A global multiplicity result for \( N \)-Laplacian with critical nonlinearity of concave-convex type

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Received 3 May 2006; revised 14 September 2006
Available online 20 October 2006

Abstract

Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded domain. We consider the following quasilinear problem depending on a real parameter \( \lambda > 0 \):

\[
(P_{\lambda}) \quad \begin{cases}
-\Delta_N u = \lambda f(u) \\
u > 0 \\
u = 0 \text{ on } \partial \Omega,
\end{cases} \quad \text{in } \Omega,
\]

where \( f(t) \) is a nonlinearity that grows like \( e^{t^{N/N-1}} \) as \( t \to \infty \) and behaves like \( t^\alpha \), for some \( \alpha \in (0, N-1) \), as \( t \to 0^+ \). More precisely, we require \( f \) to satisfy assumptions (A1)–(A5) in Section 1. With these assumptions we show the existence of \( \Lambda > 0 \) such that \( (P_{\lambda}) \) admits at least two solutions for all \( \lambda \in (0, \Lambda) \), one solution for \( \lambda = \Lambda \) and no solution for all \( \lambda > \Lambda \). We also study the problem \( (P_{\lambda}) \) posed on the ball \( B_1(0) \subset \mathbb{R}^N \) and show that the assumptions (A1)–(A5) are sharp for obtaining global multiplicity. We use a combination of monotonicity and variational methods to show multiplicity on general domains and asymptotic analysis of ODEs for the case of the ball.

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MSC: 34B15; 35J60

Keywords: Multiplicity; \( N \)-Laplace equation

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doi:10.1016/j.jde.2006.09.012
1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain. For $u \in W^{1,N}_0(\Omega)$ we let $\|u\|_{1,N}$ denote the $W^{1,N}_0(\Omega)$ norm $(\int_\Omega |\nabla u|^N)^{1/N}$ of $u$. We recall the following result due to Trudinger–Moser [24, 31]:

$$\sup_{\|u\|_{1,N} \leq 1} \int_\Omega e^{\alpha_N |u|^{N/(N-1)}} < \infty,$$

where $\alpha_N = N w_N^{1/(N-1)}$, $w_N$ = volume of $S^{N-1}$. It follows immediately from the above inequality that the embedding $W^{1,N}_0(\Omega) \ni u \mapsto e^{\alpha_N |u|^{N/(N-1)}} \in L^1(\Omega)$ is compact for all $\beta \in (0, \frac{N}{N-1})$ and is continuous for $\beta = \frac{N}{N-1}$. The fact that this embedding is not compact for $\beta = \frac{N}{N-1}$ can be shown using a sequence of functions (called “Moser functions,” see Eq. (6.5)) that are suitable truncations and dilations of the fundamental solution of $-\Delta u$ on $W^{1,N}_0(\Omega)$. Thus the growth given by the map $t \mapsto e^{\alpha_N |t|^{N/(N-1)}}$ represents the critical growth for functions $u \in W^{1,N}_0(\Omega)$.

In this work, we would like to consider the following problem:

$$(P_\lambda) \begin{cases} -\Delta u = \lambda f(u) \\ u > 0 \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

where $f$ is a “perturbation” of the critical growth nonlinearity $e^{\alpha_N |t|^{N/(N-1)}}$ as $t \to \infty$ and behaves like $t^\alpha$ for $\alpha \in (0, N-1)$ as $t \to 0^+$. We note that when $N = 2$, $f$ is a concave function near $t = 0$ and a convex function for all large $t > 0$. Problems of the form $(P_\lambda)$ where the nonlinearity has a concave-convex structure are expected to have at least two solutions for $\lambda > 0$ belonging to a maximal interval in $\mathbb{R}$. Historically, the role played by such concave-convex nonlinearities in producing multiple solutions was investigated first in the work [7]. They studied the following problem:

$$\begin{cases} -\Delta u = u^{N+2} + \lambda u^q \\ u > 0 \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

for $0 < q < 1$ and showed the existence of $\Lambda > 0$ such that (1.2) admits at least two solutions for all $\lambda \in (0, \Lambda)$ and no solutions for $\lambda > \Lambda$. Subsequently, in the works [17,18], the corresponding quasilinear version was studied:

$$\begin{cases} -\Delta_p u = u^{p^*-1} + \lambda u^q \\ u > 0 \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

where $1 < p < N$, $p^* = \frac{Np}{N-p}$ and $0 < q < p - 1$. They obtained results similar to the results of [7] above, but only for some ranges of the exponents $p$ and $q$. We summarize their results below:
Theorem. [17,18] Assume that either \( \frac{2N}{N+2} < p < 3 \) or \( p \geq 3 \), \( p - 1 > q > p^* - \frac{2}{p-1} - 1 \). Then there exists \( \Lambda > 0 \) such that (1.3) admits at least two solutions for all \( \lambda \in (0, \Lambda) \) and no solution for \( \lambda > \Lambda \).

It is possible to get complete multiplicity result for problem (1.3) if \( \Omega \) is taken to be a ball in \( \mathbb{R}^N \). This was the approach taken in [26]. In this case, due to well-known symmetry results (see [10,12]), every solution of (1.3) posed on \( \Omega = B_1(0) \) is radially symmetric about the origin and radially nonincreasing. Hence it is enough to study the ODE associated to (1.3) and by using shooting techniques and asymptotic analysis the following result was obtained:

Theorem. [26] Let \( \Omega = B_1(0) \), \( 1 < p < N \), \( 0 < q < p - 1 \). Then there exists \( \Lambda > 0 \) such that (1.3) admits at least two solutions for all \( \lambda \in (0, \Lambda) \) and no solution for \( \lambda > \Lambda \). Additionally, if \( 1 < p < 2 \), then (1.3) admits exactly two solutions for all small \( \lambda > 0 \).

Thus in the quasilinear version of problem (1.2), viz. (1.3), with \( 1 < p < N \), involving critical growth nonlinearity on a general domain we do not have complete multiplicity results. We also remark that the above multiplicity result on a general domain is technically difficult involving uniform Hölder estimates (see [18]). The main difficulty is to analyze Palais–Smale sequences concentrating at the mountain pass level for the energy functional associated to the problem (1.3). Due to the quasilinear nature of the problem, the energy cannot “split” easily in the sense of Brezis–Lieb (see [9]) unless the exponents \( p, q \) are restricted suitably as above.

In this study we would like to consider the case \( p = N \) and show existence of multiple solutions to \((P_\lambda)\) both on general domains and on a ball using ODE techniques. In striking contrast to the results for the case \( 1 < p < N \), we will be able to find a sharp condition on the nonlinearity \( f \) (see assumption (A5)) that determines the existence of multiple solutions to \((P_\lambda)\) (see Theorem 1.1). By “sharp” above, we mean that we can prove uniqueness of solutions to \((P_\lambda)\) for all small \( \lambda > 0 \) on the ball \( B_1(0) \subset \mathbb{R}^N \) for a large representative class of nonlinearities that violate (A5) (see Theorem 1.2). To simplify the presentation we consider the following three classes of model nonlinearities \( f(t) = h(t)e^{t^{N/(N-1)}} \), \( t > 0 \), classified based on the “strength” of the perturbation \( h \):

Class I. \( h(t) = t^\alpha (1 + t)^m e^{-t^\beta} \), \( \alpha \in (0, N-1), m \geq 0, \frac{1}{N-1} < \beta < \frac{N}{N-1} \).

Class II. \( h(t) = t^\alpha (1 + t)^m e^{-t^\beta} \), \( \alpha \in (0, N-1), m \geq 0, 0 < \beta < \frac{1}{N-1} \).

Class III. \( h(t) = t^\alpha (1 + t)^m e^{t^\beta} \), \( \alpha \in (0, N-1), m \geq 0, 0 < \beta < \frac{N}{N-1} \).

First, on a general domain \( \Omega \) in \( \mathbb{R}^N \), we show that for nonlinearities \( f \) from Class II and III, we have global multiplicity result for \((P_\lambda)\). More generally we list the following assumptions on \( f(t) = h(t)e^{t^{N/(N-1)}} \):

(A1) \( h \in C^1((0, \infty)); h(t) = 0, \forall t \leq 0, h(t) > 0 \, \forall t > 0 \).

(A2) The map \( t \mapsto f(t) \) is nondecreasing for \( t \in (0, t_*) \cup (\frac{1}{t_*}, \infty) \).
Let \( \Omega \subset \mathbb{R}^N, N \geq 2, \) be any bounded domain. Let \( f(t) = h(t)e^{t^N/(N-1)} \) satisfy the assumptions (A1)–(A5), then:

(i) there exists \( \Lambda > 0 \) such that \((P_\lambda)\) admits at least two solutions for all \( \lambda \in (0, \Lambda) \), say \( u_\lambda \) and \( v_\lambda \), one solution for \( \lambda = \Lambda \) and no solutions for \( \lambda > \Lambda \);
(ii) as \( \lambda \to 0, u_\lambda \to 0 \) in \( C^1(\overline{\Omega}) \) (bifurcation from 0) and \( J_\lambda(v_\lambda) \to \frac{1}{N}(N\alpha)^{N-1} \) (concentration).

Theorem 1.2. Let \( \Omega = B_1(0) \subset \mathbb{R}^N \) and \( f \) belong to Class 1. Then \((P_\lambda)\) has a unique solution for all small \( \lambda > 0 \).

