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Some multiple hypergeometric transformations and associated reduction formulas

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Abstract

The main object of the present paper is to derive various classes of double-series identities and to show how these general results would apply to yield some (known or new) reduction formulas for the Appell, Kampé de Fériet, and Lauricella hypergeometric functions of several variables. A number of closely-related linear generating functions for the classical Jacobi polynomials are also investigated.
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1. Introduction and definitions

As usual, we denote by ${}_pF_q$ a generalized hypergeometric function with p numerator and q denominator parameters, defined by [4, Chapter 4]

$$ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) = {}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right] $$

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$$:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!}$$
$$\big(p, q \in \mathbb{N}_0;\ p \leqq q+1;\ p \leqq q \text{ and } |z| < \infty;$$
$$p = q+1 \text{ and } |z| < 1;\ p = q+1,\ |z| = 1, \text{ and } \Re(\omega) > 0\big), \tag{1}$$

where (*and in what follows*) $(\lambda)_\nu$ is the Pochhammer symbol (or the *shifted* factorial, since $(1)_n = n!$ for $n \in \mathbb{N}_0$) given (for $\lambda, \nu \in \mathbb{C}$ and in terms of the familiar gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{2}$$

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ $(\mathbb{N} := \{1, 2, 3, \dots\})$, and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$
$$\big(\beta_j \notin \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\};\ \mathbb{Z}^- := \{-1, -2, -3, \dots\};\ j = 1, \dots, q\big). \tag{3}$$

We also let

$$F^{p:\, r;u}_{q:\, s;v} \quad (p, q, r, s, u, v \in \mathbb{N}_0)$$

denote a general (Kampé de Fériet's) double hypergeometric function defined by (cf., e.g., [9, Eq. 1.3(28), p. 27]; see also [1, p. 150 et seq.])

$$F^{p:\, r;u}_{q:\, s;v}\left[\begin{matrix} \alpha_1, \dots, \alpha_p\colon a_1, \dots, a_r;\ c_1, \dots, c_u; \\ \\ \beta_1, \dots, \beta_q\colon b_1, \dots, b_s;\ d_1, \dots, d_v; \end{matrix}\ x, y\right]$$
$$:= \sum_{l,m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_{l+m} \prod_{j=1}^{r} (a_j)_l \prod_{j=1}^{u} (c_j)_m}{\prod_{j=1}^{q} (\beta_j)_{l+m} \prod_{j=1}^{s} (b_j)_l \prod_{j=1}^{v} (d_j)_m} \frac{x^l}{l!} \frac{y^m}{m!}, \tag{4}$$

where, for convergence of the double hypergeometric series,

$$p + r \leqq q + s + 1 \quad \text{and} \quad p + u \leqq q + v + 1,$$

with equality only when $|x|$ and $|y|$ are appropriately constrained (see, for details, [9, Eq. 1.3(29), p. 27]).

The main object of this paper is first to present several classes of double-series identities and to show how these general results would apply to yield some (known or new) reduction formulas for the Kampé de Fériet function defined by (4). We then consider a (presumably new) reduction formula for the multivariable Lauricella function $F_D^{(n)}$ defined by (cf. [6, p. 113] and [9, Eq. 1.4(4), p. 33]; see also [1, Eq. (4), p. 114])

$$F_D^{(n)}[a, b_1, \dots, b_n; c; z_1, \dots, z_n]$$
$$:= \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!}$$
$$\big(\max\{|z_1|, \dots, |z_n|\} < 1;\ c \notin \mathbb{Z}_0^-\big), \tag{5}$$

so that, obviously,

$$F_D^{(2)}[\alpha,\beta,\beta';\gamma;x,y] =: F_1[\alpha,\beta,\beta';\gamma;x,y] \quad \bigl(\max\bigl\{|x|,|y|\bigr\}<1\bigr), \tag{6}$$

where F_1 denotes the relatively more familiar Appell function of the first kind [1, p. 14]. We also investigate several closely-related linear generating functions for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of order (or indices) (α,β) and degree n in x, defined by (cf., e.g., [11, Chapter 4])

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \tag{7}$$

or, equivalently, by

$$P_n^{(\alpha,\beta)}(x) := \binom{n+\alpha}{n}\,{}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2}\right) \tag{8}$$

in terms of the Gauss hypergeometric ${}_2F_1$ function defined by (1) with

$$p-1=q=1.$$

2. Double-series identities

We begin by recalling the following general double-series identity:

Lemma 1 (Buschman and Srivastava [3, Theorem 3, p. 437]). *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega(m+n)(\lambda)_m(\lambda)_n(\mu)_m(\mu)_n \frac{x^m}{m!}\frac{y^n}{n!} = \sum_{m,n=0}^{\infty} \Omega(m+2n)\frac{(\lambda+\mu)_{m+2n}(\lambda)_{m+n}(\mu)_{m+n}}{(\lambda+\mu)_{m+n}}\frac{(x+y)^m}{m!}\frac{(-xy)^n}{n!} \quad \bigl(\lambda+\mu\notin\mathbb{Z}_0^-\bigr), \tag{9}$$

provided that each of the double series involved is absolutely convergent.

Upon replacing x and y in (9) by x/μ and y/μ, respectively, if we let $|\mu|\to\infty$, we obtain the double-series identity given by

Lemma 2 (cf. Buschman and Srivastava [3, Eq. (2.13), p. 438]). *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega(m+n)(\lambda)_m(\lambda)_n \frac{x^m}{m!}\frac{y^n}{n!}$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+2n)(\lambda)_{m+n} \frac{(x+y)^m}{m!}\frac{(-xy)^n}{n!}, \tag{10}$$

provided that each of the double series involved is absolutely convergent.

Lemma 2 would follow also as a *confluent* case of another double-series identity of Buschman and Srivastava [3, Theorem 2, p. 437]:

$$\sum_{m,n=0}^{\infty} \Omega(m+n)\frac{(\lambda)_m(\lambda)_n}{(\nu)_m(\nu)_n} \frac{x^m}{m!}\frac{y^n}{n!}$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+2n)\frac{(\lambda)_{m+n}(\nu-\lambda)_n}{(\nu)_{m+2n}(\nu)_n} \frac{(x+y)^m}{m!}\frac{(-xy)^n}{n!} \quad \big(\nu \notin \mathbb{Z}_0^-\big) \tag{11}$$

upon replacing x and y by νx and νy, respectively, and letting $|\nu| \to \infty$.

