphi(s)}{\rho(s)} \), we have \( q(y(t)) \in Q(y'(t)) \), or, equivalently,
\[
y'(t) \in T_{q(y(t))}, \quad \text{a.e. on } [a, b],
\]
(with \( T_q \) defined as in (6)). Define, for each \( r \in \{1, 2, \ldots\} \), the measurable set 
\[
E_{q_r} := \{t \in [a, b]: y'(t) \in T_{q_r}\}
\]
and the level set 
\[
L_{q_r} := \{s \in [A, B]: q(s) = q_r\}.
\]

Then, apart from null sets, we have \( y^{-1}(L_{q_r}) \subset E_{q_r} \). Since, for each \( r \neq r' \) in \( \{1, 2, \ldots\} \), the level sets \( L_{q_r}, L_{q_{r'}} \), are disjoint, the corresponding open sets 
\[
y^{-1}(\text{interior } L_{q_r}), \quad y^{-1}(\text{interior } L_{q_{r'}})
\]
are also disjoint. Write the nonempty open set \( O_r := y^{-1}(\text{interior } L_{q_r}) \) as a countable union of nonempty pairwise disjoint open intervals \((a^r_k, b^r_k)\), \( k = 1, 2, \ldots \); and consider the countable union \( O \) of pairwise disjoint intervals,
\[
O := \bigcup_{r=1}^{+\infty} O_r = \bigcup_{r=1}^{+\infty} \bigcup_{k=1}^{+\infty} (a^r_k, b^r_k). \]
Actually we may write
\[ O := \bigcup_{r=1}^{r_1} O_r = \bigcup_{r=1}^{r_1} \bigcup_{k=1}^{k_r} (\alpha_r^k, b_r^k). \]
Indeed, the number \( r_1 \) is finite, since \( \text{epi} h(\cdot) \) has finitely many 1-dim faces on a bounded interval \((\xi_-, \xi_+)\) where the values of \( y(\cdot) \) essentially remain. Similarly, for each \( r \in \{1, 2, \ldots, r_1\} \), the number \( k_r \) is finite because each level set \( \{ s \in [A, B] : q(s) = q_r \} \) has finite boundary, and \( y(\cdot) \) increases strictly. The same happens, by the hypotheses in Remark 1, if we write \( q^d \) in place of \( q_r \) in these reasonings, where \( q^d \) is any point in \( Q_D \setminus Q_T \); therefore we may assume (and will do so, from now on) that the open set \( O \) already includes also the inverse images \( y^{-1}(\text{interior } L_{q^d}) \) for all those such points \( q^d \in Q_D \setminus Q_T \).

For \( r \in \{1, 2, \ldots, r_1\} \) and \( k \in \{1, 2, \ldots, k_r\} \), define the measurable functions
\[
\chi^k_r(t) := \begin{cases} 1 & \text{for } t \in (\alpha^k_r, b^k_r), \\ 0 & \text{for other } t \in [a, b]; \end{cases}
\]
while for \( r = 0 \) define \( k_0 = 1 \),
\[
\chi^0_1(t) := \begin{cases} 0 & \text{for } t \in O, \\ 1 & \text{for other } t \in [a, b]. \end{cases}
\]
Let us fix our attention on one of the intervals \((\alpha^k_r, b^k_r)\). Assume, to fix ideas, that
\[
T_{q_r} = [\alpha^-_r, \beta^-_r] \cup [\alpha^+_r, \beta^+_r], \quad \alpha^-_r \leq \beta^-_r < 0 < \alpha^+_r < \beta^+_r,
\]
\[
h(\xi) = \begin{cases} q_r + m^-_r \xi & \text{on } [\alpha^-_r, \beta^-_r], \\ q_r + m^+_r \xi & \text{on } [\alpha^+_r, \beta^+_r]; \end{cases}
\]
and define \( h_\xi(\xi) := h(\xi) - q_r, \quad \phi_r(s) := \psi(s) + q_r \rho(s), \) so that \( \rho(s)h(\xi) + \psi(s) = \rho(s)h_\xi(\xi) + \phi_r(s) \),
\[
\int_{a^k_r}^{b^k_r} \rho(y(t))h'(y(t)) dt = c(b^k_r - a^k_r) + \int_{a^k_r}^{b^k_r} \rho(y(t))h_r(y(t)) dt.
\]
Since \( y(\cdot) \) increases strictly on \([a, b]\), we have that \( y^{-1}(L_{q_r}) \subset E_{q_r} \) a.e. and \( y'(t) \notin \text{interior}((\partial h(\cdot))^{-1}(\partial h(0))) \) a.e., we may assume \( \alpha^+_r > 0 \), \( y'(t) \in [\alpha^+_r, \beta^+_r] \) a.e. in \((\alpha^k_r, b^k_r)\) and
\[
0 < \xi^k_r := \frac{y(b^k_r) - y(a^k_r)}{b^k_r - a^k_r} \in [\alpha^+_r, \beta^+_r].
\]
Define the affine function
\[
x^k_r(t) := y(a^k_r) + \xi^k_r(t - a^k_r), \quad t \in [a^k_r, b^k_r].
\]
Then
\[
\int_{a^k}^{b^k} x^k_r(t) \, dt = \int_{a^k}^{b^k} y(t) \, dt,
\]
\[
x^k_r((a^k_r, b^k_r)) = y((a^k_r, b^k_r)) \subset \text{interior } L_{qr},
\]
\[
x^k_r(t) \in T_{qr} \quad \text{for a.e. } t \in (a^k_r, b^k_r),
\]
\[
\int_{a^k}^{b^k} \rho(x^k_r(t)) h_r(x^k_r(t)) \, dt = m_r^+ \int_{a^k}^{b^k} \rho(y(t)) y'(t) \, dt = b^k_r \int_{a^k_r}^{b^k_r} \rho(y(t)) h_r(y'(t)) \, dt,
\]
\[
hence
\int_{a^k}^{b^k} \left\{ \rho(y(t)) h(y'(t)) + \phi(y(t)) \right\} dt = \int_{a^k}^{b^k} \left\{ \rho(x^k_r(t)) h(x^k_r(t)) + \phi(x^k_r(t)) \right\} dt,
\]