To our knowledge, even when \( N = 2 \) only few uniqueness results are known concerning the class of problems like \((P_\lambda)\) and these are for the cases where \( f(t) \) is \( C^1 \) near \( t = 0 \) and grows like a power function (see Corollary 2.32 and Section 2.4 in [25] and [3]) or behaves like \( te^t \) (see [2] and the extension for more general nonlinearities in [29]). At this point, we want to stress that Pohozaev Identities which are used to prove uniqueness or exact number of solutions when \( N \geq 3 \) (see for instance [5,16,26,28]) do not work when \( N = 2 \). Therefore, the case \( N = 2 \) is of independent interest and we restate the above uniqueness theorem in this case as:

Theorem 1.2'. Let \( N = 2 \) and \( B_1(0) \subset \mathbb{R}^2 \) be the open unit ball. Let \( f(t) = t^\alpha(1 + t)^m e^{-t^\beta}e^t \) with \( \alpha \in (0, 1), m \geq 0, 1 < \beta < 2 \). Then the corresponding problem \((P_\lambda)\) posed on \( B_1(0) \) admits a unique solution for all small \( \lambda > 0 \).

From Theorems 1.1 and 1.2, we get that the borderline condition between uniqueness and nonuniqueness is given by (A5). We point out that this condition is different from the borderline condition between existence and nonexistence obtained in [6] (when \( h(t) \) is \( C^1 \) near \( t = 0 \)): \( \liminf_{t \to \infty} h(t) t = \infty \). Therefore a larger class of nonlinearities \( h \) is involved for existence of multiple solutions to \((P_\lambda)\) in respect to the result in [6]. Furthermore, we see that in the subcritical case, global multiplicity holds since the Palais–Smale condition is satisfied. Concerning the supercritical case, an early work on multiplicity by Ni and Nussbaum [25] (pp. 91–92) show that when \( N = 2 \) there exists a nonlinearity \( f(t) \) of “supercritical” growth and a ball \( B \subset \mathbb{R}^2 \) of suitable radius such that the problem \((P_1)\) posed on \( B \) has at least two solutions (see, in particular, the assumptions \((f_1)\)–\((f_3)\) Section 4.2 of [25]). In more concrete terms, such an \( f(t) \) will be required to be smooth at \( t = 0 \), convex in \( t \in [0, \infty) \) and grow like \( e^{\alpha t} \) for some \( \alpha > 2 \) as \( t \to \infty \). Clearly, it is the “supercritical” growth of \( f \) at infinity that is responsible for multiple solutions in this work. These results highlight the more complex structure of the set of solutions to \((P_\lambda)\) in the critical case where both uniqueness and multiplicity hold (with the borderline determined by condition (A5)).
Our approach to proving Theorem 1.1 will be to use a combination of monotone iteration techniques and a generalized version of mountain pass theorem as in [4]. We will prove Theorem 1.2 by employing shooting methods and asymptotic analysis as in [8]. We also remark that we really prove Theorem 1.2 for general nonlinearities that share the structure of those in Class I. These general assumptions are listed at the beginning of the proof for this theorem.

2. Existence of local minimum for \( J_{\lambda}, \lambda > 0, \text{small} \)

In the following sections upto Section 6, we will assume (unless otherwise stated) that the nonlinearity \( f \) satisfies the above assumptions (A1)–(A5). In this section we show the existence of a local minimum for \( J_{\lambda} \) in a small neighborhood of the origin in \( W^{1,N}_{0}(\Omega) \).

Lemma 2.1. We can find \( \lambda_0 > 0, R_0 \in (0, 1/2) \) and \( \delta > 0 \) such that \( J_{\lambda}(u) \geq \delta \) for all \( \|u\|_{1,N} = R_0 \), and all \( \lambda \in (0, \lambda_0) \).

Proof. Using (A2), we have, for some \( C_1, C_2 > 0 \) independent of \( u \in W^{1,N}_{0}(\Omega) \),

\[
\int_{\Omega} F(u) = \int_{\Omega} \left( \int_{0}^{u} h(t)e^{t|t|^{N/(N-1)}} \, dt \right) \, dx 
\leq C_1 + \int_{\Omega} h(u)|u|e^{u|u|^{N/(N-1)}}
\leq C_1 + C_2 \int_{\Omega} e^{2|u|^{N/(N-1)}} \quad \text{(using (A4))}.
\]

Hence, from (1.1) and the above inequality we obtain that

\[
\sup_{\|u\|_{1,N} \leq \frac{1}{2} \Omega} \int_{\Omega} F(u) \leq C \quad \text{for some} \ C > 0.
\]

Hence for \( R_0 \in (0, \frac{1}{2}) \) we have

\[
\inf_{\|u\|_{1,N} = R_0} J_{\lambda}(u) \geq \frac{R_0^N}{N} - \lambda C.
\]

The conclusion of the lemma now follows by taking \( \lambda_0 = \frac{R_0^N}{2CN} \).

Lemma 2.2. \( J_{\lambda} \) possesses a local minimum close to the origin in \( W^{1,N}_{0}(\Omega) \) for all \( \lambda \in (0, \lambda_0) \).

Proof. Let \( R_0, \lambda_0 \) and \( \delta \) be as in Lemma 2.1. We fix \( \lambda \in (0, \lambda_0) \). Thanks to (A3) we note that \( J_{\lambda}(tu) < 0 \) for all \( t > 0 \) small enough and any \( u \in W^{1,N}_{0}(\Omega) \). In particular \( \inf_{\|u\|_{1,N} \leq R_0} J_{\lambda}(u) < 0 \) and if this infimum is achieved at some \( u_\lambda \), necessarily (by Lemma 2.1) \( \|u_\lambda\|_{1,N} < R_0 \) and hence \( u_\lambda \) is a local minimum of \( J_{\lambda} \). We now show that this infimum is indeed achieved. Let \( \{u_n\} \subset
{\|u\|_{1,N} \leq R_0}$ be a minimizing sequence and let $u_n \rightharpoonup u_\lambda$ in $W^{1,N}_0(\Omega)$. Clearly $\|u_\lambda\|_{1,N} \leq R_0 < 1/2$. Hence using (1.1) and Vitali’s convergence theorem, we obtain $\int_\Omega F(u_n) \to \int_\Omega F(u)$. Thus we get that $u_\lambda$ is a minimum for $J_\lambda$ in $\{\|u\|_{1,N} \leq R_0\}$ for all $\lambda \in (0, \lambda_0)$. □

3. Behavior of small norm solutions to $(P_\lambda)$

In this section we show that for all $\lambda > 0$ small enough, $(P_\lambda)$ admits a unique solution $u_\lambda$ with $\|u_\lambda\|_{L^\infty(\Omega)}$ small enough. We first recall the following well-known result:

**Lemma 3.1.** Let $\rho: [0, \infty) \to [0, \infty)$ be such that $t^{1-N} \rho(t)$ is nonincreasing. Let $v, w \in W^{1,N}_0(\Omega)$ be weak sub- and super-solutions respectively of the problem $-\Delta_N u = \rho(u)$ in $\Omega$. Then $w \geq v$ a.e. in $\Omega$.

Define $\tilde{f}: \mathbb{R} \to \mathbb{R}$ as:

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t < t_*, \\ f(t_*) & \text{if } t_* \leq t. \end{cases}$$

Then, by (A3), $t^{1-N} \tilde{f}(t)$ is a nonincreasing function on $\mathbb{R}$. Consider the following problem:

$$\begin{cases} -\Delta_N u = \lambda \tilde{f}(u) \\ u > 0 \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Clearly $(\tilde{P}_\lambda)$ admits a solution $\tilde{u}_\lambda$ and the solution is unique (thanks to Lemma 3.1). Let $w$ denote the unique solution to the following problem:

$$\begin{cases} -\Delta_N w = w^\alpha \\ w > 0 \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

Choose $\xi > 0$ such that $\tilde{f}(t) \leq \xi t^\alpha \forall t \in \mathbb{R}$. We now have the following

**Lemma 3.2.** Let $\lambda_0 = \xi^{-1}(t_* \|w\|_{L^\infty(\Omega)})^{N-\alpha-1}$. For any $\lambda \in (0, \lambda_0)$, the problem $(P_\lambda)$ admits exactly one solution $u_\lambda$ with the property $\|u_\lambda\|_{L^\infty(\Omega)} \leq t_*$.

**Proof.** Let $v_\lambda$ denote the unique solution of the problem:

$$\begin{cases} -\Delta_N v_\lambda = \lambda \xi v_\lambda^\alpha \\ v_\lambda > 0 \\ v_\lambda = 0 \text{ on } \partial\Omega. \end{cases}$$

Now, $(\lambda \xi)^{\frac{1+\alpha}{\alpha}} v_\lambda$ solves (3.1) and hence we obtain $\|v_\lambda\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\Omega)}(\lambda \xi)^{\frac{1}{\alpha-1}}$. It now follows that $\|v_\lambda\|_{L^\infty(\Omega)} \leq t_* \forall \lambda \in (0, \lambda_0)$. Now, $v_\lambda$ is a super solution to $(\tilde{P}_\lambda)$ and hence by
Lemma 3.1 we get $\|\tilde{u}_\lambda\|_{L^\infty(\Omega)} \leq \|v_\lambda\|_{L^\infty(\Omega)} \leq t_\ast \forall \lambda \in (0, \lambda_0)$. The conclusion now follows by noting that $\tilde{u}_\lambda$ solves $(P_\lambda)$.

Finally we show the following result on continuous dependence.

**Lemma 3.3.** For $\lambda \in (0, \lambda_0)$, choose a unique solution $u_\lambda$ of $(P_\lambda)$ with $\|u_\lambda\|_{L^\infty(\Omega)} \leq t_\ast$ using the above lemma. Then for any $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is continuous on $(0, \lambda_0)$.

**Proof.** Suppose not. Then for some $x \in \Omega$ there exists a sequence $\lambda_n \in (0, \lambda_0)$ with $\lambda_n \to \lambda \in (0, \lambda_0)$, but $u_{\lambda_n}(x)$ does not converge to $u_\lambda(x)$. Since $\|u_{\lambda_n}\|_{L^\infty(\Omega)} \leq t_\ast \forall n$, by elliptic regularity we obtain that (upto a subsequence) $u_{\lambda_n} \to u_0$ in $C^1(\overline{\Omega})$ and $u_0$ solves $(P_\lambda)$, $u_\lambda \neq u_0$. This contradicts the uniqueness assertion in Lemma 3.2.