A *further* special case of the double-series identity (10) when $y = -x$ yields the following simpler form:

$$\sum_{m,n=0}^{\infty} \Omega(m+n)(\lambda)_m(\lambda)_n \frac{x^m}{m!}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \Omega(2n)(\lambda)_n \frac{x^{2n}}{n!}, \tag{12}$$

which was given earlier by Srivastava [7, Eq. (17), p. 297]. An interesting companion of the double-series identity (10), which is asserted by Lemma 2, is contained in

Lemma 3. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega(m+n)\frac{(\alpha)_{m+2n}}{(\gamma)_{m+2n}} \frac{x^m}{m!}\frac{y^n}{n!}$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+n)\frac{(\alpha)_{m+2n}(\gamma-\alpha)_m}{(\gamma)_{2m+2n}} \frac{x^m}{m!}\frac{(x+y)^n}{n!} \quad \big(\gamma \notin \mathbb{Z}_0^-\big), \tag{13}$$

provided that each of the double series involved is absolutely convergent.

Proof. Denoting, for convenience, the first member of the assertion (13) by $\mathcal{S}(x, y)$, if we set

$$m = N - n \quad (0 \leqq n \leqq N;\ N \in \mathbb{N}_0),$$

we readily obtain

$$\mathcal{S}(x, y) = \sum_{N=0}^{\infty} \Omega(N)\frac{(\alpha)_N}{(\gamma)_N}\frac{x^N}{N!}\, {}_2F_1\left(-N, \alpha+N; \gamma+N; -\frac{y}{x}\right). \tag{14}$$

Now we apply the following polynomial identity [4, Eq. 2.10(3) with $a = -n$ $(n \in \mathbb{N}_0)$, p. 109]:

$$
{}_2F_1(-n, b; c; z) = \frac{(b)_n}{(c)_n}(1-z)^n \, {}_2F_1\left(-n, c-b; 1-b-n; \frac{1}{1-z}\right) \qquad (n \in \mathbb{N}_0) \tag{15}
$$

to the hypergeometric polynomial occurring in (14). We thus find eventually that

$$
\mathcal{S}(x, y) = \sum_{N=0}^{\infty} \Omega(N) \sum_{m=0}^{N} \frac{(\alpha)_{2N-m}(\gamma-\alpha)_m}{(\gamma)_{2N}} \frac{x^m}{m!} \frac{(x+y)^{N-m}}{(N-m)!}, \tag{16}
$$

which, upon inversion of the order of summation, immediately yields the second member of the assertion (13) by setting $N = m + n$.

Alternatively, since (cf., e.g., [10, Eq. 1.4(3), p. 42])

$$
{}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; z\right] = \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n}(-z)^n \, {}_{q+1}F_p\left[\begin{matrix} -n, 1-\beta_1-n, \ldots, 1-\beta_q-n; \\ 1-\alpha_1-n, \ldots, 1-\alpha_p-n; \end{matrix} \; \frac{(-1)^{p+q}}{z}\right], \tag{17}
$$

where we have merely reversed the order of the terms of the hypergeometric polynomial by appealing to the following simple consequence of the definition (2):

$$
(\lambda)_{n-k} = \frac{(-1)^k (\lambda)_n}{(1-\lambda-n)_k} \qquad (k = 0, 1, \ldots, n;\; n \in \mathbb{N}_0), \tag{18}
$$

it is readily seen from (14) that

$$
\mathcal{S}(x, y) = \sum_{N=0}^{\infty} \Omega(N) \frac{(\alpha)_{2N}}{(\gamma)_{2N}} \frac{y^N}{N!} \, {}_2F_1\left(-N, 1-\gamma-2N; 1-\alpha-2N; -\frac{x}{y}\right). \tag{19}
$$

Thus, by applying the following Pfaff–Kummer transformation [4, Eq. 2.1.4(22), p. 64]:

$$
{}_2F_1(a, b; c; z) = (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \qquad \left(\left|\arg(1-z)\right| \leqq \pi - \varepsilon;\; 0 < \varepsilon < \pi;\; c \notin \mathbb{Z}_0^-\right), \tag{20}
$$

instead of the polynomial identity (15), we find from (19) that

$$
\mathcal{S}(x, y) = \sum_{N=0}^{\infty} \Omega(N) \frac{(\alpha)_{2N}}{(\gamma)_{2N}} \frac{(x+y)^N}{N!} \, {}_2F_1\left(-N, \gamma-\alpha; 1-\alpha-2N; \frac{x}{x+y}\right), \tag{21}
$$

which would yield the second member of the assertion (13) just as we have already indicated in connection with (16) above. □

For $y=-x$, Lemma 3 immediately yields the following companion of the known reduction formula (12):

$$\sum_{m,n=0}^{\infty} \Omega(m+n)\frac{(\alpha)_{m+2n}}{(\gamma)_{m+2n}}\frac{x^m}{m!}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \Omega(n)\frac{(\alpha)_n(\gamma-\alpha)_n}{(\gamma)_{2n}}\frac{x^n}{n!} \quad \left(\gamma \notin \mathbb{Z}_0^-\right), \tag{22}$$

which does not seem to have been recorded earlier.

In terms of generalized hypergeometric functions defined by (1), Lemmas 2 and 3 with

$$\Omega(n) = \frac{(\rho_1)_n \dots (\rho_p)_n}{(\sigma_1)_n \dots (\sigma_q)_n} \quad (n \in \mathbb{N}_0) \tag{23}$$

can be reduced to the following hypergeometric forms:

$$F^{p:\,1;1}_{q:\,0;0}\left[\begin{matrix}\rho_1,\dots,\rho_p\colon & \lambda; & \lambda; \\ & & & x,y \\ \sigma_1,\dots,\sigma_q\colon & \text{---}; & \text{---}; \end{matrix}\right] = \sum_{m=0}^{\infty} \frac{(\rho_1)_m \dots (\rho_p)_m(\lambda)_m}{(\sigma_1)_m \dots (\sigma_q)_m}\frac{(x+y)^m}{m!} \cdot {}_{2p+1}F_{2q}\left[\begin{matrix}\lambda+m,\Delta(2;\rho_1+m),\dots,\Delta(2;\rho_p+m); \\ & -4^{p-q}xy \\ \Delta(2;\sigma_1+m),\dots,\Delta(2;\sigma_q+m); \end{matrix}\right], \tag{24}$$

which, in the special case when $y=-x$, yields the following known result [3, Eq. (3.3), p. 439]:

$$F^{p:\,1;1}_{q:\,0;0}\left[\begin{matrix}\rho_1,\dots,\rho_p\colon & \lambda; & \lambda; \\ & & & x,-x \\ \sigma_1,\dots,\sigma_q\colon & \text{---}; & \text{---}; \end{matrix}\right] = {}_{2p+1}F_{2q}\left[\begin{matrix}\lambda,\Delta(2;\rho_1),\dots,\Delta(2;\rho_p); \\ & 4^{p-q}x^2 \\ \Delta(2;\sigma_1),\dots,\Delta(2;\sigma_q); \end{matrix}\right], \tag{25}$$

