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Transcendental infinite sums

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ABSTRACT

We show that it follows from results on linear forms in logarithms of algebraic numbers that numbers such as

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}, \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n}, \quad \sum_{n=1}^{\infty} \frac{F_n}{n2^n}$$

where χ is any non-principal Dirichlet character and $(F_n)_{n=0}^{\infty}$ the Fibonacci sequence, are transcendental.

1. INTRODUCTION

In the mathematical literature the transcendency of infinite sums like

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}, \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n}, \quad \sum_{n=1}^{\infty} \frac{F_n}{n2^n}$$

where χ is a non-principal Dirichlet character and $(F_n)_{n=0}^{\infty}$ the Fibonacci sequence, has received little attention. In this paper, we show that these numbers are transcendental and we give approximation measures for them.

By a computable number, we mean a number which can be explicitly determined as a function of its defining parameters. The proofs of our results depend on Baker's theory on linear forms in logarithms. By this approach most results say that the considered infinite sum has either a computable algebraic value (which is often 0 or some rational number) or is transcendental. In practice, it will be often easy to exclude the former option. We write $\mathbb{Z}$ for the set of integers. Further we denote by $\mathbb{Q}$ the field of rational numbers and by $\overline{\mathbb{Q}}$ the field of algebraic numbers. For an algebraic number α we denote the degree $[\mathbb{Q}(\alpha):\mathbb{Q}]$ by d_α and the absolute logarithmic height (cf. Section 2) by h_α or $h(\alpha)$. For a function $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ which is periodic mod q we denote $[\mathbb{Q}(f(0),f(1),\cdots,f(q-1)):\mathbb{Q}]$ by d_f and $\sum_{j=0}^{q-1} h(f(j))$ by h_f. At several places in the paper the rearrangement of terms in the infinite series requires justification which is left to the reader. We begin with the statement of Theorem 1 which deals with the case when the denominator is n. This will be followed by a corollary and corollaries of Theorems 2-4. We refer to sections 4-6 for the full statements of Theorems 2-4.

Theorem 1. *Let* $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ *be periodic* mod q *and such that*

$$S=\sum_{n=1}^{\infty}\frac{f(n)}{n} \tag{1}$$

converges. Then $S=0$ *or* $S\notin\overline{\mathbb{Q}}$. *In the latter case we have*

$$\log\ |S-\alpha|\geq -c^q q^{3q}(d_\alpha d_f)^{q+3}\max\ (h_\alpha,h_f)$$

for any algebraic number α, *where* c *is some computable absolute constant.*

The example

$$1-\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{3}{6}+\frac{1}{7}+\frac{1}{8}+\cdots=0$$

with $q=4$ shows that the case $S=0$ cannot be excluded. In many instances it is simple to check that $S\neq 0$. It may be cumbersome to give a general criterion. The question whether $S=0$ can be excluded under certain general conditions has been the subject of conjectures of Chowla [6] and Erdős, see [10]. In this connection we refer to Baker, Birch and Wirsing [3], Okada [11] and Tijdeman [12]. In the former paper the theory of linear forms in logarithms has been applied to answer a question of Chowla, see Lemma 3. On combining Theorem 1 with Dirichlet's result that $L(1,\chi)\neq 0$ for an arbitrary non-principal Dirichlet character χ, we immediately obtain the following:

Corollary 1.1. *Let* $q\geq 2$ *be an integer and* χ *a non-principal Dirichlet character* mod q. *Then* $L(1,\chi)$ *is transcendental.*

In particular, if $\chi(n)=(\frac{d}{n})$ where d is the discriminant of a quadratic field and $(\frac{d}{n})$ denotes the Kronecker symbol, then

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}$$

is transcendental. This is a well known fact by the class number formula for quadratic fields. Since we do not know whether $\sum_{n=1}^{\infty} n^{-k}$ is transcendental for any odd integer k, it will be difficult to replace the denominator in (1) by the value at n of an arbitrary polynomial. However, by using partial fractions, we can deal with the case that the denominator is $Q(n)$ where $Q(X) \in \mathbb{Q}[X]$ has only simple rational zeros. We call the polynomial $Q(X)$ reduced if $Q(X) \in \mathbb{Q}[X]$ and it has only simple rational zeros which are all in the interval $[-1, 0)$. The following consequence of Theorem 2 is an extension of Theorem 1.

Corollary 2.1. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be periodic* mod q. *Let $Q(X) \in \mathbb{Q}[X]$ have simple rational zeros. If*

$$S = \sum_{n=0}^{\infty} \frac{f(n)}{Q(n)}$$

converges, then S equals a computable algebraic number or $S \notin \overline{\mathbb{Q}}$. In the latter case we have

$$\log \mid S - \alpha \mid \geq -c_{f,Q} d_{\alpha}^{q+3} h_{\alpha}$$

for any algebraic number α, where $c_{f,Q}$ is a computable number depending only on f and Q.