4. Non-existence for large $\lambda > 0$

Let $A = \sup\{\lambda > 0: (P_\lambda)$ has a solution$\}$.

**Lemma 4.1.** $0 < A < \infty$.

**Proof.** By Lemma 2.2 it is clear that $A > 0$. We show that $A < \infty$. Suppose, by way of contradiction, that there exists a sequence $\lambda_n \to \lambda \in (0, \lambda_0)$, such that $(P_{\lambda_n})$ admits a solution $u_n$. Let $\lambda_1$ be the first eigenvalue of $-\Delta_N$ on $W_0^{1,N}(\Omega)$ with $\phi_1$ the corresponding normalized eigenfunction. Let $\epsilon \in (0, 1)$. By (A3)–(A4) there exists $\lambda_\ast > 0$ such that $\lambda f(t) > (\lambda_1 + \epsilon)t^{N-1}$ for all $\lambda > \lambda_\ast$, $t > 0$. Choose $\lambda_n > \lambda_\ast$. Clearly, $u_n$ is a super solution of

$$
\begin{align*}
-\Delta_N u &= (\lambda_1 + \epsilon)u^{N-1} \\
0 &> u \quad \text{in } \Omega, \\
0 &= u \quad \text{on } \partial \Omega,
\end{align*}
$$

(4.1)

and $\mu \phi_1$ a subsolution for $\mu < \lambda_1 + \epsilon$. We now choose $\mu > 0$ small enough so that $\mu \phi_1(x) \leq u_n(x) \forall x \in \Omega$. By monotone iteration procedure we obtain a solution $\phi_\epsilon > 0$ of (4.1) for all $\epsilon \in (0, 1)$. This contradicts the fact that $\lambda_1$ is an isolated point in the spectrum of $-\Delta_N$ on $W_0^{1,N}(\Omega)$. Hence $A < \infty$.

5. Existence of a local minimum for $J_\lambda$, $\lambda \in (0, A)$

Some of the arguments in this section are inspired by the ideas contained in the works [17, 18]. Let $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial \Omega\}$. Denote $C = C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. For $u \in C$ we let $\|u\|_C = \|u\|_{C^1(\overline{\Omega})}$. We start by the following strong comparison principle when $N \geq 3$.

**Lemma 5.1.** Let $u, v \in C^{1,\theta}(\overline{\Omega})$, for some $0 < \theta < 1$, satisfy $0 \leq u \leq v$, $u \neq 0$, and solve the following equation in $\Omega$

$$
\begin{align*}
-\Delta_N u + Ku &= \rho_u, \\
-\Delta_N v + Kv &= \rho_v,
\end{align*}
$$

(5.1) (5.2)
with \( u = v = 0 \) on \( \partial \Omega \), where \( K \geq 0 \), \( \rho_u, \rho_v \in C(\overline{\Omega}) \), locally Lipschitz in \( \Omega \), are such that \( 0 < \rho_u \leq \rho_v \) in \( \Omega \) and \( \rho_u \not\equiv \rho_v \) in every small neighborhood of \( \partial \Omega \). Then, the following strong comparison principle holds:

\[
0 < u < v \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial v} < \frac{\partial u}{\partial v} < 0 \quad \text{on } \partial \Omega.
\]  

(5.3)

**Proof.** First, note that from the strong maximum principle of Vázquez (see Theorem 5 in [32]), we have that \( u, v > 0 \) in a small neighborhood of \( \partial \Omega \) and \( \frac{\partial u}{\partial v} < 0 \) on \( \partial \Omega \). Now, following the ideas in [11], we show that the inequalities in (5.3) hold near the boundary. As in the proof of Proposition 2.4 in [11] (see p. 729), there exists a small \( \eta > 0 \) such that in the open \( \eta \)-neighborhood \( \Omega_\eta \subset \Omega \) of the boundary \( \partial \Omega \),

\[
\Omega_\eta := \{ x \in \Omega : d(x) := \text{dist}(x, \partial \Omega) < \eta \},
\]

we have for \( w := v - u, 0 \leq w \in C^{1,\theta}(\overline{\Omega}) \) with \( w = 0 \) on \( \partial \Omega \) and

\[
- \text{div}(A(x)\nabla w) + Kw = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial w}{\partial x_j} \right) + Kw = \rho_v - \rho_u \geq 0 \quad \text{for } x \in \Omega_\eta.
\]

(5.5)

The coefficients \( a_{ij} \) belong to \( C^{0,\theta}(\overline{\Omega}) \) and form a uniformly elliptic operator in \( \Omega_\eta \). Then, (5.5) satisfies the conditions of the Hopf Lemma. Thus, (5.3) is satisfied near the boundary. Now, \( u < v \) follows from Theorem 1.4 in [13]. This completes the proof of Lemma 5.1. \( \square \)

**Remark.** When \( K = 0 \), a simpler proof based on generalized integration by parts is given in [23].

**Lemma 5.2.** \( J_{\lambda|\mathcal{C}} \) (restriction of \( J_{\lambda} \) to \( \mathcal{C} \)) has a local minimum for all \( \lambda \in (0, \Lambda) \).

**Proof.** Fix \( \lambda \in (0, \Lambda) \). Choose \( \bar{\lambda} \in (\lambda, \Lambda) \) such that \( (P_{\bar{\lambda}}) \) has a solution, say \( \bar{u} \). Let \( \bar{\lambda} = \lambda \inf_{t>0} f(t)t^{-\alpha} \). Thanks to assumptions (A3)–(A4), \( \bar{\lambda} > 0 \). Let \( u \) denote the unique solution of

\[
\begin{aligned}
-\Delta_N u &= \bar{\lambda} u^\alpha \\
u &> 0 \quad \text{in } \Omega,
\end{aligned}
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]  

(5.6)

Clearly \( \bar{u} \) is a super solution of (5.6) and hence by Lemma 3.1 and strong comparison principle in Lemma 5.1, we have that \( u < \bar{u} \) in \( \Omega \), \( \frac{\partial u}{\partial v} < \frac{\partial u}{\partial v} < 0 \) on \( \partial \Omega \). Thanks to (A2), we can choose \( K > 0 \) large enough so that \( f(t) + Kt \) is nondecreasing for all \( t \in \mathbb{R} \). Define the cut-off nonlinearity:

\[
(x \in \Omega, t \in \mathbb{R}) \quad g(x, t) = \begin{cases} f(\bar{u}(x)) + Ku(x) & \text{if } t > \bar{u}(x), \\
f(t) + Kt & \text{if } u(x) \leq t \leq \bar{u}(x), \\
f(\bar{u}(x)) + Ku(x) & \text{if } t < u(x). \end{cases}
\]
Let \( G(x,u) = \int_0^u g(x,t) \, dt \). Define the functional \( I_\lambda : W^{1,N}_0(\Omega) \to \mathbb{R} \) by

\[
I_\lambda(u) = \frac{1}{N} \int_\Omega |\nabla u|^N + \frac{\lambda K}{2} \int_\Omega |u|^2 - \lambda \int_\Omega G(x,u) \, dx.
\]

Clearly, \( I_\lambda \) is bounded on \( W^{1,N}_0(\Omega) \) and weakly lower semicontinuous. Let \( u_\lambda \) denote the global minimum of \( I_\lambda \). Then, \( u_\lambda \) solves the equation \(-\Delta_N u_\lambda + Ku_\lambda = \lambda g(x,u_\lambda) \) in \( \Omega \) and hence by standard elliptic regularity \( u_\lambda \in C^{1,\theta}(\Omega) \) for some \( \theta \in (0,1) \). Again, by strong comparison principle (see Lemma 5.1), we conclude that \( u < u_\lambda < \bar{u} \) in \( \Omega \) and \( \frac{\partial}{\partial v}(u_\lambda - \bar{u}) < 0 \) on \( \partial \Omega \). This means that \( u_\lambda \) solves \((P_\lambda)\) and also that we can find \( \delta > 0 \) small enough so that if \( v \in W^{1,N}_0(\Omega) \) \( |u| \leq v \leq \bar{u} \) in \( \Omega \), we get that \( u_\lambda \) is a local minimum for \( J_\lambda |_{C^1} \).

**Lemma 5.3.** \( u_\lambda \) is a local minimum for \( J_\lambda \) in \( W^{1,N}_0(\Omega) \) \( \forall \lambda \in (0,\Lambda) \).

**Proof.** We first show the following

**Claim.** If \( \epsilon > 0 \) is small enough,

(i) \( v \mapsto J_\lambda(u_\lambda + v) \) is bounded on \( \{v; \|v\|_{1,N} \leq \epsilon\} \).

(ii) \( \inf_{\|v\|_{1,N} \leq \epsilon} J_\lambda(u_\lambda + v) \) is achieved, say at \( v_\lambda \).