where, *and elsewhere in this paper*, $\Delta(N;\lambda)$ denotes the array of N parameters

$$\frac{\lambda+j-1}{N} \quad (j=1,\dots,N;\ N\in\mathbb{N});$$

$$\sum_{m=0}^{\infty} \frac{(\alpha)_m(\rho_1)_m \dots (\rho_p)_m}{(\gamma)_m(\sigma_1)_m \dots (\sigma_q)_m}\frac{x^m}{m!}\, {}_{p+2}F_{q+2}\left[\begin{matrix}\Delta(2;\alpha+m),\rho_1+m,\dots,\rho_p+m; \\ & y \\ \Delta(2;\gamma+m),\sigma_1+m,\dots,\sigma_q+m; \end{matrix}\right] = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\gamma-\alpha)_m(\rho_1)_m \dots (\rho_p)_m}{(\gamma)_{2m}(\sigma_1)_m \dots (\sigma_q)_m}\frac{x^m}{m!}$$

$$\cdot\ {}_{p+2}F_{q+2}\left[\begin{matrix}\Delta(2;\alpha+m),\rho_1+m,\ldots,\rho_p+m;\\ \\ \Delta(2;\gamma+m),\sigma_1+m,\ldots,\sigma_q+m;\end{matrix}\ x+y\right], \tag{26}$$

which, in the special case when $y=-x$, yields the following (presumably new) reduction formula:

$$\sum_{m=0}^{\infty}\frac{(\alpha)_m(\rho_1)_m\ldots(\rho_p)_m}{(\gamma)_m(\sigma_1)_m\ldots(\sigma_q)_m}\frac{x^m}{m!}\,{}_{p+2}F_{q+2}\left[\begin{matrix}\Delta(2;\alpha+m),\rho_1+m,\ldots,\rho_p+m;\\ \\ \Delta(2;\gamma+m),\sigma_1+m,\ldots,\sigma_q+m;\end{matrix}\ -x\right]$$
$$={}_{p+2}F_{q+2}\left[\begin{matrix}\alpha,\gamma-\alpha,\rho_1,\ldots,\rho_p;\\ \\ \Delta(2;\gamma),\sigma_1,\ldots,\sigma_q;\end{matrix}\ \frac{1}{4}x^2\right]. \tag{27}$$

3. Further applications of Lemmas 2 and 3

For

$$\lambda=\beta\quad\text{and}\quad\Omega(n)=\frac{(\alpha)_n}{(\gamma)_n}\quad(n\in\mathbb{N}_0), \tag{28}$$

Lemma 2 yields the following special case:

$$\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+2n}(\beta)_{m+n}}{(\gamma)_{m+2n}}\frac{(x+y)^m}{m!}\frac{(-xy)^n}{n!}. \tag{29}$$

On the other hand, if we set

$$\Omega(n)=(\beta)_n\quad(n\in\mathbb{N}_0), \tag{30}$$

we find from the assertion (13) of Lemma 3 that

$$\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+2n}(\beta)_{m+n}}{(\gamma)_{m+2n}}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+2n}(\beta)_{m+n}(\gamma-\alpha)_m}{(\gamma)_{2m+2n}}\frac{x^m}{m!}\frac{(x+y)^n}{n!}. \tag{31}$$

Thus, by combining the double-series identities (29) and (31), we obtain

$$\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+2n}(\beta)_{m+n}(\gamma-\alpha)_m}{(\gamma)_{2m+2n}}\frac{(x+y)^m}{m!}\frac{(x+y-xy)^n}{n!} \tag{32}$$

or, equivalently,

$$F_1[\alpha,\beta,\beta;\gamma;x,y]=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m(\gamma-\alpha)_m}{(\gamma)_{2m}}\frac{(x+y)^m}{m!}\cdot{}_3F_2\left[\begin{matrix}\Delta(2;\alpha+m),\beta+m;\\ \Delta(2;\gamma+2m);\end{matrix}\,x+y-xy\right] \tag{33}$$

in terms of the Appell function F_1 defined by (6).

A *direct* proof of (33), based upon the following quadratic transformation [4, Eq. 2.11 (34), p. 113]:

$${}_2F_1(a,b;a-b+1;z)=(1+z)^{-a}\,{}_2F_1\left(\Delta(2;a);a-b+1;\frac{4z}{(1+z)^2}\right), \tag{34}$$

was attributed to the referee of their paper by Ismail and Pitman [5, Eq. (95), p. 979].

By setting

$$y=\frac{x}{x-1}\quad(\text{so that } x+y-xy=0),$$

we find from (33) that (cf. [5, Eq. (96), p. 979])

$$F_1\left[\alpha,\beta,\beta;\gamma;x,\frac{x}{x-1}\right]={}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma-\alpha;\\ \Delta(2;\gamma);\end{matrix}\,\frac{x^2}{4(x-1)}\right], \tag{35}$$

whose *further* special cases when $\gamma=2\alpha$ and when $\gamma=2\alpha-1$ were actually proven and applied by Ismail and Pitman [5] in their algebraic evaluations of some symmetric Euler integrals related to Brownian variations.

For $y=-x$, (33) reduces immediately to the following simple reduction formula:

$$F_1[\alpha,\beta,\beta;\gamma;x,-x]={}_3F_2\left[\begin{matrix}\Delta(2;\alpha),\beta;\\ \Delta(2;\gamma);\end{matrix}\,x^2\right], \tag{36}$$

which indeed is contained, as a very specialized case, in each of the known results (12) and (25) (see also [10, Paragraph 1, p. 56]).

Next, for the multivariable Lauricella function $F_D^{(n)}$ defined by (5), we derive the following transformation formula:

$$F_D^{(2n)}[\alpha,\beta_1,\dots,\beta_n,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n,y_1,\dots,y_n]$$
$$=\sum_{l_1,m_1,\dots,l_n,m_n=0}^{\infty}\frac{(\alpha)_{L+2M}(\gamma-\alpha)_L(\beta_1)_{l_1+m_1}\dots(\beta_n)_{l_n+m_n}}{(\gamma)_{2L+2M}}\cdot\prod_{j=1}^{n}\left\{\frac{(x_j+y_j)^{l_j}}{l_j!}\frac{(x_j+y_j-x_jy_j)^{m_j}}{m_j!}\right\}$$
$$\left(\gamma\notin\mathbb{Z}_0^-;\ L:=l_1+\dots+l_n;\ M:=m_1+\dots+m_n;\ n\in\mathbb{N}\right) \tag{37}$$