Suppose S is algebraic. It follows from the proof of Corollary 2.1 that $S \in \mathbb{Q}$ if f assumes only rational values. Moreover it follows from Theorem 2 that $S = 0$ if Q is reduced. We now state a similar result if f in Theorem 1 is replaced by a polynomial. It follows from Theorem 3.

Corollary 3.1. *Let $P(X) \in \overline{\mathbb{Q}}[X]$. Let $Q(X) \in \mathbb{Q}[X]$ have simple rational zeros. If*

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$$

converges, then S equals a computable algebraic number or $S \notin \overline{\mathbb{Q}}$. In the latter case we have

$$\log \mid S - \alpha \mid \geq -c_{P,Q} d_{\alpha}^{c_Q} h_{\alpha}$$

for any algebraic number α, where $c_{P,Q}$ is a computable number depending only on P and Q and $c_{\mathbb{Q}}$ a computable number depending only on Q.

Suppose S is algebraic. Then it follows from the proof of Corollary 3.1 that $S \in \mathbb{Q}$ if P assumes only rational values. Moreover it follows from Theorem 3 that $S = 0$ if Q is reduced. Thus

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}$$

is transcendental, but not a Liouville number. Finally, we state a result in which the f in Theorem 1 is replaced by an exponential polynomial. It follows from Theorem 4.

Corollary 4.1. *Let* $P_1(X), \cdots, P_\ell(X) \in \overline{\mathbb{Q}}[X], \alpha_1, \cdots, \alpha_\ell \in \overline{\mathbb{Q}}$. *Put* $g(X) = \sum_{\lambda=1}^{\ell} P_\lambda(X)\alpha_\lambda^X$. *Let* $Q(X) \in \mathbb{Q}[X]$ *have simple rational zeros. If*

$$S = \sum_{n=0}^{\infty} \frac{g(n)}{Q(n)}$$

converges, then S is a computable algebraic number or $S \notin \overline{\mathbb{Q}}$. *In the latter case we have*

$$\log \mid S - \alpha \mid \geq -c_{g,Q} d_\alpha^{\ell c_Q} h_\alpha$$

for any algebraic number α, *where* $c_{g,Q}$ *is a computable number depending only on g and Q, and* c_Q *a computable number depending only on Q.*

Suppose S is algebraic. Then $S \in \mathbb{Q}$ if $P_1(X), \cdots, P_\ell(X) \in \mathbb{Q}[X]$ and $\alpha_1, \cdots, \alpha_\ell \in \mathbb{Q}$ and $S = 0$ if Q is reduced. It therefore follows from Corollary 4.1 that the positive number

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n}$$

is transcendental in view of $F_n = (1/\sqrt{5})\{(\frac{1}{2}+\frac{1}{2}\sqrt{5})^n - (\frac{1}{2}-\frac{1}{2}\sqrt{5})^n\}$. On the other hand, it is well known that $\sum_{n=1}^{\infty} (F_n/2^n) = 2$. There are several results on the arithmetic nature of sums involving Fibonacci numbers in the literature, see [1],[4], [5] and [8], but they concern sums where F_n appears in the denominator.

2. AUXILIARY RESULTS

One of the earliest results of Baker on linear forms in logarithms reads as follows.

Lemma 1. (Baker [2]). *If* $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}} - \{0\}, \beta_1, \cdots, \beta_n \in \overline{\mathbb{Q}}$ *and* $\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$, *then*

$$\Lambda = 0 \textit{ or } \Lambda \textit{ is transcendental.}$$

Here and later we shall read log as the principal value of the logarithm with the argument in $(-\pi, \pi]$. Baker's method is effective and several lower bounds for non-zero Λ have been given. We shall use an estimate due to Waldschmidt in simplified form. For an algebraic number α with minimal polynomial over $\mathbb{Z}$ given by $a_0 \prod_{j=1}^{D} (X - \alpha_j)$, we define the Mahler measure of α by

$$M(\alpha) = a_0 \prod_{j=1}^{D} \max(1, \mid \alpha_j \mid)$$

and the absolute logarithmic height by

$$h(\alpha) = \frac{1}{D} \log M(\alpha).$$

We have

(2) $h(\alpha\beta) \leq h(\alpha) + h(\beta)$

and

(3) $h(\alpha_1 + \cdots + \alpha_n) \leq h(\alpha_1) + \cdots + h(\alpha_n) + \log n,$

see [13,(2.1),(2.3)].