**Proof of the claim.** We have using (A2) and (A4), for any \( v \in W^{1,N}_0(\Omega) \),

\[
|F(u_\lambda + v)| \leq |f(u_\lambda + v)(u_\lambda + v)| \leq C_1 e^{C_2|v|^{N/(N-1)}}
\]

for some constants \( C_1, C_2 > 0 \). Let \( \epsilon_0 = \frac{1}{2C_2} \), choose \( \epsilon \in (0,\epsilon_0) \). Then from (1.1) and the above inequality, we get that

\[
\sup_{\|v\|_{1,N} \leq \epsilon} \int_\Omega |F(u_\lambda + v)|^2 < \infty.
\]

(5.7)

This proves (i). Let \( \{v_n\} \) be a minimizing sequence for \( J_\lambda(u_\lambda + \cdot) \) on \( \{v; \|v\|_{1,N} \leq \epsilon\} \). Let \( v_n \to v_\lambda \) in \( W^{1,N}_0(\Omega) \). From (5.7) using Vitali’s convergence theorem we obtain that \( \int_\Omega F(u_\lambda + v_n) \to \int_\Omega F(u_\lambda + v_\lambda) \). Hence, \( J_\lambda(u_\lambda + v_n) \leq \liminf_{n \to \infty} J_\lambda(u_\lambda + v_n) = \inf_{\|v\|_{1,N} \leq \epsilon} J_\lambda(u_\lambda + v) \). This proves (ii) and hence the claim. \( \square \)

**Proof of Lemma 5.3 completed.** Suppose the conclusion of lemma is false and we derive a contradiction. Thanks to the claim above, for every \( \epsilon \in (0,\epsilon_0) \) we obtain \( v_\epsilon \) such that \( 0 < \|v_\epsilon\|_{W^{1,N}_0(\Omega)} < \epsilon \) and

\[
J_\lambda(u_\lambda + v_\epsilon) < J_\lambda(u_\lambda), \quad J_\lambda(u_\lambda + v_\epsilon) = \inf_{\|v\| \leq \epsilon} J_\lambda(u_\lambda + v).
\]

(5.8)
By the Lagrange multiplier rule we obtain $\mu_\varepsilon \leq 0$ such that
\[
\left\langle J'_\lambda(u_\lambda + v_\varepsilon), \phi \right\rangle = \mu_\varepsilon \int_\Omega |\nabla v_\varepsilon|^{N-2} \nabla v_\varepsilon \cdot \nabla \phi, \quad \forall \phi \in W^{1,N}_0(\Omega).
\]
That is, in the weak sense,
\[
-\Delta_N(u_\lambda + v_\varepsilon) - \lambda f(u_\lambda + v_\varepsilon) = -\mu_\varepsilon \Delta_N v_\varepsilon.
\] (5.9)
Define the maps $A_\varepsilon : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ as
\[
A_\varepsilon(x, p) = |\nabla u_\lambda(x) + p|^{N-2} (\nabla u_\lambda(x) + p) - |\nabla u_\lambda(x)|^{N-2} \nabla u_\lambda(x) - \mu_\varepsilon |p|^{N-2} p,
\]
\[
b(x, s) = \lambda (f(u_\lambda(x) + s) - f(u_\lambda(x))).
\]
Then (5.9) can be written as
\[
\begin{cases}
-\text{div}(A_\varepsilon(x, \nabla v_\varepsilon)) = b(x, v_\varepsilon), & \text{in } \Omega, \\
v_\varepsilon = 0, & \text{on } \partial \Omega.
\end{cases}
\] (5.10)
Since $\mu_\varepsilon \leq 0$, it is easy to check that
\[
\rho = \inf_{(x, p) \in \bar{\Omega} \times \mathbb{R}^N} \frac{(A_\varepsilon(x, p), p)}{|p|^N} > 0.
\]
Therefore, we can employ the Moser iteration technique (see Theorem 15.7 in [20]) to conclude that
\[
\exists \beta \in (0, 1): \lim_{\varepsilon \to 0} \|v_\varepsilon\|_{C^{0,\beta}(\Omega)} < \infty.
\] (5.11)
Since $b$ is a locally Holder continuous function of $s$, it follows from the regularity results in DiBenedetto [15], Tolksdorf [30] that
\[
\lim_{\varepsilon \to 0} \|v_\varepsilon\|_{C^{1,\beta}(\Omega)} < \infty \quad \text{for some } \beta \in (0, 1).
\]
Since $\|v_\varepsilon\|_{W^{1,N}_0(\Omega)} < \varepsilon$, this means that $v_\varepsilon \to 0$ in $C^{1}(\Omega)$. This gives a contradiction in view of (5.8) since $u_\lambda$ is a local minimum of $J_\lambda$ in $\mathcal{C}$.

Lemma 5.4. There exists a solution $u_\Lambda$ of $(P_\Lambda)$.

Proof. Let $v_\lambda$ be as in claim (ii) of Lemma 5.3. It is clear that $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) \leq I_\lambda(v_\lambda) < 0$ as $u_\lambda$ is the global minimum of $I_\lambda$. Now suppose $[\lambda_n]$ be a sequence such that $\lambda_n \to \Lambda$ and $u_n$ be the corresponding solutions of $(P_{\lambda_n})$ obtained in Lemmas 5.2 and 5.3. Then from above discussion it is easy to see
\[
J_{\lambda_n}(u_n) < 0, \quad J'_{\lambda_n}(u_n) = 0.
\] (5.12)
Since \( \lambda_n \) is bounded, (5.12) implies that \( \|u_n\|_{W^{1,N}_0(\Omega)} \leq C \), for some \( C > 0 \). Then there exists \( u_A \) such that \( u_n \rightharpoonup u_A \) in \( W^{1,N}_0(\Omega) \). Now it is easy to verify that \( u_A \) is a weak solution of \( (P_\Lambda) \). \( \square \)

6. Mountain pass solution for \( \lambda \in (0, \Lambda) \) (Proof of Theorem 1.1)

We assume, without loss of generality that \( 0 \in \Omega \). We use a generalized version of mountain pass theorem for a modified version of \( J_\lambda \) (called \( \tilde{J}_\lambda \) here) to show the existence of a second solution to \( (P_\lambda) \). We adapt the techniques used in [1,14] to show that the mountain pass critical value is below a certain threshold energy level.

Choose \( K > 0 \) so that \( f(t) + Kt \) is nondecreasing for all \( t > 0 \) (possible thanks to (A2)). Let \( u \) be as in (5.6). Define

\[
\tilde{f}(x,s) = \begin{cases} 
 f(s) + Ks & \text{if } s > u(x), \\
 f(u(x)) + Ku(x) & \text{if } s \leq u(x).
\end{cases}
\]

Let \( \tilde{F}(x,s) = \int_0^s \tilde{f}(x,t) \, dt \). Define \( \tilde{J}_\lambda : W^{1,N}_0(\Omega) \to \mathbb{R} \) by

\[
\tilde{J}_\lambda(u) = \frac{1}{N} \int_\Omega |\nabla u|^N + \frac{\lambda K}{2} \int_\Omega |u|^2 - \lambda \int_\Omega \tilde{F}(x,u).
\]

Now, as in the proof of Lemma 5.2 we obtain that if \( v \in W^{1,N}_0(\Omega) \cap C \) is close enough to \( u_\lambda \) in the space \( C \) then \( I_\lambda(v) = \tilde{J}_\lambda(v) \), i.e., \( u_\lambda \) is a local minimum for \( \tilde{J}_\lambda|_{C_\sigma} \). Arguing as in Lemma 5.1 we can conclude that \( u_\lambda \) is a local minimum for \( \tilde{J}_\lambda \) in \( W^{1,N}_0(\Omega) \). Then, as in the proof of Lemma 5.2 we can check that if \( v_\lambda \) is a critical point of \( \tilde{J}_\lambda \), then in fact \( v_\lambda > u \) in \( \Omega \) and hence \( v_\lambda \) solves (\( P_\lambda \)).

Also, \( \tilde{J}_\lambda \) has only nontrivial critical points. Hence to prove Theorem 1.1 it is enough to show (which we will do) that \( \tilde{J}_\lambda \) has a critical point \( v_\lambda \) different from \( u_\lambda \). We first define a generalized notion of Palais–Smale sequence for \( \tilde{J}_\lambda \).

**Definition 6.1.** Let \( \mathcal{F} \subset W^{1,N}_0(\Omega) \) be a closed set. We say that a sequence \( \{u_n\} \subset W^{1,N}_0(\Omega) \) is a Palais–Smale sequence for \( \tilde{J}_\lambda \) at the level \( \rho \) around \( \mathcal{F} \) (a \((PS)_{\mathcal{F},\rho}\) sequence, for short) if

\[
\lim_{n \to \infty} \text{dist}(u_n, \mathcal{F}) = 0, \quad \lim_{n \to \infty} \tilde{J}_\lambda(u_n) = \rho, \quad \text{and} \quad \lim_{n \to \infty} \|\tilde{J}_\lambda'(u_n)\|_{W^{-1,N}} = 0.
\]

We have the following compactness result for \((PS)_{\mathcal{F},\rho}\) sequences for \( \tilde{J}_\lambda \):

**Lemma 6.2.** Let \( \mathcal{F} \subset W^{1,N}_0(\Omega) \) be a closed set, \( \rho \in \mathbb{R} \). Let \( \{u_n\} \subset W^{1,N}_0(\Omega) \) be a \((PS)_{\mathcal{F},\rho}\) sequence for \( \tilde{J}_\lambda \). Then there exists a subsequence \( \{u_n\} \) of \( \{u_n\} \) and \( u_0 \in W^{1,N}_0(\Omega) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( W^{1,N}_0(\Omega) \), and \( \lim_{n \to \infty} \int_\Omega \tilde{F}(x,u_n) = \int_\Omega \tilde{F}(x,u_0) \).

**Proof.** Since \( \{u_n\} \) is a \((PS)_{\mathcal{F},\rho}\) sequence for \( \tilde{J}_\lambda \) we have the following relations as \( n \to \infty \):

\[
\frac{1}{N} \int_\Omega |\nabla u_n|^N + \frac{\lambda K}{2} \int_\Omega |u_n|^2 - \lambda \int_\Omega \tilde{F}(x,u_n) = \rho + o_n(1), \quad (6.1)
\]
\[
\left| \int_{\Omega} |\nabla u_n|^{N-2} \nabla u_n \cdot \nabla \phi + \lambda K \int_{\Omega} u_n \phi - \lambda \int_{\Omega} \tilde{f}(x, u_n) \phi \right| \leq o_n(1) \|\phi\|, \quad \forall \phi \in W^{1,N}_0(\Omega). \quad (6.2)
\]

We divide the proof into two steps.