because \( \phi(\cdot) := \rho(y(\cdot)) y'(\cdot) \in L^1(a^k_r, b^k_r) \) implies \( \rho(\cdot) \in L^1(y(a^k_r), y(b^k_r)) \).

Notice also that, since in any component \((a^d_r, b^d_r)\) of \( y^{-1}(\text{interior } L_{qd}) \), for those \( q^d \in QD \setminus QT \) has already to be affine (with slope \( \xi_d \)), we may set here \( x^k_d(\cdot) := y(\cdot) \).

(b) Define \( x^0_1(\cdot) := y(\cdot) \) on \([a, b]\).

Define the new function \( x(\cdot) \in A \) by
\[
x(t) := A + \int_a^t \sum_{r=0}^{r_1} \sum_{k=1}^{k_r} \chi^k_r(\tau) x^k_r(\tau) \, d\tau.
\]
Clearly, \( x(a) = A \) and this definition makes sense:
\[
x(b) - A = \int_a^b \sum_{r=0}^{r_1} \sum_{k=1}^{k_r} \chi^k_r(t) x^k_r(t) \, dt = \int_a^b \chi^0_1(t) y'(t) \, dt + \sum_{r=1}^{r_1} \sum_{k=1}^{k_r} \int_a^b x^k_r(t) \, dt
\]
\[
= \int_a^b \chi^0_1(t) y'(t) \, dt + \sum_{r=1}^{r_1} \sum_{k=1}^{k_r} \int_a^b y'(t) \, dt = \int_a^b \sum_{r=0}^{r_1} \sum_{k=1}^{k_r} \chi^k_r(t) y'(t) \, dt
\]
\[
= B - A,
\]
so that \( x(\cdot) \) is absolutely continuous and \( x(b) = B \). One easily checks that
\[
\phi(x(t)) + \rho(x(t)) h(x'(t)) = \phi(y(t)) + \rho(y(t)) h(y'(t))
\]
a.e. on \([a, b] \setminus O\), hence \( x(\cdot) \) minimizes the integral (1) on the class \( A \):
\[
\int_a^b \{ \varphi(x(t)) + \rho(x(t))h(x'(t)) \} \, dt \\
= \int_a^b \chi^1_0(t) \{ \varphi(x(t)) + \rho(x(t))h(x'(t)) \} \, dt \\
+ \int_a^b \sum_{r=1}^{r_1} \sum_{k=1}^{k_r} \chi^k_r(t) \{ \varphi(x^k_r(t)) + \rho(x^k_r(t))h(x^k_r(t)) \} \, dt \\
= \int_a^b \chi^1_0(t) \{ \varphi(y(t)) + \rho(y(t))h(y'(t)) \} \, dt \\
+ \sum_{r=1}^{r_1} \sum_{k=1}^{k_r} \int_{a^k_r}^{b^k_r} \{ \varphi(y(t)) + \rho(y(t))h(y'(t)) \} \, dt \\
= \int_a^b \{ \varphi(y(t)) + \rho(y(t))h(y'(t)) \} \, dt.
\]