Lemma 2. (Waldschmidt [13]). *Let $n \geq 1$. Let K be a number field of degree D over $\mathbb{Q}$. Let $\alpha_1, \cdots, \alpha_n$ be non-zero elements of K. Let $\beta_0, \beta_1, \cdots, \beta_n \in K$. Put $\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$. Let the real numbers $V_1, \cdots, V_n, W$ satisfy*

$$D^{-1} \leq V_1 \leq \cdots \leq V_n,\ V_{n-1} \geq 1,$$
$$V_j \geq \max \{h(\alpha_j), |\log \alpha_j| / D\} \text{ for } 1 \leq j \leq n,$$
$$W \geq \max_{1 \leq j \leq n} h(\beta_j).$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| > -C(n) D^{n+2} V_1 \cdots V_n (W + \log (eDV_n)) \log(eDV_{n-1})$$

with $C(n) = 2^{8n+51} n^{2n}$.

The next lemma provides a possibility to exclude the case $\Lambda = 0$.

Lemma 3. (Baker, Birch, Wirsing [3]). *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}, f \neq 0$ be periodic* mod q *such that*

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges. If

(i) $f(r) = 0$ for $1 <$ gcd $(q, r) < q$

and

(ii) the cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(0), f(1), \cdots, f(q-1))$, then $S \neq 0$.

A crucial link between infinite sums and linear forms in logarithms is given by the following lemma.

Lemma 4. *Let q be a positive integer. Let ζ denote a primitive q th root of unity and β any algebraic number with $|\beta| \leq 1, \beta^q \neq 1$. Then we have, for $0 < r \leq q$*

$$\sum_{\substack{m=1 \\ m \equiv r (\mathrm{mod}\ q)}}^{\infty} \frac{\beta^m}{m} = -\frac{1}{q} \sum_{j=0}^{q-1} \zeta^{-jr} \log (1 - \beta\zeta^j).$$

Proof. Note that by Dirichlet's convergence criterion and $\beta^q \neq 1$, we have that

$$\sum_{\substack{m=1 \\ m \equiv \lambda (\mathrm{mod}\ q)}}^{\infty} \frac{\beta^m}{m} = \beta^{\lambda} \sum_{n=0}^{\infty} \frac{(\beta^q)^n}{nq + \lambda}$$

converges for $0 < \lambda \le q$. On the one hand, using $\beta\zeta^j \neq 1$ for $0 \le j < q$, we have

$$\sum_{j=0}^{q-1} \sum_{\lambda=0}^{q-1} \zeta^{(\lambda-r)j} \sum_{\substack{m=1 \\ m\equiv\lambda(\text{mod } q)}}^{\infty} \frac{\beta^m}{m} = \sum_{j=0}^{q-1} \zeta^{-jr} \sum_{m=1}^{\infty} \frac{(\beta\zeta^j)^m}{m}$$
$$= -\sum_{j=0}^{q-1} \zeta^{-jr} \log (1-\beta\zeta^j).$$

On the other hand,

$$\sum_{j=0}^{q-1} \sum_{\lambda=0}^{q-1} \zeta^{(\lambda-r)j} \sum_{\substack{m=1 \\ m\equiv\lambda(\text{mod } q)}}^{\infty} \frac{\beta^m}{m} = \sum_{\lambda=0}^{q-1} \sum_{\substack{m=1 \\ m\equiv\lambda(\text{mod } q)}}^{\infty} \frac{\beta^m}{m} \sum_{j=0}^{q-1} \zeta^{(\lambda-r)j}$$
$$= q \sum_{\substack{m=1 \\ m\equiv r(\text{mod } q)}}^{\infty} \frac{\beta^m}{m}. \quad \square$$

In case β is a q th root of unity we use the following substitute for Lemma 4.

Lemma 5. *Let q be a positive integer. Let $c_1, \cdots, c_m \in \mathbb{C}$. Let $k_1, \cdots, k_m$, $r_1, \cdots, r_m$ be integers with $0 < r_\mu \le k_\mu$ for $\mu = 1, \cdots, m$. If the double sum on the left hand side converges, then*

$$\sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu - 1} \frac{c_\mu}{k_\mu}(1-\zeta_\mu^{-jr_\mu}) \log (1-\zeta_\mu^j) \tag{4}$$

where ζ_μ is some primitive k_μ–th root of unity for $\mu = 1, \cdots, m$.