or, equivalently,

$$\begin{aligned}
&F_D^{(2n)}[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n, y_1, \ldots, y_n] \\
&\quad = \sum_{l_1,\ldots,l_n=0}^{\infty} \frac{(\alpha)_L(\gamma-\alpha)_L(\beta_1)_{l_1}\ldots(\beta_n)_{l_n}}{(\gamma)_{2L}} \prod_{j=1}^{n} \left\{ \frac{(x_j+y_j)^{l_j}}{l_j!} \right\} \\
&\qquad \cdot F_{2:\,0;\ldots;0}^{2:\,1;\ldots;1} \left[\begin{matrix} \Delta(2;\alpha+L): & \beta_1+l_1; & \ldots; & \beta_n+l_n; \\ \Delta(2;\gamma+2L): & \text{———}; & \ldots; & \text{———}; \end{matrix} \; z_1, \ldots, z_n \right] \\
&\quad \bigl(\gamma \notin \mathbb{Z}_0^-;\ L := l_1 + \cdots + l_n;\ z_j := x_j + y_j - x_j y_j;\ j = 1, \ldots, n\bigr),
\end{aligned} \tag{38}$$

where the notation used for the generalized Lauricella function in n variables (occurring on the right-hand side) is analogous to that in the two-variable case (4) (see, for details, [9, Eq. 1.4(24), p. 38]).

Proof. By applying the definitions (5) and (6), we readily have

$$\begin{aligned}
&F_D^{(2n)}[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n, y_1, \ldots, y_n] \\
&\quad = \sum_{l_2,m_2,\ldots,l_n,m_n=0}^{\infty} \frac{(\alpha)_{L^*+M^*}(\beta_2)_{l_2}(\beta_2)_{m_2}\ldots(\beta_n)_{l_n}(\beta_n)_{m_n}}{(\gamma)_{L^*+M^*}} \\
&\qquad \cdot F_1\bigl(\alpha+L^*+M^*, \beta_1, \beta_1; \gamma+L^*+M^*; x_1, y_1\bigr) \prod_{j=2}^{n} \left\{ \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\},
\end{aligned} \tag{39}$$

where, for convenience,

$$L^* := L - l_1 = l_2 + \cdots + l_n \quad \text{and} \quad M^* := M - m_1 = m_2 + \cdots + m_n.$$

Now, making use of (32) or (33) in (39), we obtain

$$\begin{aligned}
&F_D^{(2n)}[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; x_1, \ldots, x_n, y_1, \ldots, y_n] \\
&\quad = \sum_{l,m=0}^{\infty} \frac{(\alpha)_{l+2m}(\beta_1)_{l+m}(\gamma-\alpha)_l}{(\gamma)_{2l+2m}} \frac{(x_1+y_1)^l}{l!} \frac{(x_1+y_1-x_1y_1)^m}{m!} \\
&\qquad \cdot F_D^{(2n-2)}[\alpha+l+2m, \beta_2, \ldots, \beta_n, \beta_2, \ldots, \beta_n; \\
&\qquad\qquad \gamma+2l+2m; x_2, \ldots, x_n, y_2, \ldots, y_n].
\end{aligned} \tag{40}$$

Finally, the desired transformation formula (37) would result upon iterating this process to the other variable pairs $(x_2, y_2), \ldots, (x_n, y_n)$.

Alternatively, since (32) is precisely the case $n = 1$ of the transformation formula (37), it is fairly straightforward to construct a proof of (37) by the principle of mathematical induction on $n \in \mathbb{N}$. Indeed, by assuming (37) to hold true for some fixed positive integer n, we find from the definition (5) that

$$F_D^{(2n+2)}[\alpha, \beta_1, \ldots, \beta_{n+1}, \beta_1, \ldots, \beta_{n+1}; \gamma; x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]$$
$$= \sum_{l,m=0}^{\infty} \frac{(\alpha)_{l+m}(\beta_{n+1})_l(\beta_{n+1})_m}{(\gamma)_{l+m}} \frac{x_{n+1}^l}{l!} \frac{y_{n+1}^m}{m!}$$
$$\cdot F_D^{(2n)}[\alpha + l + m, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; \gamma + l + m; x_1, \ldots, x_n, y_1, \ldots, y_n]$$
$$= \sum_{l,m=0}^{\infty} \frac{(\alpha)_{l+m}(\beta_{n+1})_l(\beta_{n+1})_m}{(\gamma)_{l+m}} \frac{x_{n+1}^l}{l!} \frac{y_{n+1}^m}{m!}$$
$$\cdot \sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \frac{(\alpha + l + m)_{L+2M}(\gamma - \alpha)_L(\beta_1)_{l_1+m_1} \ldots (\beta_n)_{l_n+m_n}}{(\gamma + l + m)_{2L+2M}}$$
$$\cdot \prod_{j=1}^{n} \left\{ \frac{(x_j + y_j)^{l_j}}{l_j!} \frac{(x_j + y_j - x_j y_j)^{m_j}}{m_j!} \right\}, \tag{41}$$

where we have also applied the transformation formula (37).

By rearranging the multiple series in (41) and using the definition (6), we get

$$F_D^{(2n+2)}[\alpha, \beta_1, \ldots, \beta_{n+1}, \beta_1, \ldots, \beta_{n+1}; \gamma; x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]$$
$$= \sum_{l_1,m_1,\ldots,l_n,m_n=0}^{\infty} \frac{(\alpha)_{L+2M}(\gamma - \alpha)_L(\beta_1)_{l_1+m_1} \ldots (\beta_n)_{l_n+m_n}}{(\gamma)_{2L+2M}}$$
$$\cdot \prod_{j=1}^{n} \left\{ \frac{(x_j + y_j)^{l_j}}{l_j!} \frac{(x_j + y_j - x_j y_j)^{m_j}}{m_j!} \right\}$$
$$\cdot \sum_{l_{n+1},m_{n+1}=0}^{\infty} \frac{(\alpha + L + 2M)_{l_{n+1}+2m_{n+1}}(\beta_{n+1})_{l_{n+1}+m_{n+1}}(\gamma - \alpha + L)_{l_{n+1}}}{(\gamma + 2L + 2M)_{2l_{n+1}+2m_{n+1}}}$$
$$\cdot \frac{(x_{n+1} + y_{n+1})^{l_{n+1}}}{l_{n+1}!} \frac{(x_{n+1} + y_{n+1} - x_{n+1}y_{n+1})^{m_{n+1}}}{m_{n+1}!}, \tag{42}$$

by means of the double-series identity (32).