(c) By construction, the function \( x(\cdot) \) is piecewise affine along each interval of the compact set

\[
\mathcal{D} := \bigcup_{r=1}^{r_1} \mathcal{D}_r = \bigcup_{r=1}^{r_1} \bigcup_{k=1}^{k_r} [a^k_r, b^k_r].
\]

Define the finite sets

\[
\mathcal{D}_- := \mathcal{D} \cap (-\infty, 0) \cap [\xi_-, \xi_+] = \{ \xi_d : d \in \Delta_- \}, \\
\mathcal{D}_+ := \mathcal{D} \cap (0, +\infty) \cap [\xi_-, \xi_+] = \{ \xi_d : d \in \Delta_+ \},
\]

and consider the multifunction \( Q(\cdot) \) defined on \( \mathcal{D}_- \cup \mathcal{D}_+ \) with values

\[
Q(\xi) := h(\xi) - \xi \partial h(\xi).
\]

Define the open set

\[
L_{Q_d} := \{ s \in (A, B) : q(s) \in \text{interior} Q(\xi_d) \} \quad \text{for} \ d \in \Delta := \Delta_- \cup \Delta_+.
\]

Assuming first \( \varphi(\cdot) \), \( \rho(\cdot) \) to be continuous, \( L_{Q_d} \) is a finite union of open intervals, because

\[
\partial \{ s \in (A, B) : q(s) \in (q_-, q_+) \} \subset \partial \{ s \in (A, B) : q(s) \in \partial (q_-, q_+) \} \cup \{A, B\},
\]

and the right-hand side is finite, by the hypotheses, \( \forall q_-, q_+ \). Since \( x(\cdot) \) increases strictly along \( [a, b] \), the DuBois–Reymond differential inclusion (4) yields:

\[
x'(t) = \xi_d \quad \text{for a.e.} \ t \in x^{-1}(L_{Q_d}), \quad \text{for any} \ d \in \Delta_+.
\]
Hence \( x(\cdot) \) is affine along each one of the finitely many intervals of the open set

\[
\mathcal{O}':=x^{-1}\left(\bigcup_{d\in \Delta_+} L_{Q_d}\right).
\]

Let \((a_1, b_1)\) be any one of the finitely many connected components of the open set \(\mathcal{O}'':= (a, b) \setminus \mathcal{O} \setminus \mathcal{O}'\), and define the sets

\[
T_+:=\left(\bigcup_{r=1}^{r_1} \text{interior } T_q\right) \cap (0, +\infty) \cap (\xi_-, \xi_+),
\]

\[
N_+:=\{t \in (a_1, b_1): x'(t) \in T_+ \cup D_+\}.
\]

For a.e. \( t \) on \( N_+ \), we have, by the DuBois–Reymond differential inclusion (4): either \( q(x(t)) = q_r \) for some \( r \in \{1, \ldots, r_1\} \) or else \( q(x(t)) \in Q(\xi_d) \) for some \( d \) in \( \Delta_+\). In the first case we must have \( x(t) \in \bigcup_{r=1}^{r_1} \partial L_{q_r} \), while in the second \( x(t) \in \bigcup_{d\in \Delta_+} \partial q^{-1}(\partial Q(\xi_d)) \). Since both these sets are finite, we must have \( x'(t) = 0 \) a.e. on \( N_+ \). But because, on the other hand, \( x'(t) > 0 \) a.e. on \([a, b]\) by the assumption made at the beginning of (a), \( N_+ \) must be a null set. (In particular, \( \mathcal{O}'' \) is empty in the situation of Corollary 3, since \( x'(t) \) is a.e. either in the vertical projection of the interior of a face (i.e. in \( T_+ \)), or of a vertex (i.e. in \( D_+ \)).)