Proof. Since the double sum on the left hand side of (4) converges we have

$$\sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu} = 0. \tag{5}$$

We follow the arguments of Lehmer [9]. We have

$$\sum_{n=0}^{N} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} c_\mu \sum_{n=0}^{N} \frac{1}{k_\mu n + r_\mu}$$
$$= \sum_{\mu=1}^{m} c_\mu \{ \sum_{n=0}^{N} \frac{1}{k_\mu n + r_\mu} - \frac{\log (k_\mu N)}{k_\mu} \} + \sum_{\mu=1}^{m} \frac{c_\mu \log k_\mu}{k_\mu}$$

by (5). Note that

$$\gamma(r_\mu, k_\mu) := \lim_{N\to\infty} \{ \sum_{n=0}^{N} \frac{1}{k_\mu n + r_\mu} - \frac{\log (k_\mu N)}{k_\mu} \}$$

exists. According to Theorem 1 of [9], for any $0 \le r < k$,

$$k\gamma(r,k) = \gamma - \sum_{j=1}^{k-1} \zeta^{-jr} \log (1-\zeta^j)$$

where ζ is a primitive kth root of unity and γ is Euler's constant. Note that the above equation is valid for $0 < r \leq k$. We prefer to work with r in this range. We obtain

$$\sum_{n=0}^{\infty}\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} c_\mu \gamma(r_\mu, k_\mu) + \sum_{\mu=1}^{m}\frac{c_\mu \log k_\mu}{k_\mu}$$

$$= \gamma \sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu} - \sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu}\sum_{j=1}^{k_\mu - 1}\zeta_\mu^{-jr_\mu}\log(1-\zeta_\mu^j) + \sum_{\mu=1}^{m}\frac{c_\mu \log k_\mu}{k_\mu}.$$

Using (5), $\prod_{j=1}^{k_\mu - 1}(1-\zeta_\mu^j) = k_\mu$, and that $\sum_{j=1}^{k_\mu - 1}\log(1-\zeta_\mu^j)$ is real we obtain

$$\sum_{n=0}^{\infty}\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m}\sum_{j=1}^{k_\mu - 1}\frac{c_\mu}{k_\mu}(1-\zeta_\mu^{-jr_\mu})\log(1-\zeta_\mu^j). \quad \square$$

3. THE PERIODIC CASE WITH DENOMINATOR N

Proof of Theorem 1. By applying Lemma 5 with $m = k_\mu = q, r_\mu = \mu, c_\mu = f(\mu)$ for $\mu = 1, \cdots, q$, we obtain

$$S = \sum_{n=1}^{\infty}\frac{f(n)}{n} = \sum_{n=0}^{\infty}\sum_{\mu=1}^{q}\frac{f(\mu)}{nq+\mu} = \sum_{j=1}^{q-1}\beta_j \log(1-\zeta_q^j)$$

where the algebraic number β_j is given by

$$\beta_j = \sum_{\mu=1}^{q-1}\frac{f(\mu)}{q}(1-\zeta_q^{-j\mu}) \text{ for } j = 1, \cdots, q-1. \tag{6}$$

The above expression for S is given in [3] and [9]. By Lemma 1 we have either $S = 0$ or S is transcendental. We apply Lemma 2 to the linear form representation for S. We take $n = q-1, \alpha_j = 1 - \zeta_q^j$ for $1 \leq j \leq q-1$; $\beta_0 = -\alpha$ and β_j for $1 \leq j \leq q-1$ as in (6). Then $D \leq q d_f d_\alpha$ and for $1 \leq j \leq q-1$, $h(\alpha_j) \leq \log 2$, $V_j = 4$. Further, by (2) and (3),

$$h(\beta_j) \leq \log q + \sum_{\mu=1}^{q-1}\{h(f(\mu)) + h(q) + h(1-\zeta_q^{-j\mu})\}$$

$$\leq \log q + h_f + q \log q + q \log 2.$$

Thus $W \leq \max(h_\alpha, h_f + 3q \log q)$. Hence we obtain

$$\log|S-\alpha| > -2^{8q+51}q^{2q}(4qd_\alpha d_f)^{q+2}\{\max(h_\alpha, h_f + 3q\log q) + \log(4eqd_\alpha d_f)\}\log(4eqd_\alpha d_f)$$

$$> -c^q q^{3q}(d_\alpha d_f)^{q+3}\max(h_\alpha, h_f). \quad \square$$

By combining Theorem 1 and Lemma 3, the following result is immediate.

Corollary 1.2. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ with $f \neq 0$ be a periodic function* mod q. *Suppose f*

satisfies conditions (i) and (ii) of Lemma 3 *and* $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ *is convergent. Then* $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ *is transcendental.*

Theorem 3 of [3] states that if $S=0$ and (i) holds, then f is an odd function. It follows that in Corollary 1.2 we can replace condition (ii) by $(ii)^*$: f is not odd. Theorem 2 of [3] provides a simple characterisation of all odd functions $f:\mathbb{Z}\to\overline{\mathbb{Q}}$, periodic mod q, for which $S=0$.