**Step 1.** \(\sup_n \|u_n\|_{W^{1,N}_0(\Omega)} < \infty, \sup_n \int_{\Omega} \tilde{f}(x, u_n) u_n < \infty.\)

Given any \(\epsilon > 0\), there exists \(s_\epsilon > 0\) such that

\[
\frac{Ks^2}{2} + \tilde{F}(x, s) \leq \epsilon \tilde{f}(x, s)s, \quad \forall |s| \geq s_\epsilon.
\]

Using (6.1) together with this relation, we get,

\[
\frac{1}{N} \int_{\Omega} |\nabla u_n|^N \leq \frac{\lambda K}{2} \left( \int_{|u_n| \leq s_\epsilon} |u_n|^2 + \int_{|u_n| \leq s_\epsilon} \tilde{F}(x, u_n) + \lambda \epsilon \int_{\Omega} \tilde{f}(x, u_n) u_n + \rho + o_n(1) \right)
\]

\[
\leq C_\epsilon + \lambda \epsilon \int_{\Omega} \tilde{f}(x, u_n) u_n. \quad (6.3)
\]

From (6.2) with \(\phi = u_n\) there exists \(C_K > 0\) such that

\[
\lambda \int_{\Omega} \tilde{f}(x, u_n) u_n \leq C_K \int_{\Omega} |\nabla u_n|^N + o_n(1) \|u_n\|_{W^{1,N}_0(\Omega)}. \quad (6.4)
\]

Hence, by choosing \(\epsilon\) small, we obtain from (6.3)

\[
\|u_n\|^N \leq \frac{NC_\epsilon}{1 - N\epsilon} + o_n(1) \|u_n\|.
\]

Substituting the above inequality in (6.4) we obtain that \(\sup_n \int_{\Omega} \tilde{f}(x, u_n) u_n < +\infty.\) This finishes Step 1.

Since \(\{u_n\} \subset W^{1,N}_0(\Omega)\) is bounded, up to a subsequence, \(u_n \rightharpoonup u_0\) in \(W^{1,N}_0(\Omega)\), for some \(u_0 \in W^{1,N}_0(\Omega)\).

**Step 2.** \(\lim_{n \to \infty} \int_{\Omega} \tilde{f}(x, u_n) = \int_{\Omega} \tilde{f}(x, u_0), \lim_{n \to \infty} \int_{\Omega} \tilde{F}(x, u_n) = \int_{\Omega} \tilde{F}(x, u_0).\)

We first show that \(\{\tilde{f}(x, u_n)\}\) is an equi-integrable family in \(L^1(\Omega)\), i.e., given \(\epsilon > 0\) there exists \(\delta > 0\) such that for any \(A \subset \Omega\) with \(|A| < \delta\), we have \(\sup_n \int_A |\tilde{f}(x, u_n)| \leq \epsilon.\) Then the conclusion follows from Vitali’s convergence theorem and \(\tilde{F}(x, s) \leq C \tilde{f}(x, s)s\) for all \(x \in \Omega, s \in \mathbb{R}.\)

Let \(\bar{C} = \sup_n \int_{\Omega} |\tilde{f}(x, u_n) u_n|\). By Step 1, \(\bar{C} < \infty.\) Given \(\epsilon > 0\), define

\[
\mu_\epsilon = \sup_{x \in \Omega, |s| \leq \frac{\epsilon}{2\mu_\epsilon}} |\tilde{f}(x, s)|.
\]

Then for any \(A \subset \Omega\) with \(|A| \leq \frac{\epsilon}{{2\mu_\epsilon}}\), we get
We define the following sequence

\[ \rho_n = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \tilde{J}_\lambda(t \gamma(t)). \]

It follows that \( \rho_0 \geq \tilde{J}_\lambda(u_\lambda) \). Let \( R_0 = \| e - u_\lambda \| \). If \( \rho_0 = \tilde{J}_\lambda(u_\lambda) \) we have \( \inf \{ \tilde{J}_\lambda(v) : \| v - u_\lambda \| = R \} = \rho_0 \) for all \( R \in (0, R_0) \). We now let \( \mathcal{F} = W^{1,N}_0(\Omega) \) if \( \rho_0 > \tilde{J}_\lambda(u_\lambda) \) and \( \mathcal{F} = \{ \| v - u_\lambda \| = R_0 \} \) if \( \rho_0 = \tilde{J}_\lambda(u_\lambda) \). We have the following upper bound on \( \rho_0 \).

**Lemma 6.3.** \( \rho_0 < \tilde{J}_\lambda(u_\lambda) + \frac{1}{N} (\alpha_N)^{N-1} \).

**Proof.** We define the following sequence \( \tilde{\phi}_n \) on \( \mathbb{R}^N \):

\[
\tilde{\phi}_n(x) = \frac{1}{ \nu_{\frac{1}{N}}^{1/N} } \left\{ \begin{array}{ll}
(\log n)^{(N-1)/N}, & 0 \leq |x| \leq \frac{1}{n}, \\
(\log \frac{1}{|x|})/(\log n)^{1/N}, & \frac{1}{n} \leq |x| \leq 1, \\
0, & |x| \geq 1.
\end{array} \right.
\] (6.5)

The above functions were constructed by Moser (see [24]). It can be checked that \( \| \tilde{\phi}_n \|_{W^{1,N}_0(\mathbb{R}^N)} = 1, \forall n \). Let \( \delta_n > 0 \) be such that \( B_{\delta_n}(0) \subset \Omega \), \( \delta_n \rightarrow 0 \) as \( n \rightarrow +\infty \) (we will fix later the behavior of \( \delta_n \)) and let \( \phi_n(x) = \tilde{\phi}_n(\frac{x}{\delta_n}) \). Then \( \phi_n \) has support in \( B_{\delta_n}(0) \) and \( \| \phi_n \| = 1 \) for all \( n \). We prove the lemma by contradiction argument. Suppose for each \( n \) there exists \( t_n > 0 \) such that

\[ \sup_{t > 0} \tilde{J}_\lambda(t \phi_n + u_\lambda) = \tilde{J}_\lambda(u_\lambda + t_n \phi_n) \geq \tilde{J}_\lambda(u_\lambda) + \frac{(\alpha_N)^{N-1}}{N}. \] (6.6)

From (6.6) we get \( t_n \) is bounded, otherwise \( \tilde{J}_\lambda(t_n \phi_n + u_\lambda) \rightarrow -\infty \). We have:

\[ |\nabla (u_\lambda + t \phi_n)|^N = (|\nabla u_\lambda|^2 + 2t \nabla u_\lambda \nabla \phi_n + t^2 |\nabla \phi_n|^2)^{N/2}. \] (6.7)

Since \( \nabla u_\lambda \in L^\infty \) (thanks to \( u_\lambda \in C^{1,\theta}(\overline{\Omega}) \) for some \( 0 < \theta < 1 \), from (6.7) and the one-dimensional inequality:

\[ (1 + t^2 + 2t \cos \alpha)^{N/2} \leq 1 + t^N + N t \cos \alpha + C(t^2 + t^{N-1}) \]

for \( t \geq 0 \), uniformly in \( \alpha \),

we estimate \( \tilde{J}_\lambda(t_n \phi_n + u_\lambda) \):
\[ J_{\lambda}(t_n\phi_n + u_\lambda) \leq \frac{t_n^N}{N} + \frac{1}{N} \int_{\Omega} |\nabla u_\lambda|^N + t_n \int_{\Omega} |\nabla u_\lambda|^{N-2} \nabla u_\lambda \nabla \phi_n \]

\[ + O\left( t_n^2 \int_{\Omega} |\nabla \phi_n|^2 + t_n^{N-1} \int_{\Omega} |\nabla \phi_n|^{N-1} \right) + \frac{\lambda K}{2} \int_{\Omega} |u_\lambda + t_n \phi_n|^2 - \lambda \int_{\Omega} \tilde{F}(x, u_\lambda + t_n \phi_n) \]

\[ = \frac{t_n^N}{N} + \tilde{J}_{\lambda}(u_\lambda) + O\left( t_n^2 \int_{\Omega} |\nabla \phi_n|^2 + \int_{\Omega} |\phi_n|^2 \right) + t_n^{N-1} \int_{\Omega} |\nabla \phi_n|^{N-1} \]

\[ - \lambda \int_{\Omega} \left[ \tilde{F}(x, u_\lambda + t_n \phi_n) - \tilde{F}(x, u_\lambda) - t_n (f(u_\lambda) + Ku_\lambda) \phi_n \right] \]

\[ = \frac{t_n^N}{N} + \tilde{J}_{\lambda}(u_\lambda) + O\left( t_n^2 \left( \frac{\delta_n^{-2}(\log n)^{-2/N} + t_n^{N-1} \delta_n (\log n)^{-(N-1)/N} \right) \right) \]

\[ - \lambda \int_{\Omega} \left[ \tilde{F}(x, u_\lambda + t_n \phi_n) - \tilde{F}(x, u_\lambda) - t_n \tilde{f}(x, u_\lambda) \phi_n \right] + t_n \int_{\Omega} \left[ \tilde{f}(x, u_\lambda) - f(u_\lambda) - Ku_\lambda \phi_n \right] \phi_n \]

\[ \leq \frac{t_n^N}{N} + \tilde{J}_{\lambda}(u_\lambda) + t_n^2 O\left( \delta_n^{-2}(\log n)^{-2/N} + t_n^{N-1} O\left( \delta_n (\log n)^{-(N-1)/N} \right) \right) \]