Now we rearrange the multiple series in (42). We thus find from (42) that

$$F_D^{(2n+2)}[\alpha, \beta_1, \ldots, \beta_{n+1}, \beta_1, \ldots, \beta_{n+1}; \gamma; x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]$$
$$= \sum_{l_1,m_1,\ldots,l_{n+1},m_{n+1}=0}^{\infty} \frac{(\alpha)_{L^\dagger+2M^\dagger}(\gamma - \alpha)_{L^\dagger}(\beta_1)_{l_1+m_1} \ldots (\beta_{n+1})_{l_{n+1}+m_{n+1}}}{(\gamma)_{2L^\dagger+2M^\dagger}}$$
$$\cdot \prod_{j=1}^{n+1} \left\{ \frac{(x_j + y_j)^{l_j}}{l_j!} \frac{(x_j + y_j - x_j y_j)^{m_j}}{m_j!} \right\}$$
$$\big(L^\dagger := L + l_{n+1} = l_1 + \cdots + l_{n+1};$$
$$M^\dagger := M + m_{n+1} = m_1 + \cdots + m_{n+1}\big), \tag{43}$$

which is precisely the transformation formula (37) with n replaced by $n+1$ ($n \in \mathbb{N}$). This evidently completes our alternative proof of (37) by the principle of mathematical induction on $n \in \mathbb{N}$. □

By setting

$$y_j = \frac{x_j}{x_j - 1} \quad (\text{so that } x_j + y_j - x_j y_j = 0)\ (j = 1, \dots, n)$$

in (37), we obtain

$$F_D^{(2n)}\left[\alpha, \beta_1, \dots, \beta_n, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n, \frac{x_1}{x_1 - 1}, \dots, \frac{x_n}{x_n - 1}\right]$$
$$= \sum_{l_1, \dots, l_n = 0}^{\infty} \frac{(\alpha)_L (\gamma - \alpha)_L (\beta_1)_{l_1} \dots (\beta_n)_{l_n}}{(\gamma)_{2L}} \prod_{j=1}^{n} \left\{ \frac{[x_j^2/(x_j - 1)]^{l_j}}{l_j!} \right\}$$
$$(L := l_1 + \dots + l_n), \tag{44}$$

that is,

$$F_D^{(2n)}\left[\alpha, \beta_1, \dots, \beta_n, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n, \frac{x_1}{x_1 - 1}, \dots, \frac{x_n}{x_n - 1}\right]$$
$$= F_{2:\,0;\dots;0}^{2:\,1;\dots;1}\left[\begin{matrix} \alpha, \gamma - \alpha: & \beta_1; & \dots; & \beta_n; \\ \Delta(2; \gamma): & \text{---}; & \dots; & \text{---}; \end{matrix} \ \frac{x_1^2}{4(x_1 - 1)}, \dots, \frac{x_n^2}{4(x_n - 1)}\right] \tag{45}$$

in terms of a generalized Lauricella function in n variables, which occurs already in our general result (38).

For

$$\gamma = 2\alpha - 1, \qquad \gamma = 2\alpha, \quad \text{and} \quad \gamma = 2\alpha + 1,$$

each of our results (44) and (45) yields the following *further* special cases:

$$F_D^{(2n)}\left[\alpha, \beta_1, \dots, \beta_n, \beta_1, \dots, \beta_n; 2\alpha - 1; x_1, \dots, x_n, \frac{x_1}{x_1 - 1}, \dots, \frac{x_n}{x_n - 1}\right]$$
$$= F_D^{(n)}\left[\alpha - 1, \beta_1, \dots, \beta_n; \alpha - \frac{1}{2}; \frac{x_1^2}{4(x_1 - 1)}, \dots, \frac{x_n^2}{4(x_n - 1)}\right], \tag{46}$$

$$F_D^{(2n)}\left[\alpha, \beta_1, \dots, \beta_n, \beta_1, \dots, \beta_n; 2\alpha; x_1, \dots, x_n, \frac{x_1}{x_1 - 1}, \dots, \frac{x_n}{x_n - 1}\right]$$
$$= F_D^{(n)}\left[\alpha, \beta_1, \dots, \beta_n; \alpha + \frac{1}{2}; \frac{x_1^2}{4(x_1 - 1)}, \dots, \frac{x_n^2}{4(x_n - 1)}\right], \tag{47}$$

and

$$F_D^{(2n)}\left[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha+1; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right]$$
$$= F_D^{(n)}\left[\alpha, \beta_1, \ldots, \beta_n; \alpha+\frac{1}{2}; \frac{x_1^2}{4(x_1-1)}, \ldots, \frac{x_n^2}{4(x_n-1)}\right], \tag{48}$$

respectively.

Furthermore, since (cf. [6, p. 148]; see also [1, p. 116])

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; z_1, \ldots, z_n]$$
$$= (1-z_1)^{-b_1} \ldots (1-z_n)^{-b_n} F_D^{(n)}\left[c-a, b_1, \ldots, b_n; c; \frac{z_1}{z_1-1}, \ldots, \frac{z_n}{z_n-1}\right], \tag{49}$$

the reduction formula (48) can easily be rewritten in its *equivalent* form (cf. Eq. (46)):

$$F_D^{(2n)}\left[\alpha+1, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha+1; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right]$$
$$= F_D^{(n)}\left[\alpha, \beta_1, \ldots, \beta_n; \alpha+\frac{1}{2}; \frac{x_1^2}{4(x_1-1)}, \ldots, \frac{x_n^2}{4(x_n-1)}\right], \tag{50}$$

which, when compared with (47), yields the following multiple hypergeometric identity:

$$F_D^{(2n)}\left[\alpha+1, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha+1; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right]$$
$$= F_D^{(2n)}\left[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right]. \tag{51}$$

In fact, by comparing (47) and (48), we similarly obtain the following multiple hypergeometric identity:

$$F_D^{(2n)}\left[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right]$$
$$= F_D^{(2n)}\left[\alpha, \beta_1, \ldots, \beta_n, \beta_1, \ldots, \beta_n; 2\alpha+1; x_1, \ldots, x_n, \frac{x_1}{x_1-1}, \ldots, \frac{x_n}{x_n-1}\right], \tag{52}$$

which, in view of (49), is the same as (51).

The *special* reduction formula (47) was derived earlier by Ismail and Pitman [5, Eq. (21), p. 965], who also gave the simpler cases of the identities (50) and (51) when $n = 2$ (cf. [5, Eqs. (93) and (94), p. 978]).

For a *direct* proof of the equivalent multiple hypergeometric identities (51) and (52), *without* using such reduction formulas as (47), (48), and (50), we choose to make use of the following recurrence relation for the Lauricella function $F_D^{(n)}$ defined by (5):

$$\begin{aligned}(\gamma-\alpha)F_D^{(n)}&[\alpha,\beta_1,\ldots,\beta_n;\gamma+1;z_1,\ldots,z_n]\\ &=\gamma F_D^{(n)}[\alpha,\beta_1,\ldots,\beta_n;\gamma;z_1,\ldots,z_n]\\ &\quad-\alpha F_D^{(n)}[\alpha+1,\beta_1,\ldots,\beta_n;\gamma+1;z_1,\ldots,z_n],\end{aligned}\tag{53}$$

which can be derived fairly easily by appealing to the familiar result [4, Eq. 2.8(35), p. 103]:

$$(c-a)\,{}_2F_1(a,b;c+1;z)=c\,{}_2F_1(a,b;c;z)-a\,{}_2F_1(a+1,b;c+1;z)\tag{54}$$

for the Gauss hypergeometric function. The case $n=2$ of (53) was indeed recorded by, for example, Appell and Kampé de Fériet [1, p. 33].