4. THE GENERAL PERIODIC CASE

Corollary 2.1 is a consequence of the following result.

Theorem 2. *Let* $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ *be periodic* mod q. *Let* $Q(X)\in\mathbb{Q}[X]$ *be reduced. If*

$$S=\sum_{n=0}^{\infty}\frac{f(n)}{Q(n)}$$

converges, then $S=0$ *or* $S\notin\overline{\mathbb{Q}}$. *Further, in the latter case we have*

$$\log|S-\alpha|\geq -d_{\alpha}^{c^*_{f,Q}}h_{\alpha}$$

for any algebraic number α *where* $c^*_{f,Q}$ *is a computable number depending only on* f *and* Q.

Proof. We may write

$$Q(X)=a_0\prod_{\mu=1}^{m}(k_\mu X+r_\mu) \text{ with } a_0\in\mathbb{Q}, k_\mu, r_\mu\in\mathbb{Z}, \tag{7}$$
$$\gcd(k_\mu, r_\mu)=1, 0<r_\mu\leq k_\mu \text{ for } 1\leq\mu\leq m$$

and

$$\frac{1}{Q(X)}=\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu X+r_\mu} \text{ with } c_\mu\in\mathbb{Q} \text{ for } 1\leq\mu\leq m.$$

Then

$$1=\sum_{\mu=1}^{m}\frac{c_\mu Q(X)}{k_\mu X+r_\mu}$$

from which it follows that $\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu}=0$, by comparing the coefficients of X^{m-1} on both the sides. Thus

$$S=\sum_{n=0}^{\infty}\frac{f(n)}{Q(n)}=\sum_{k=0}^{\infty}\sum_{\lambda=0}^{q-1}\frac{f(kq+\lambda)}{Q(kq+\lambda)}=\sum_{\lambda=0}^{q-1}f(\lambda)\sum_{k=0}^{\infty}\frac{1}{Q(kq+\lambda)}$$
$$=\sum_{\lambda=0}^{q-1}f(\lambda)\sum_{k=0}^{\infty}\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu kq+k_\mu\lambda+r_\mu}.$$

Each of the inner infinite sums is convergent since $\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu}=0$. Now we have by Lemma 5,

$$S = \sum_{\lambda=0}^{q-1} f(\lambda) \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu - 1} \frac{c_\mu}{k_\mu q}(1 - \zeta_\mu^{-j(k_\mu \lambda + r_\mu)}) \log(1 - \zeta_\mu^j)$$

where ζ_μ is a primitive $k_\mu q$-th root of unity for $1 \le \mu \le m$. Hence

$$S = \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu - 1} \beta_{\mu,j} \log(1 - \zeta_\mu^j)$$

where the algebraic number $\beta_{\mu,j}$ is given by

$$\beta_{\mu,j} = \frac{c_\mu}{k_\mu q} \sum_{\lambda=0}^{q-1} f(\lambda)(1 - \zeta_\mu^{-j(k_\mu \lambda + r_\mu)}).$$

It follows from Lemma 1 that $S = 0$ or $S \notin \overline{\mathbb{Q}}$. As in the proof of Theorem 1, we find by applying Lemma 2, that

$$| S - \alpha | \ge \exp(-d_\alpha^{c_{f,Q}^*} h_\alpha). \quad \square$$

Proof of Corollary 2.1. Let $c_{f,Q,1}, c_{f,Q,2}$, and $c_{f,Q,3}$ be computable numbers depending only on f and Q. Without loss of generality we may assume that

$$(8) \qquad \begin{gathered} Q(X) = a_0 \prod_{\mu=1}^{m} (k_\mu X + s_\mu) \text{ with } a_0 \in \mathbb{Q},\ k_\mu, s_\mu \in \mathbb{Z}, \\ \gcd(k_\mu, s_\mu) = 1 \text{ for } 1 \le \mu \le m. \end{gathered}$$

Put

$$\frac{1}{Q(X)} = \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu X + s_\mu}$$

with $c_\mu \in \mathbb{Q}$ for $1 \le \mu \le m$. Then

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{f(n)}{Q(n)} = \sum_{k=0}^{\infty} \sum_{\lambda=0}^{q-1} \frac{f(kq+\lambda)}{Q(kq+\lambda)} = \sum_{\lambda=0}^{q-1} f(\lambda) \sum_{k=0}^{\infty} \frac{1}{Q(kq+\lambda)} \\ &= \sum_{\lambda=0}^{q-1} f(\lambda) \sum_{k=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu kq + k_\mu \lambda + s_\mu}. \end{aligned}$$