(since \( \tilde{f}' \geq 0 \) and \( u_\lambda > u \) in \( \Omega \)). Using (6.6), we get

\[ t_n^N \geq (\alpha_N)^{N-1} - C\left( \delta_n^{-2}(\log n)^{-2/N} + \delta_n (\log n)^{-(N-1)/N} \right), \quad C > 0. \quad (6.8) \]

Now \( t_n \) is a point of maximum for the one-dimensional map \( t \mapsto \tilde{J}_{\lambda}(u_\lambda + t \phi_n) \) and hence, \( \frac{d}{dt} \tilde{J}_{\lambda}(t \phi_n + u_\lambda)|_{t=t_n} = 0 \). That is,

\[ \int_{\Omega} |\nabla(u_\lambda + t_n \phi_n)|^{N-2} \nabla(u_\lambda + t_n \phi_n) \nabla \phi_n + \lambda K \int_{\Omega} u_\lambda \phi_n = \lambda \int_{\Omega} \tilde{f}(x, u_\lambda + t_n \phi_n) \phi_n. \quad (6.9) \]

We estimate the RHS of (6.9) from below. Let \( c_n = \min_{|x| \leq \delta_n/n} u_\lambda(x) \). Then,

\[ \int_{\Omega} \tilde{f}(x, u_\lambda + t_n \phi_n) \phi_n \geq \int_{|x| \leq \delta_n/n} h(t_n \phi_n(0)) \phi_n(0) e^{(c_n + t_n \phi_n(0))^{N/(N-1)}} \]

\[ = (\delta_n/n)^N h(t_n \phi_n(0)) \phi_n(0) e^{(c_n + t_n \phi_n(0))^{N/(N-1)}}. \quad (6.10) \]

Using Taylor’s expansion we estimate:

\[ (c_n + t_n \phi_n(0))^{N/(N-1)} \geq (t_n \phi_n(0))^{N/(N-1)} + \frac{c_n N}{N-1} (t_n \phi_n(0))^{1/(N-1)}. \]

Now using (6.6), the explicit value of \( \phi_n(0) \) and the fact that \( c_n \to u_\lambda(0) \) as \( n \to \infty \), the above inequality becomes for all large \( n \),
\[(c + t_n \phi_n(0))^{N/(N-1)} \geq N \left(1 - C \left( \delta_n^{N-2}(\log n)^{-2/N} + \delta_n(\log n)^{-(N-1)/N} \right) \right) \log n \]
\[+ K_0(\log n)^{1/N},\]
for some \(C, K_0 > 0\). Hence, from (6.10) and choosing \(\delta_n = (\log n)^{-1/N}\) we get from the last inequality (for some \(\eta > 0\))
\[
\int_\Omega \tilde{f}(x, u_\lambda + t_n \phi_n) \phi_n \geq \left( (\log n)^{-1/N} / n \right)^N \left( h(t_n \phi_n(0)) \right) \phi_n(0) e^{\eta(\log n)^{1/N}}
\]
\[= \left( (\log n)^{-1/N} / t_n \right) h(t_n \phi_n(0)) t_n \phi_n(0) e^{\eta(\log n)^{1/N}} \]
\[\geq h(t_n \phi_n(0)) t_n \phi_n(0) e^{\eta/2(\log n)^{1/N}}.
\]
By assumption (A5) we get that RHS of (6.10) tends to \(\infty\) as \(n \to \infty\). It is easy to see that LHS of (6.10) is bounded as \(n \to \infty\). This gives a contradiction and proves the lemma. \(\square\)

We recall the following result due to P.L. Lions (see [22])

**Proposition 6.4.** Let \(\{u_n: \|u_n\| = 1\}\) be a sequence in \(W^{1,N}_0(\Omega)\) converging weakly to a non-zero \(u\). Then, for every \(p < (1 - \|u\|^N)^{-1/(N-1)}\),
\[
\sup_n \int_\Omega \exp(p \alpha_\lambda |u_n|^{N/(N-1)}) < \infty. \tag{6.11}
\]

We can now prove the following

**Lemma 6.5.** Let \(u_\lambda\) be the local minimum for \(\tilde{J}_\lambda\) obtained in Lemma 5.3. Then there exists another solution \(v_\lambda \in W^{1,N}_0(\Omega)\) of \((P_\lambda)\) of mountain-pass type.

**Proof.** From Lemma 6.3, we know that the mountain-pass level \(\rho_0 < \tilde{J}_\lambda(u_\lambda) + \frac{1}{N} (\alpha_\lambda)^{N-1}\). Let \(\{v_n\} \subset W^{1,N}_0(\Omega)\) be a Palais–Smale sequence for \(\tilde{J}_\lambda\) at the level \(\rho_0\) around \(F\) (such a sequence always exists by the result in [19]). Then by Lemma 6.2, there exists \(v_\lambda\) such that \(v_n \rightharpoonup v_\lambda\) in \(W^{1,N}_0(\Omega)\), \(\int_\Omega \tilde{f}(x, v_n) \to \int_\Omega \tilde{f}(x, v_\lambda)\), \(\int_\Omega \tilde{F}(x, v_n) \to \int_\Omega \tilde{F}(x, v_\lambda)\). It follows that \(v_\lambda\) is a critical point of \(\tilde{J}_\lambda\) and, as noted in the beginning of this section, this implies \(v_\lambda\) solves \((P_\lambda)\). It only remains to show that \(v_\lambda \neq u_\lambda\). Let us consider the following cases.

**Case I:** \(\rho_0 = \tilde{J}_\lambda(u_\lambda)\), \(v_\lambda \equiv u_\lambda\).

We recall that \(\mathcal{F} = \{v \in W^{1,N}_0(\Omega): \|v - u_\lambda\| = \frac{R_0}{2}\}\). Also,
\[
\tilde{J}_\lambda(u_\lambda) = \rho_0 = \lim_{n \to \infty} \tilde{J}_\lambda(v_n)
\]
\[= \lim_{n \to \infty} \frac{1}{N} \int_\Omega |\nabla v_n|^N + \frac{\lambda K}{2} \int_\Omega v_n^2 - \lambda \int_\Omega \tilde{F}(x, v_n)
\]
\[= \lim_{n \to \infty} \frac{1}{N} \int_\Omega |\nabla v_n|^N + \frac{\lambda K}{2} \int_\Omega u_\lambda^2 - \lambda \int_\Omega \tilde{F}(x, u_\lambda)
\]
\[
\lim_{n \to \infty} \frac{1}{N} \int_{\Omega} |\nabla v_n|^N + \bar{J}_\lambda(u_\lambda) - \frac{1}{N} \int_{\Omega} |\nabla u_\lambda|^N.
\]

Therefore, \(v_n \to u_\lambda\) strongly in \(W^{1,N}_0(\Omega)\), which gives contradiction to the fact that \(\{v_n\}\) is a (PS)\(F,\rho_0\) sequence.

**Case II:** \(\rho_0 \geq \bar{J}_\lambda(u_\lambda), v_\lambda \equiv u_\lambda\).

First we show that

\[
\int_{\Omega} \tilde{f}(x, v_n) v_n \to \int_{\Omega} \tilde{f}(x, u_\lambda) u_\lambda.
\] (6.12)

Since \(\rho_0 < \bar{J}_\lambda(u_\lambda) + \frac{1}{N}(\alpha_N)^{N-1}\), there exists \(\epsilon > 0\) small enough such that

\[
0 < (\rho_0 - \bar{J}_\lambda(u_\lambda))(1 + \epsilon)^{N-1} < \frac{1}{N}(\alpha_N)^{N-1}.
\] (6.13)

Set \(\beta_0 := \lambda \int_{\Omega} \tilde{F}(x, u_\lambda) - \frac{\lambda K}{2} \int_{\Omega} u_\lambda^2\). Then we have

\[
\lim_{n \to \infty} \|v_n\|^N = N \lim_{n \to \infty} \lambda \int_{\Omega} \tilde{F}(x, v_n) - \frac{\lambda K}{2} \int_{\Omega} v_n^2 = N(\rho_0 + \beta_0).
\] (6.14)

Since \(\frac{v_n}{\|v_n\|} \to \frac{u_\lambda}{(N(\rho_0 + \beta_0))^{1/N}}\), choosing \(1 < p < (1 - \frac{\|u_\lambda\|^N}{N(\rho_0 + \beta_0)})^{-1/(N-1)}\), from Proposition 6.4 we get

\[
\sup_n \int_{\Omega} \exp\left(p\alpha_N\left(\frac{v_n}{\|v_n\|}\right)^{N/(N-1)}\right) < \infty.
\]

From this it follows that \(\sup_n \int_{\Omega} |\tilde{f}(x, v_n) v_n|^q < \infty\) for some \(q > 1\). Now Vitali’s convergence theorem gives the conclusion in (6.12).