Now, in view of (49), we find from (53) (with n replaced by $2n$) that

$$\begin{aligned}\gamma F_D^{(2n)}&[\alpha,\beta_1,\ldots,\beta_n,\delta_1,\ldots,\delta_n;\gamma;x_1,\ldots,x_n,y_1,\ldots,y_n]\\ &=(\gamma-\alpha)F_D^{(2n)}[\alpha,\beta_1,\ldots,\beta_n,\delta_1,\ldots,\delta_n;\gamma+1;x_1,\ldots,x_n,y_1,\ldots,y_n]\\ &\quad+\alpha\prod_{j=1}^{n}\left\{(1-x_j)^{-\beta_j}(1-y_j)^{-\delta_j}\right\}\\ &\quad\cdot F_D^{(2n)}\biggl[\gamma-\alpha,\delta_1,\ldots,\delta_n,\beta_1,\ldots,\beta_n;\gamma+1;\\ &\qquad\frac{y_1}{y_1-1},\ldots,\frac{y_n}{y_n-1},\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\biggr],\end{aligned}\tag{55}$$

where we have also made use of the obvious correspondence between the sets

$$\{b_1,\ldots,b_n\}\quad\text{and}\quad\{z_1,\ldots,z_n\}$$

in the definition (5).

Upon setting

$$\gamma=2\alpha,\qquad \delta_j=\beta_j,\quad\text{and}\quad y_j=\frac{x_j}{x_j-1}\quad(j=1,\ldots,n),$$

the recurrence relation (55) reduces immediately to the following form:

$$\begin{aligned}2\alpha F_D^{(2n)}&\left[\alpha,\beta_1,\ldots,\beta_n,\beta_1,\ldots,\beta_n;2\alpha;x_1,\ldots,x_n,\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right]\\ &=\alpha F_D^{(2n)}\left[\alpha,\beta_1,\ldots,\beta_n,\beta_1,\ldots,\beta_n;2\alpha+1;x_1,\ldots,x_n,\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right]\\ &\quad+\alpha F_D^{(2n)}\left[\alpha,\beta_1,\ldots,\beta_n,\beta_1,\ldots,\beta_n;2\alpha+1;x_1,\ldots,x_n,\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right]\\ &=2\alpha F_D^{(2n)}\left[\alpha,\beta_1,\ldots,\beta_n,\beta_1,\ldots,\beta_n;2\alpha+1;x_1,\ldots,x_n,\frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right],\end{aligned}\tag{56}$$

which evidently proves the multiple hypergeometric identity (52). And, as we have already indicated above, (52) would yield the multiple hypergeometric identity (51) by virtue of (49). □

Lastly, the general transformation formula (37) or (38) with

$$y_j = -x_j \quad (j = 1, \dots, n)$$

yields the following multivariable extension of (36):

$$\begin{aligned} &F_D^{(2n)}[\alpha, \beta_1, \dots, \beta_n, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n, -x_1, \dots, -x_n] \\ &\quad = F^{2:\,1;\dots;1}_{2:\,0;\dots;0}\left[\begin{array}{l} \Delta(2;\alpha):\ \ \beta_1;\ \dots;\ \ \beta_n; \\ \\ \Delta(2;\gamma):\ \ \text{---};\ \dots;\ \text{---}; \end{array} x_1^2, \dots, x_n^2\right]. \end{aligned} \tag{57}$$

We remark in passing that this last formula (57) was deduced, in a much more general setting, by Srivastava [8, Eq. (26), p. 3083], who also proved many multiple-series identities including (for example) the following multivariable extension of Lemma 2 [8, Eq. (17), p. 3082]:

$$\begin{aligned} &\sum_{l_1, m_1, \dots, l_n, m_n = 0}^{\infty} \Omega(L+M) \prod_{j=1}^{n} \left\{ (\lambda_j)_{l_j} (\lambda_j)_{m_j} \frac{x_j^{l_j}}{l_j!} \frac{y_j^{m_j}}{m_j!} \right\} \\ &\quad = \sum_{l_1, m_1, \dots, l_n, m_n = 0}^{\infty} \Omega(L+2M) \prod_{j=1}^{n} \left\{ (\lambda_j)_{l_j+m_j} \frac{(x_j+y_j)^{l_j}}{l_j!} \frac{(-x_j y_j)^{m_j}}{m_j!} \right\} \\ &\quad (L := l_1 + \dots + l_n;\ M := m_1 + \dots + m_n), \end{aligned} \tag{58}$$

provided that each of the multiple series involved is absolutely convergent.

4. Linear generating functions for the Jacobi polynomials

One of the earliest known generating functions for the classical Jacobi polynomials defined by (7) or (8) is the following linear generating function attributed to Carl Gustav Jacob Jacobi (1804–1851):

$$\begin{aligned} &\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n = 2^{\alpha+\beta} R^{-1} (1 - t + R)^{-\alpha} (1 + t + R)^{-\beta} \\ &\quad \big(R := (1 - 2xt + t^2)^{1/2}\big). \end{aligned} \tag{59}$$

The following *multiparameter* extension of the generated sum in the classical result (59) is an immediate consequence of the definition (8):

$$\sum_{n=0}^{\infty} \frac{(\lambda_1)_n \dots (\lambda_p)_n (\alpha+\beta+1)_n}{(\mu_1)_n \dots (\mu_q)_n (\alpha+1)_n} P_n^{(\alpha,\beta)}(x) t^n$$

$$= \sum_{l=0}^{\infty} \frac{(\lambda_1)_l \ldots (\lambda_p)_l (\alpha+\beta+1)_l}{(\mu_1)_l \ldots (\mu_q)_l} \frac{t^l}{l!} \cdot {}_{p+2}F_{q+1}\left[\begin{matrix} \lambda_1+l, \ldots, \lambda_p+l, \Delta(2;\alpha+\beta+l+1); \\ \mu_1+l, \ldots, \mu_q+l, \alpha+1; \end{matrix} 2(x-1)t\right], \tag{60}$$

which obviously corresponds to (59) when

$$p=q=1 \quad (\lambda_1=\alpha+1;\ \mu_1=\alpha+\beta+1).$$

The hypergeometric representation [10, Eq. 2.3(29), p. 111]:

$$P_{m+n}^{(\alpha,\beta)}(x) = \binom{\alpha+m+n}{m+n}\left(\frac{x+1}{2}\right)^{-\alpha-\beta-m-n-1} \cdot {}_2F_1\left(\alpha+m+n+1, \alpha+\beta+m+n+1; \alpha+1; \frac{x-1}{x+1}\right) \quad (m,n \in \mathbb{N}_0), \tag{61}$$

which follows readily from the definition (8) in conjunction with the Pfaff–Kummer transformation (20) in its *equivalent* form given below:

$${}_2F_1(a,b;c;z) = (1-z)^{-b}\, {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) \quad \left(\left|\arg(1-z)\right| \leqq \pi-\varepsilon;\ 0<\varepsilon<\pi;\ c \notin \mathbb{Z}_0^-\right), \tag{62}$$

would lead us immediately to the following *extended* linear generating function [10, Eq. 2.3(31), p. 111]:

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \frac{(\alpha+\beta+m+1)_n}{(\gamma+1)_n} P_{m+n}^{(\alpha,\beta)}(x) t^n = \binom{\alpha+m}{m}\left(\frac{x+1}{2}\right)^{-\alpha-\beta-m-1} \cdot F_4\left[\alpha+\beta+m+1, \alpha+m+1; \gamma+1, \alpha+1; \frac{2t}{x+1}, \frac{x-1}{x+1}\right] \quad \left(|x-1|^{1/2}+|2t|^{1/2}<|x+1|^{1/2};\ m \in \mathbb{N}_0\right), \tag{63}$$

where F_4 denotes the Appell function of the fourth kind, defined by (cf. [1, p. 14]; see also [10, Eq. 1.6(7), p. 53])

$$F_4[\alpha,\beta;\gamma,\gamma';x,y] := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_m} \frac{x^m}{m!}\frac{y^n}{n!} \quad \left(|x|^{1/2}+|y|^{1/2}<1\right). \tag{64}$$

More generally, in terms of the Kampé de Fériet function (4), it can be observed from the hypergeometric representation (61) that

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\lambda_1)_n\dots(\lambda_p)_n}{(\mu_1)_n\dots(\mu_q)_n}P_{m+n}^{(\alpha,\beta)}(x)t^n$$
$$=\binom{\alpha+m}{m}\left(\frac{x+1}{2}\right)^{-\alpha-\beta-m-1}$$
$$\cdot F_{0:\ q+1;1}^{2:\ \ p;0}\left[\begin{array}{l}\alpha+\beta+m+1,\alpha+m+1:\quad \lambda_1,\dots,\lambda_p;\ \text{——};\\ \text{————}:\ \alpha+\beta+m+1,\mu_1,\dots,\mu_q;\ \alpha+1;\end{array}\ \frac{2t}{x+1},\frac{x-1}{x+1}\right]$$
$$(m\in\mathbb{N}_0), \tag{65}$$

which reduces to (63) when

$$p=q=1\quad(\lambda_1=\alpha+\beta+m+1;\ \mu_1=\gamma+1). \tag{66}$$

In a similar manner, since [cf. Eq. (61)]

$$P_{m+n}^{(\alpha,\beta)}(x)=(-1)^{m+n}\binom{\beta+m+n}{m+n}\left(\frac{1-x}{2}\right)^{-\alpha-\beta-m-n-1}$$
$$\cdot {}_2F_1\left(\beta+m+n+1,\alpha+\beta+m+n+1;\beta+1;\frac{x+1}{x-1}\right)$$
$$(m,n\in\mathbb{N}_0), \tag{67}$$

which does indeed follow from (61) by virtue of the familiar relationship [11, Eq. (4.1.3), p. 59]:

$$P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x), \tag{68}$$

we obtain the following companion of the generating function (65):

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\lambda_1)_n\dots(\lambda_p)_n}{(\mu_1)\dots(\mu_q)_n}P_{m+n}^{(\alpha,\beta)}(x)t^n$$
$$=\binom{\beta+m}{m}\left(\frac{1-x}{2}\right)^{-\alpha-\beta-m-1}$$
$$\cdot F_{0:\ q+1;1}^{2:\ \ p;0}\left[\begin{array}{l}\alpha+\beta+m+1,\beta+m+1:\quad \lambda_1,\dots,\lambda_p;\ \text{——};\\ \text{————}:\ \alpha+\beta+m+1,\mu_1,\dots,\mu_q;\ \beta+1;\end{array}\ \frac{2t}{x-1},\frac{x+1}{x-1}\right]$$
$$(m\in\mathbb{N}_0), \tag{69}$$

which, in the special case given by (66), yields a companion of the linear generating function (63) in the form given below:

$$\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\alpha+\beta+m+1)_n}{(\gamma+1)_n}P_{m+n}^{(\alpha,\beta)}(x)t^n$$
$$=\binom{\beta+m}{m}\left(\frac{1-x}{2}\right)^{-\alpha-\beta-m-1}$$
$$\cdot F_4\left[\alpha+\beta+m+1,\beta+m+1;\gamma+1,\beta+1;\frac{2t}{x-1};\frac{x+1}{x-1}\right]$$
$$\left(|x+1|^{1/2}+|2t|^{1/2}<|x-1|^{1/2};\ m\in\mathbb{N}_0\right). \tag{70}$$

We now recall a very specialized case of a known family of linear generating functions for the Jacobi polynomials [10, Eq. 2.6(31), p. 145; Problem 14(ii), p. 168]:

$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_n\dots(\lambda_p)_n}{(\mu_1)_n\dots(\mu_q)_n}P_n^{(\alpha,\beta)}(x)t^n$$
$$=F^{p+2:\,0;0}_{q:\,1;1}\left[\begin{matrix}\alpha+1,\beta+1,\lambda_1,\dots,\lambda_p\colon\ \text{——};\ \text{——};\\ \mu_1,\dots,\mu_q\colon\ \alpha+1;\ \beta+1;\end{matrix}\ \frac{1}{2}(x-1)t,\frac{1}{2}(x+1)t\right]. \tag{71}$$