Choose r_μ such that $0 < r_\mu \le k_\mu, r_\mu \equiv s_\mu (\text{mod } k_\mu q)$. Put

$$S^* = \sum_{\lambda=0}^{q-1} f(\lambda) \sum_{k=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu kq + k_\mu \lambda + r_\mu}.$$

Then

$$(9) \qquad S - S^* = \sum_{\lambda=0}^{q-1} f(\lambda) \sum_{\mu=1}^{m} \sum_{k}^{(\lambda,\mu)} \frac{c_\mu}{k_\mu kq + k_\mu \lambda + r_\mu}$$

where the summation $\sum_k^{(\lambda,\mu)}$ extends over $| r_\mu - s_\mu | / k_\mu q$ terms. Hence $S - S^*$ is an algebraic number, β say, of degree at most d_f and absolute logarithmic height at most $c_{f,Q,1}$. We apply Theorem 2 to S^* to find that either S^* is 0 or S^* is transcendental. Hence by (9), we see that either S is a computable algebraic number or S is transcendental. In the latter case, by (9) and Theorem 2, we see by (3) that

$$\begin{aligned} |S-\alpha| &= |S-S^*+S^*-\alpha| = |S^*-\alpha+\beta| \\ &\geq \exp(-d_{\alpha-\beta}^{c_{f,Q,2}} h_{\alpha-\beta}) \\ &\geq \exp(-d_{\alpha}^{c_{f,Q,3}} h_{\alpha}). \quad \square \end{aligned}$$

5. THE CASE OF RATIONAL FUNCTIONS

Corollary 3.1 is a consequence of the following result. Let $Q(X)$ be given by (7). We put $k_Q = \sum_{\mu=1}^{m} k_\mu$.

Theorem 3. *Let $P(X) \in \overline{\mathbb{Q}}[X]$. Let $Q(X) \in \mathbb{Q}[X]$ be reduced. If*

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$$

converges, then $S = 0$ or $S \notin \overline{\mathbb{Q}}$. Further, we have

$$\log |S-\alpha| \geq c_{P,Q} d_\alpha^{k_Q+3} h_\alpha$$

for every algebraic number α where $c_{P,Q}$ is a computable number depending only on P and Q.

Proof. Since $\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$ converges, we may write

$$\frac{P(X)}{Q(X)} = \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu X + r_\mu} \quad \text{with } c_\mu \in \overline{\mathbb{Q}} \text{ for } 1 \leq \mu \leq m$$

such that $\sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu} = 0$. We have, by Lemma 5,

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} = \sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu - 1} \beta_{\mu,j} \log(1-\zeta_\mu^j)$$

where

$$\beta_{\mu,j} = \frac{c_\mu}{k_\mu}(1-\zeta_\mu^{-jr_\mu}).$$

The above expression for S is given in Lehmer [9]. By Lemma 1, we see that $S = 0$ or S is transcendental. The number of logarithms in the linear form representation of S is at most k_Q. Now we apply Lemma 2 as in the proof of Theorem 2 to get the approximation measure for S. $\square$

Proof of Corollary 3.1. Let $Q(X)$ be given by (8). We put

$$\frac{P(X)}{Q(X)} = \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu X + s_\mu} \text{ with } c_\mu \in \overline{\mathbb{Q}} \text{ for } 1 \leq \mu \leq m.$$

Then choosing r_μ such that $r_\mu \equiv s_\mu (\text{mod } k_\mu)$ and $0 < r_\mu \leq k_\mu$, we get

$$S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} = \sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + s_\mu} = \sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu} + \sum_{\mu=1}^{m} \sum_{n}^{(\mu)} \frac{c_\mu}{k_\mu n + r_\mu}$$

where the summation $\sum_n^{(\mu)}$ extends over $| r_\mu - s_\mu | / k_\mu$ values of n. Now we apply Theorem 3 to

$$S^* := \sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu}.$$

Then $S^* = 0$ or $S^* \notin \overline{\mathbb{Q}}$ and in the latter case

$$\log | S^* - \alpha | \geq -c_{P,Q} d_\alpha^{k_Q + 3} h_\alpha$$

for any algebraic number α where $c_{P,Q}$ and the subsequent $c_{P,Q,1}$ are computable numbers depending only on P and Q. On the other hand, $S - S^*$ is an algebraic number of degree and absolute logarithmic height bounded by $c_{P,Q,1}$. Now the result follows as in the proof of Corollary 2.1.