(d) In the situation of Theorem 2, if \( \mathcal{O}'' \) is nonempty then along \((a_1, b_1)\) the DuBois–Reymond inclusion (4) becomes an implicit differential equation:

\[
q_{+}(x'(t)) = q(x(t)) \quad \text{a.e. on } [a_1, b_1].
\]

The function

\[
q_{+}:(\xi_-, \xi_+) \cap h^{-1}(\mathbb{R}) \cap (0, +\infty) \setminus T_+ \setminus D_+ \rightarrow (\{-|c| - M, h(0)\}),
\]

\[
q_{+}(\xi) := h(\xi) - \xi h'(\xi)
\]

is well-defined, decreases strictly (by the strict convexity), and is uniformly continuous along each one of the finitely many intervals that form its domain. In particular, the inverse function \( f_{+}(\cdot) \) of \( q_{+}(\cdot) \),

\[
f_{+}:q_{+}\left((\xi_-, \xi_+) \cap h^{-1}(\mathbb{R}) \cap (0, +\infty) \setminus T_+ \setminus D_+\right)
\]

\[
\rightarrow(\xi_-, \xi_+) \cap h^{-1}(\mathbb{R}) \cap (0, +\infty) \setminus T_+ \setminus D_+,
\]

\[
\xi = f_{+}(r) \iff r = q_{+}(\xi).
\]

is well-defined and continuously decreases strictly along each one of the finitely many intervals of its domain, hence is uniformly continuous there. Equation (7) may therefore be written in the equivalent form (of a canonical differential equation, with \( x'(\cdot) \) explicitly as a function of \( x(\cdot) \)):

\[
x'(t) = f_{+}\left(q_x(t)\right) \quad \text{a.e. on } [a_1, b_1].
\]
Since $x(\cdot)$ is uniformly continuous on $(a_1, b_1)$, the function $q(\cdot)$ is uniformly continuous on $x([a_1, b_1])$ and $f_+(\cdot)$ is continuous and monotone—hence uniformly continuous—along the compact interval which is the image of $[a_1, b_1]$ by $q(x(\cdot))$, $x(\cdot)$ is actually $C^1$ on $[a_1, b_1]$.

The same happens in case $\varphi(\cdot)$ and/or $\rho(\cdot)$ is only piecewise-continuous: if $(s_1, s_2)$ is an interval where $q(\cdot)$ is uniformly continuous, and $(a_2, b_2)$ is $x^{-1}((s_1, s_2))$, then $q(x([a_1, b_1] \cap [a_2, b_2]))$ is an interval where $f_+(\cdot)$ is uniformly continuous. This completes the proof. □

**Remark 2.** The hypotheses of Theorem 2 are satisfied by the usual functions $\varphi(\cdot)$, $\rho(\cdot)$, $h(\cdot)$ appearing in real-world applications, at least when $h(\cdot)$ takes only finite values—e.g., whenever these functions are piecewise analytic over each bounded interval.

The hypothesis $h^{-1}(\mathbb{R})$ open is not needed in case one can guarantee the existence of a minimizer $y(\cdot)$ of the integral (1) satisfying the DuBois–Reymond differential inclusion (4); and this is guaranteed provided $\gamma(\cdot) \in \text{interior } h^{-1}(\mathbb{R})$ a.e.

By locally piecewise affine (respectively locally piecewise continuous) we mean a function satisfying the property: each bounded interval has a partition, by finitely many points, such that the function is affine (respectively uniformly continuous) inside each open subinterval of the partition. And piecewise-$C^1$ means to have a piecewise continuous derivative.

**Remark 3.** We imposed, in the above theorems, $\varphi(\cdot)$ bounded below just for simplicity. Actually, as is usual in the application of the direct method, it suffices to ask for

$$\varphi(s) \geq -\gamma_1 - \gamma_2 |s|^p, \quad p \geq 1,$$

provided a stronger growth condition is imposed on $h(\cdot)$ in case $p > 1$:

$$h(\xi) \geq -\gamma_3 + \gamma_4 |\xi|^q,$$

with either $q > p$ and $\gamma_4 = 1$ or $q = p$ and $\gamma_4 > 0$ large enough relative to $b - a$ and $\gamma_2 > 0$ (see, e.g., [3–5]). Also $h(\cdot)$ is supposed $\geq 0$ for the same reason; otherwise define $h_+(\xi) := h(\xi) - \min h(\mathbb{R})$, $\varphi_+(s) := \varphi(s) + \rho(s) \min h(\mathbb{R})$, so that $h_+(\cdot) \geq 0$ and $\rho(s)h(\xi) + \varphi(s) = \rho(s)h_+(\xi) + \varphi_+(s)$; and assume $\varphi_+(\cdot)$ lower semicontinuous and bounded below. Or else assume $\varphi_+(\cdot)$ lower semicontinuous and $\varphi_+(\cdot)$, $h_+(\cdot)$ to satisfy the above inequalities.

**References**