We note that

\[
\rho_0 = \lim_{n \to \infty} \left[\bar{J}_\lambda(v_n) - \frac{1}{N}[\bar{J}_\lambda'(v_n), v_n]\right]
\]

\[
= \lim_{n \to \infty} \left[\frac{\lambda}{N} \int_{\Omega} \tilde{f}(x, v_n) v_n - \lambda \int_{\Omega} \tilde{F}(x, v_n) + \frac{\lambda K(N-2)}{2N} \int_{\Omega} v_n^2\right]
\]

\[
= \left[\frac{\lambda}{N} \int_{\Omega} \tilde{f}(x, u_\lambda) u_\lambda - \lambda \int_{\Omega} \tilde{F}(x, u_\lambda) + \frac{\lambda K(N-2)}{2N} \int_{\Omega} u_\lambda^2\right]
\]

\[
= \bar{J}_\lambda(u_\lambda) - \frac{1}{N}[\bar{J}_\lambda'(u_\lambda), u_\lambda] = \bar{J}_\lambda(u_\lambda).
\]

This contradicts the assumption that \(\rho_0 > \bar{J}_\lambda(u_\lambda)\). In either case the assumption \(u_\lambda \equiv v_\lambda\) leads to a contradiction, thereby proving the lemma. \(\square\)
Proof of Theorem 1.1. (i) follows from Lemmas 5.3, 6.5, 4.1 and 5.4. (ii) follows from (i) and Lemma 3.2. □

7. Preliminary reductions for ODE analysis

Thanks to the symmetry result of Gidas–Ni–Nirenberg type (see [10,12,21]) any solution to \((P_{\lambda})\) on \(B_1(0)\) is radially symmetric about the origin and radially nonincreasing. Hence we can study the following equivalent ODE formulation of \((P_{\lambda})\) on \(B_1(0)\):

\[
\begin{align*}
-\left(r^{N-1}|u'|^{N-2}u\right)' &= \lambda r^{N-1} f(u) \quad r \in (0, 1), \\
u' &> 0 \\
u'(0) &= u(1) = 0.
\end{align*}
\] (7.1)

Changing the variables as:

\[
\lambda = R^N, \quad s = Rr
\] (7.2)

and letting \(v(s) = u(r)\), we see that if \(u\) solves (7.1) then \(v\) solves the following set of equations and vice-versa:

\[
\begin{align*}
-\left(s^{N-1}|v'|^{N-2}v\right)' &= s^{N-1} f(v) \quad s \in (0, R), \\
v' &> 0 \\
v'(0) &= v(R) = 0.
\end{align*}
\] (7.3)

Thus the problem (7.1) has exactly the same number of solutions for fixed \(\lambda > 0\) as the problem (7.3) for fixed \(R > 0\). We study (7.3) for varying values of \(R > 0\) via the following initial-value problem parametrized by a parameter \(\gamma > 0\):

\[
\begin{align*}
-\left(s^{N-1}|w'|^{N-2}w\right)' &= s^{N-1} f(w), \\
w(0) &= \gamma, \quad w'(0) = 0.
\end{align*}
\] (7.4)

Let \(R_0(\gamma)\) denote the first zero of the solution \(w(., \gamma)\) of (7.4). It can be shown that \(R_0(\gamma)\) is a continuous function of \(\gamma > 0\) and that \(\lim_{\gamma \to 0^+} R_0(\gamma) = 0\) (see [27]).

Clearly, in order to prove Theorem 1.2 it is enough to show the following two properties:
(i) \(R_0(\gamma)\) is a strictly increasing function of \(\gamma > 0\) for all small \(\gamma > 0\) and (ii) \(\liminf_{\gamma \to \infty} R_0(\gamma) > 0\).

To do this, it will be convenient to make the following singular transformation:

\[
r = Ne^{-t/N}, \quad y(t) = w(r).
\] (7.5)

Then it can be checked that \(w\) is a solution of (7.4) if and only if the corresponding \(y\) (defined via (7.5)) is a solution of the following Emden–Fowler type equation:

\[
\begin{align*}
-\left(|y'|^{N-2}y\right)' &= e^{-t} f(y), \\
y(\infty) &= \gamma, \quad y'(\infty) = 0.
\end{align*}
\] (7.6)

Let $T_0(\gamma) = N \log(N R_0(\gamma))$. Clearly $T_0(\gamma)$ is the first zero of the solution $y(., \gamma)$ of (7.6). Therefore in order to prove Theorem 1.2 it is enough to show

\begin{align*}
&\text{(i) } R_0(\gamma) \text{ is a strictly increasing function of } \gamma \text{ for all small } \gamma > 0, \\
&\text{(ii) } \lim_{\gamma \to \infty} T_0(\gamma) < \infty. \quad (7.7)
\end{align*}

8. Uniqueness in the radial case: Proof of Theorem 1.2

In this section we prove Theorem 1.2. It can be easily checked that if $f$ belongs to Class I, then $g(s) = \log f(s) = s^{N/(N-1)} + g_1(s) + \alpha \log s$ satisfies the following hypotheses:

\begin{enumerate}
\item[(H1)] $g \in C^3(0, \infty)$ with $g_1(0) = 0$.
\item[(H2)] $g_1(s) < 0 \forall$ large $s > 0$.
\item[(H3)] $\lim_{s \to \infty} g(s) = (0, 0)$.
\item[(H4)] $\lim_{s \to \infty} s g''(s) - (g'(s))^2 > 0$.
\end{enumerate}

As discussed in the previous section we will show the properties in (7.7) hold for any $f$ satisfying the above hypotheses thereby proving Theorem 1.2.

Lemma 8.1. $R_0(\gamma)$ is a strictly increasing function for all small $\gamma > 0$.

Proof. Choose $\gamma_0 > 0$ so that $[0, \gamma_0) \subset [0, t_*) \cap \{t: t \geq 0, \ R_0(t)^N \in [0, \lambda_0)\}$ where $\lambda_0$ is as in the statement of Lemma 3.2. Suppose there exist $0 < \gamma_1 < \gamma_2 < \gamma_0$ such that $R_0(\gamma_1) = R_0(\gamma_2)$. Let $w_1, w_2$ denote the corresponding solutions of (7.4) with $w_1(0) = \gamma_1, w_2(0) = \gamma_2$. Then $w_1, w_2$ are distinct solutions to (7.3) with $R = R_*$. If we define $u_i(r) = w_i(R_* r), r \in (0, 1), i = 1, 2$, we have that $u_1, u_2$ are distinct solutions to (7.1) with $\lambda = R_*^{N}$. Furthermore we have that $\|u_i\|_{L^\infty((0,1))} = u_i(0) = \gamma_i < t_*$ and $\lambda < \lambda_0$. This contradicts the conclusion of Lemma 3.2. \qed

We now prove the required asymptotic behavior of $T_0(\gamma)$ as $\gamma \to \infty$. For this purpose we adapt the approach in [6] for the general case $N \geq 3$. First, we choose $s_0$ as large as it is required in the subsequent arguments and in particular such that $\log f(s)$ is convex for $s \geq s_0$. Let us define $t_0 = t_0(\gamma) > T_0(\gamma)$ by $y(t_0, \gamma) = s_0$ where $y(., \gamma)$ solves the problem (7.6).

Let

$$g(s) = \log f(s). \quad (8.1)$$

The following function

$$\theta_1(t) = \frac{g(N-1)/N(y(t)))}{t} \quad \text{for } t \geq T_0(\gamma) \quad (8.2)$$
will play an important role. Note that
\[
\theta'_1(t) = \frac{N - 1}{N} y'(t) g'(y(t)) g^{-1/N}(y(t)) t^{-1} - g^{(N-1)/N}(y(t)) t^{-2}.
\] (8.3)

Now, we state the following lemma whose proof will be given subsequently:

**Lemma 8.2.** Let \( f \) satisfy the assumptions (H1)–(H4). Suppose for a sequence \( \gamma_k \to +\infty \), \( T_0(\gamma_k) \to \infty \). Then \( \theta'(t_0) \leq 0 \) for large \( \gamma_k \).

Assuming the above lemma, we can prove the following:

**Proposition 8.3.** Let \( f \) satisfy the hypotheses (H1)–(H4). Then, \( \limsup_{\gamma \to \infty} T_0(\gamma) < \infty \).

**Proof.** We consider only the case \( N > 2 \) since the case \( N = 2 \) is done in [6] (see Main Lemma there). It is easy to show that there exists \( c > 0 \) such that
\[
\frac{N}{N - 1} \frac{g(s)}{g'(s)} \leq s \left( 1 - \frac{c}{s^{N/(N-1)}} \right)
\] for large \( s > 0 \). (8.4)

Since \( y \) solves the problem (7.6) with \( \gamma = \gamma_k \), we have
\[
y(t_0) \leq y'(t_0)(t_0 - T_0(\gamma_k)) + \int_{T_0(\gamma_k)}^{t_0} ds \left( \int_s^{t_0} e^{-u} f(y(u)) \right)^{1/(N-1)}.
\]

Now for any \( u \in [T_0, t_0] \), since \( y(u) \leq y(t_0) = s_0 \), we get \( (f(y(u)))^{1/(N-1)} \leq C_0 = f(s_0)^{1/(N-1)} \). Using (8.4) in the above inequality we get,
\[
s_0 = y'(t_0)(t_0 - T_0(\gamma_k)) + C_0 \int_{T_0(\gamma_k)}^{t_0} ds \left( \int_s^{t_0} e^{-u} \right)^{1/(N-1)}
\leq y'(t_0)t_0 + C_0 e^{-1/(N-1)T_0(\gamma_k)}
\leq s_0 \left( 1 - \frac{c}{s_0^{N/(N-1)}} \right) + C_0 e^{-1/(N-1)T_0(\gamma_k)}
\]
which contradicts the fact that \( T_0(\gamma_k) \to \infty \) as \( k \to \infty \). This completes the proof of Proposition 8.3. \( \Box \)

Let \( y \) be the solution of the problem (7.6) and \( E \) the associated energy function given by:
\[
E(t) = (y')^{N-1} - \frac{N - 1}{N} (y')^N g'(y) - e^{g(y)-t}.
\] (8.5)

It follows that
\[
E'(t) = -\frac{(N - 1)}{N} (y')^{N+1} g''(y).
\] (8.6)
From (8.6), we get \( E'(t) < 0 \) \( \forall \ t \geq t_0 \). Also, (8.5) implies that \( \lim_{t \to \infty} E(t) = 0 \). Hence, we get \( E(t) > 0 \) \( \forall \ t \geq t_0 \). This gives \( \forall \ t \geq t_0 \),

\[
(y')^{N-1} - \frac{(N - 1)}{N} (y')^{N} g'(y) \geq e^{g(y)} - t > 0
\]

and hence,

\[
y' g'(y) < \frac{N}{N - 1}, \quad t \geq t_0. \quad (8.7)
\]

Also, writing

\[
E(t) = (y')^{N-2} \left[ -\frac{N - 1}{N} g'(y) \left( y' - \frac{N}{2(N - 1)g'(y)} \right)^2 + \frac{N}{4(N - 1)g'(y)} \right] - e^{g(y)} - t,
\]

and using \( E(t) > 0 \) for \( t \geq t_0 \) and (8.7) we get,

\[
4(g'(y))^N e^{g(y)} - t \leq \left( \frac{N}{N - 1} \right)^N, \quad \forall t \geq t_0. \quad (8.8)
\]

Taking logarithm on both sides of (8.8) we get

\[
g(y) - t \leq -N \log(g'(y)) + \log c_N, \quad c_N = \frac{1}{4} \left( \frac{N}{N - 1} \right)^N, \quad \forall t \geq t_0. \quad (8.9)
\]

Similarly as in [6], we need the following preliminary results. Since these results are straightforward extensions from \( N = 2 \) to the general case \( N \geq 3 \), we omit their proofs.