Upon replacing $(1/2)(x-1)t$ and $(1/2)(x+1)t$ simply by x and y, respectively, if we compare the generating function (71) with the special cases of the results (65) and (69) when $m=0$, we are led *eventually* to the following general hypergeometric transformations:

$$F^{p+2:\,0;0}_{q:\,1;1}\left[\begin{matrix}\alpha+1,\beta+1,\lambda_1,\dots,\lambda_p\colon\ \text{——};\ \text{——};\\ \mu_1,\dots,\mu_q\colon\ \alpha+1;\ \beta+1;\end{matrix}\ x,y\right]$$
$$=\left(1-\frac{y}{x}\right)^{\alpha+\beta+1}$$
$$\cdot F^{2:\quad p;0}_{0:\,q+1;1}\left[\begin{matrix}\alpha+\beta+1,\beta+1\colon\ \lambda_1,\dots,\lambda_p;\ \text{——};\\ \text{————}\colon\ \alpha+\beta+1,\mu_1,\dots,\mu_q;\ \beta+1;\end{matrix}\ \frac{(x-y)^2}{x},\frac{y}{x}\right] \tag{72}$$

and

$$F^{p+2:\,0;0}_{q:\,1;1}\left[\begin{matrix}\alpha+1,\beta+1,\lambda_1,\dots,\lambda_p\colon\ \text{——};\ \text{——};\\ \mu_1,\dots,\mu_q\colon\ \alpha+1;\ \beta+1;\end{matrix}\ x,y\right]$$
$$=\left(1-\frac{x}{y}\right)^{\alpha+\beta+1}$$
$$\cdot F^{2:\quad p;0}_{0:\,q+1;1}\left[\begin{matrix}\alpha+\beta+1,\alpha+1\colon\ \lambda_1,\dots,\lambda_p;\ \text{——};\\ \text{————}\colon\ \alpha+\beta+1,\mu_1,\dots,\mu_q;\ \alpha+1;\end{matrix}\ \frac{(y-x)^2}{y},\frac{x}{y}\right], \tag{73}$$

where x and y (and the various parameters involved) are so constrained that each member of (72) and (73) exists.

In its *further* special case when $m = 0$, the familiar generating function (70) immediately yields

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\gamma+1)_n} P_n^{(\alpha,\beta)}(x)t^n = \left(\frac{1-x}{2}\right)^{-\alpha-\beta-1} F_4\left[\alpha+\beta+1, \beta+1; \gamma+1, \beta+1; \frac{2t}{x-1}, \frac{x+1}{x-1}\right]$$
$$\left(|x+1|^{1/2} + |2t|^{1/2} < |x-1|^{1/2}\right). \tag{74}$$

Since (cf. [2, Example 20(ii), p. 102]; see also [10, Problem 20(ii), p. 92])

$$F_4[a, b; c, b; x, y] = (1-u)^a(1-v)^a F_1[a, c-b, a-c+1; c; u, uv], \tag{75}$$

where F_1 denotes the Appell function of the first kind, defined by (6), and

$$x = -\frac{u}{(1-u)(1-v)} \quad \text{and} \quad y = -\frac{v}{(1-u)(1-v)}, \tag{76}$$

by setting

$$\frac{2t}{x-1} = -\frac{u}{(1-u)(1-v)} \quad \text{and} \quad \frac{x+1}{x-1} = -\frac{v}{(1-u)(1-v)}, \tag{77}$$

so that

$$(x+1)u^2 - 2(1+t)u + 2t = 0 \tag{78}$$

and

$$v = \frac{(x+1)u}{2t}, \tag{79}$$

we find that

$$u = \frac{1+t-R}{x+1} \quad \text{and} \quad v = \frac{1+t-R}{2t}, \tag{80}$$

where $u = u(t)$ is so chosen that $u(0) = 0$, R being given (as before) with (59). Thus, by applying the transformation (75), the generating function (74) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\gamma+1)_n} P_n^{(\alpha,\beta)}(x)t^n = \left(\frac{2}{1+t+R}\right)^{\alpha+\beta+1} F_1\left[\alpha+\beta+1, \gamma-\beta, \alpha+\beta-\gamma+1; \gamma+1; \frac{1+t-R}{x+1}, \frac{(1+t-R)^2}{2(x+1)t}\right]. \tag{81}$$

In precisely the same manner, by applying the transformation (75) to the *further* special case of the known generating function (63) when $m = 0$, we can deduce the following *equivalent* form of (81):

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\gamma+1)_n} P_n^{(\alpha,\beta)}(x) t^n = \left(\frac{2}{1-t+R}\right)^{\alpha+\beta+1} F_1\left[\alpha+\beta+1, \gamma-\alpha, \alpha+\beta-\gamma+1; \gamma+1; \frac{t-1+R}{x-1}, \frac{(t-1+R)^2}{2(x-1)t}\right], \tag{82}$$

which, in view of the relationship (68), is easily deducible from (81) by letting

$$x \mapsto -x \quad \text{and} \quad t \mapsto -t.$$

Each of the generating functions (81) and (82) can be shown to reduce, when $\gamma = \alpha + \beta$, to the classical result (59). The *equivalent* form (81) of the *special* case of the familiar generating functions (63) and (70) when $m = 0$ was derived earlier by Wimp [12, Eq. (49), p. 412].

The following pair of linear generating functions for the classical Jacobi polynomials were proven recently by Ismail and Pitman [5, Theorem 2, p. 977]:

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha+1/2)_n} P_n^{(\beta-1/2,-1/2)}\left(\frac{2(y+z-yz)^2}{yz(2-y)(2-z)} - 1\right)\left[\frac{yz}{(2-y)(2-z)}\right]^n = \left[\left(1-\frac{y}{2}\right)\left(1-\frac{z}{2}\right)\right]^{\beta} F_1[\alpha, \beta, \beta; 2\alpha; y, z], \tag{83}$$

whose generated series corresponds to that in (81) or (82) when

$$\beta = -\frac{1}{2}, \qquad \alpha \mapsto \beta - \frac{1}{2}, \quad \text{and} \quad \gamma = \alpha - \frac{1}{2};$$

$$\sum_{n=0}^{\infty} \frac{(\beta+1)_n}{(\alpha+3/2)_n} P_n^{(\beta-1/2,1/2)}\left(\frac{2(y+z-yz)^2}{yz(2-y)(2-z)} - 1\right)\left[\frac{yz}{(2-y)(2-z)}\right]^n = \frac{(\alpha+1/2)[(2-y)(2-z)]^{\beta+1}}{4^{\beta}(y+z-yz)\beta} \cdot \left\{F_1[\alpha+1, \beta, \beta; 2\alpha+1; y, z] - F_1[\alpha, \beta, \beta; 2\alpha; y, z]\right\}, \tag{84}$$

whose generated series corresponds to that in (81) or (82) when

$$\beta = \frac{1}{2}, \qquad \alpha \mapsto \beta - \frac{1}{2}, \quad \text{and} \quad \gamma = \alpha + \frac{1}{2},$$

it being understood in *each* case that

$$x = \frac{2(y+z-yz)^2}{yz(2-y)(2-z)} - 1 \quad \text{and} \quad t = \frac{yz}{(2-y)(2-z)}. \tag{85}$$

Finally, upon multiplying each member of the generating function (84) by $y + z - yz$, if we set

$$z = \frac{y}{y-1} \quad \text{(so that } y + z - yz = 0\text{)},$$

we immediately arrive at the aforementioned simpler case of the multiple hypergeometric identity (51) when $n = 2$.

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