6. THE CASE OF EXPONENTIAL POLYNOMIALS

Corollary 4.1 is a consequence of the following result. Let Q be given by (7) and we define k_Q as in section 5.

Theorem 4. *Let $P_1(X), \cdots, P_\ell(X) \in \overline{\mathbb{Q}}[X]$ and $\alpha_1, \cdots, \alpha_\ell \in \overline{\mathbb{Q}}$. Put $g(X) = \sum_{\lambda=1}^{\ell} P_\lambda(X) \alpha_\lambda^X$. Let $Q(X) \in \mathbb{Q}[X]$ be reduced. If*

$$S = \sum_{n=0}^{\infty} \frac{g(n)}{Q(n)}$$

converges, then $S = 0$ or $S \notin \overline{\mathbb{Q}}$. Further we have

$$\log | S - \alpha | \geq -c_{g,Q} d_\alpha^{\ell k_Q + 3} h_\alpha$$

for any algebraic number α where $c_{g,Q}$ is a computable number depending only on g and Q.

Proof. We may assume without loss of generality that α_λ's are distinct. Let $Q(X)$ be given by (7). For $1 \leq \lambda \leq \ell$, we put

$$\frac{P_\lambda(X)}{Q(X)} = P_\lambda^*(X) + \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}}{k_\mu X + r_\mu}.$$

with $P_\lambda^*(X) \in \overline{\mathbb{Q}}[X], c_{\lambda,\mu} \in \overline{\mathbb{Q}}$ for $1 \leq \mu \leq m$. Then

$$S = \sum_{n=0}^{\infty} \frac{g(n)}{Q(n)} = \sum_{n=0}^{\infty} \sum_{\lambda=1}^{\ell} (P_\lambda^*(n) + \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}}{k_\mu n + r_\mu}) \alpha_\lambda^n.$$

Let U be the set of λ with $1 \leq \lambda \leq \ell$ such that $P_\lambda^* \neq 0$ and $|\alpha_\lambda| \geq 1$. Let $\{\lambda_1, \cdots, \lambda_s\} \subseteq U$ be the set of all j such that

$$|\alpha_j| = \max_\lambda\{|\alpha_\lambda| \ : P_\lambda \neq 0\} =: A \text{ and}$$
$$\deg P_j = \max\{\deg P_\lambda : \ |\alpha_\lambda| = A\} =: B.$$

Let a_λ be the leading coefficient of P_λ^* for $1 \leq \lambda \leq \ell$. Then

$$\sum_{\lambda=1}^{\ell} (P_\lambda^*(n) + \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}}{k_\mu n + r_\mu})\alpha_\lambda^n = \sum_{j=1}^{s} a_{\lambda_j} n^B \alpha_{\lambda_j}^n + o(n^B A^n)$$

as $n \to \infty$. Since $\frac{g(n)}{Q(n)} \to 0$ as $n \to \infty$, we have

$$\sum_{j=1}^{s} a_{\lambda_j} \alpha_{\lambda_j}^n = o(A^n) \text{ as } n \to \infty.$$

Let

$$\begin{array}{ccc} a_{\lambda_1}\alpha_{\lambda_1}^{n+1} & + \cdots + & a_{\lambda_s}\alpha_{\lambda_s}^{n+1} = \epsilon_{n+1} \\ \vdots & & \vdots \\ a_{\lambda_1}\alpha_{\lambda_1}^{n+s} & + \cdots + & a_{\lambda_s}\alpha_{\lambda_s}^{n+s} = \epsilon_{n+s} \end{array}$$

Then $\epsilon_{n+i} = o(A^{n+i})$ as $n \to \infty$ and by Cramer's rule

$$a_{\lambda_j} = \frac{1}{\alpha_{\lambda_j}^{n+1}} \frac{\begin{vmatrix} 1 & \cdots & \epsilon_{n+1} & \cdots & 1 \\ \alpha_{\lambda_1} & \cdots & \epsilon_{n+2} & \cdots & \alpha_{\lambda_s} \\ \vdots & & \vdots & & \vdots \\ \alpha_{\lambda_1}^{s-1} & \cdots & \epsilon_{n+s} & \cdots & \alpha_{\lambda_s}^{s-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \alpha_{\lambda_1} & \cdots & \alpha_{\lambda_j} & \cdots & \alpha_{\lambda_s} \\ \vdots & & \vdots & & \vdots \\ \alpha_{\lambda_1}^{s-1} & \cdots & \alpha_{\lambda_j}^{s-1} & \cdots & \alpha_{\lambda_s}^{s-1} \end{vmatrix}}.$$

The determinant in the numerator is bounded from above by

$$\max_{1 \leq i \leq s} |\epsilon_{n+i}| \, C^{(1)}_{\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_s}}$$

where $C^{(1)}_{\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_s}}$ is a computable number depending only on $\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_s}$. The determinant in the denominator is a van der Monde determinant whose absolute value is bounded from below by a positive number $C^{(2)}_{\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_s}}$ depending only on $\alpha_{\lambda_1}, \cdots, \alpha_{\lambda_s}$. We have $|\alpha_{\lambda_j}|^{-n-1} = A^{-n-1} \leq 1$. Thus

$$a_{\lambda_j} = o(A^s) \text{ as } n \to \infty$$

which implies that $a_{\lambda_j} = 0$ for $1 \leq j \leq s$. Thus $P_\lambda^* = 0$ for every λ with $|\alpha_\lambda| \geq 1$. It follows that

$$(10) \qquad \sum_{n=0}^{\infty} \sum_{\lambda=1}^{\ell} P_\lambda^*(n)\alpha_\lambda^n$$

is absolutely convergent. We write

$$P_\lambda^*(X) = \sum_{k=1}^{t_\lambda} b_k(X+1)\cdots(X+k).$$

Thus

$$(11) \qquad \sum_{n=0}^{\infty} \sum_{\lambda=1}^{\ell} P_\lambda^*(n)\alpha_\lambda^n = \sum_{\lambda=1}^{\ell} \sum_{k=1}^{t_\lambda} b_k \sum_{n=0}^{\infty} (n+1)\cdots(n+k)\alpha_\lambda^n.$$

Since

$$\sum_{n=0}^{\infty} (n+1)\cdots(n+k)\alpha_\lambda^n = (\sum_{n=0}^{\infty} x^{n+k})^{(k)}_{x=\alpha_\lambda} = (\frac{x^k}{1-x})^{(k)}_{x=\alpha_\lambda}$$

represents an algebraic number when $|\alpha_\lambda| < 1$, we obtain from (11) that (10) is an algebraic number ς. Thus we conclude that

$$S = \varsigma + \sum_{n=0}^{\infty} \sum_{\lambda=1}^{\ell} \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}\alpha_\lambda^n}{k_\mu n + r_\mu}.$$

It follows that $|\alpha_\lambda| \leq 1$ when $c_{\lambda,\mu} \neq 0$ for some μ. For λ with $\alpha_\lambda \neq 1$ we choose $\beta_{\lambda,\mu}$ such that $\beta_{\lambda,\mu}^{k_\mu} = \alpha_\lambda$. By Lemma 4 we get, for such λ,

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}\alpha_\lambda^n}{k_\mu n + r_\mu} &= \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}}{\beta_{\lambda,\mu}^{r_\mu}} \sum_{\substack{n=1 \\ n \equiv r_\mu (\mathrm{mod}\, k_\mu)}}^{\infty} \frac{\beta_{\lambda,\mu}^n}{n} \\
&= -\sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}}{k_\mu \beta_{\lambda,\mu}^{r_\mu}} \sum_{j=0}^{k_\mu - 1} \zeta_{k_\mu}^{-jr_\mu} \log(1 - \beta_{\lambda,\mu}\zeta_{k_\mu}^j) \\
&= \sum_{\mu=1}^{m} \sum_{j=0}^{k_\mu - 1} \beta_{j,\lambda,\mu} \log(1 - \beta_{\lambda,\mu}\zeta_{k_\mu}^j)
\end{aligned}$$

with $\beta_{j,\lambda,\mu} = -\frac{c_{\lambda,\mu}}{k_\mu}\beta_{\lambda,\mu}^{-r_\mu}\zeta_{k_\mu}^{-jr_\mu}$ where ζ_{k_μ} is a primitive k_μ-th root of unity. If $\alpha_\lambda = 1$, we apply Lemma 5 to obtain

$$\sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_{\lambda,\mu}\alpha_\lambda^n}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} \sum_{j=0}^{k_\mu - 1} \beta_{j,\lambda,\mu} \log(1 - \zeta_{k_\mu}^j)$$

where $\beta_{j,\lambda,\mu} = (c_{\lambda,\mu}/k_\mu)(1 - \zeta_{k_\mu}^{-jr_\mu})$. Thus $S - \varsigma$ can be written as a linear form in at most $\sum_{\lambda=1}^{\ell} \sum_{\mu=1}^{m} k_\mu = \ell k_Q$ logarithms of algebraic numbers with algebraic coefficients. By Lemma 1 we see that $S = 0$ or $S \notin \overline{\mathbb{Q}}$. We apply Lemma 2 as earlier to obtain the approximation measure for S. □

Proof of Corollary 4.1. The proof is similar to the proof of Corollaries 2.1 and 3.1. □

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