**Lemma 8.4.** Assume that for all large \( \gamma > 0 \) there exists \( \tau = \tau(\gamma) > t_0 \) such that \( \frac{E}{(y')^N} \) is decreasing in \( [t_0, \tau] \). Let \( \alpha_0 = \frac{1}{N-1} (1 - \frac{E(\tau)}{y(\tau)^N}) \). Then, the following relations hold in \( [t_0, \tau] \):

\[
y'(t) \geq y'(\tau)e^{\alpha_0(\tau-t) - \frac{g'(y(\tau))}{N}(y(\tau) - y(t))}, \quad (8.10)
\]

\[
y'(t) \leq y'(\tau)e^{\alpha_0(\tau-t) - \frac{1}{N}(g(y(\tau)) - g(y))}, \quad (8.11)
\]

\[
g(y) \geq g(y(\tau)) - N \log \left[ 1 + \frac{1}{N\alpha_0} y'(\tau)g'(y(\tau))(e^{\alpha_0(\tau-t)} - 1) \right], \quad (8.12)
\]

\[
y(t) \leq y(\tau) - \frac{N}{g'(y(\tau))} \log \left[ 1 + \frac{1}{N\alpha_0} y'(\tau)g'(y(\tau))(e^{\alpha_0(\tau-t)} - 1) \right], \quad (8.13)
\]

\[
t \geq \tau - \frac{1}{\alpha_0} \log \left( \frac{N\alpha_0}{y'(\tau)g'(y(\tau))} \right) - \frac{1}{\alpha_0N} g'(y(\tau))(y(\tau) - y(t))
\]

\[
- \frac{1}{\alpha_0} \log \left[ 1 + e^{-\frac{g(y(\tau))}{N}(y(\tau) - y(t))} \left( \frac{y'(\tau)g'(y(\tau))}{N\alpha_0} - 1 \right) \right]. \quad (8.14)
\]
Lemma 8.5. Assume \( \frac{E}{(y')^{N-1}} \) is decreasing in \([t_0, \tau]\), \( \delta = k \log(y(\tau)) \), \( k \) a large (but fixed) positive integer. Let
\[
S_1 = \tau - \frac{\delta}{\alpha_0} + \frac{1}{\alpha_0} \log \left( \frac{g'(y(\tau))y'(\tau)}{N\alpha_0} \right). \tag{8.15}
\]
Assume that \( S_1 \in [t_0, \tau] \). Then, the following asymptotics hold at \( S_1 \):
\[
y(S_1) = y(\tau) - \frac{N\delta}{g'(y(\tau))} + O\left( \frac{\delta^2}{(g(y(\tau))^{2/N})} \right), \tag{8.16}
\]
\[
y'(S_1) = \frac{N\alpha_0}{g'(y(\tau))} \left( 1 + O\left( \frac{\delta^2}{(g(y(\tau))^{2/N})} \right) \right), \tag{8.17}
\]
\[
g(y(S_1)) = g(y(\tau)) - N\delta + O\left( \frac{\delta^2}{(g(y(\tau))^{2/N})} \right). \tag{8.18}
\]

We are now ready to give the Proof of Lemma 8.2. Suppose there is a subsequence, again denoted by \( \gamma_k \), of \( \gamma_k \) such that
\[
\theta'_1(t_0) > 0 \quad \forall \gamma_k. \tag{8.19}
\]
We show that this assumption leads to a contradiction. Define \( \theta_2(t) = (y')^{N-1} - E(t) \).

Step 1. \( \theta'_2(t_0) > 0 \) as \( \gamma_k \to \infty \).

Suppose \( \theta'_2(t_0) \leq 0 \) for a subsequence \( \gamma_{k_i} \) of \( \gamma_k \). Then, from (8.6), we have \( \forall i \)
\[
0 \geq (N - 1)(y'(t_0))^{N-2} \left[ \frac{1}{N} (y'(t_0))^3 g''(s_0) - e^{s(s_0)-i_0} \right].
\]
The assumption (8.19) combined with (8.3) gives the following estimate:
\[
y'(t_0) > \frac{N}{N - 1} t_0^{-3} \left( \frac{g}{g'} \right)(s_0).
\]
Substituting the above estimate into (8.20) we obtain a contradiction since \( t_0(\gamma_{k_i}) \to \infty \) as \( \gamma_{k_i} \to \infty \). This contradiction proves Step 1.

Hence, at \( t_0, \theta'_1(t_0) > 0, \theta'_2(t_0) > 0 \) \( \forall \) large \( \gamma_k \). Also for each fixed \( \gamma_k \),
\[
\lim_{t \to \infty} \theta_1(t) = 0 = \lim_{t \to \infty} \theta_2(t).
\]

Hence, we can find \( \forall \) large \( \gamma_k \), points \( \tau_i = \tau_i(\gamma_k) \) \( (i = 1, 2) \) at which \( \theta_i \) attains its first maximum to the right of \( t_0 \). That is, \( \tau_i > t_0 \) and the following relations hold at \( \tau_i \), for \( i = 1, 2 \) \( \forall \) large \( \gamma_k \):
\[
\theta'_i(\tau_i) = 0, \tag{8.20}
\]
\[
\theta'_i(t) \geq 0 \quad \forall t \in [t_0, \tau_i], \tag{8.21}
\]
\[
\theta''(\tau_i) \leq 0. \tag{8.22}
\]

We now derive some important results on \( \theta_i \) \( (i = 1, 2) \) from (8.20)–(8.22).
Step 2. Results for $\theta_i$: Using (8.3), (8.20) and (8.21) give respectively:

$$y'(\tau_1) = \frac{Ng(y(\tau_1))}{(N-1)g'(y(\tau_1))\tau_1}$$  \hspace{1cm} (8.23)

and

$$y'(t) \geq \frac{Ng(y(t))}{(N-1)tg'(y(t))}, \quad \forall t \in [t_0, \tau_1].$$  \hspace{1cm} (8.24)

Computing explicitly in (8.22), using the expression for $y'(\tau_1)$ in (8.23) and the ODE for $y''$ we get

$$e^{g(y(\tau_1)) - \tau_1} \geq \left( \frac{N}{N-1} \right)^N \left( \frac{g(y(\tau_1))}{\tau_1 g'(y(\tau_1))} \right)^N \left[ \frac{Ng(y(\tau_1))g''(y(\tau_1))}{(g')^2(y(\tau_1))} - 1 \right].$$  \hspace{1cm} (8.25)

From the above inequality it follows that $y(\tau_1) \to \infty$ as $\gamma_k \to \infty$. Taking logarithm on both sides, rearranging and noting that $g(y(\tau_1)) = O((g')^N(y(\tau_1)))$ for all large $k$, we get for some constant $c' > 0$,

$$g(y(\tau_1)) - \tau_1 \geq c'(\log(g'(y(\tau_1))) + \log(\tau_1)).$$  \hspace{1cm} (8.26)

Using (8.8) and (8.26), we get

$$g(y(\tau_1)) - \tau_1 = O(\log \tau_1).$$  \hspace{1cm} (8.27)

Step 3. Results for $\theta_2$: We recall that $\theta_2 = (y')^{N-1} - E(t) = \frac{N-1}{N} (y')^N g'(y) + e^{g(y) - t}$. From (8.20), (8.21), and the ODE for $y$, we get $\forall$ large $\gamma_k$ and $t \in [t_0, \tau_2]$:

$$\frac{N-1}{N} y'(\tau_2)^{N+1} g''(y(\tau_2)) = e^{g(y(\tau_2)) - \tau_2},$$  \hspace{1cm} (8.28)

$$\frac{N-1}{N} (y'(t))^{N+1} g''(y(t)) \geq e^{g(y(t)) - t}.$$  \hspace{1cm} (8.29)

Computing $\theta_2''$ explicitly, from (8.22), we get that

$$e^{g(y(\tau_2)) - \tau_2} \left[ - \frac{N+1}{N} g''(y(\tau_2))(y')^2(\tau_2) + \left( \frac{g'''}{g''} - g' \right)(y(\tau_2))y'(\tau_2) + 1 \right] \leq 0.$$  \hspace{1cm} (8.30)

Now, for large $\gamma_k$, consider the quadratic expression

$$Q(x) = \frac{N+1}{N} g''(y(\tau_2)) x^2 - \left[ \frac{g'''}{g''}(y(\tau_2)) - g'(y(\tau_2)) \right] x - 1.$$
From Eq. (8.30) above, we have \( Q(y'(\tau_2)) \geq 0 \), \( Q(0) = -1 \) and \( \lim_{|x| \to +\infty} Q(x) = +\infty \). Then, \( Q \) has two real roots \( \alpha_{\pm} \) given explicitly by:

\[
\alpha_{\pm} = \frac{N \beta}{2(N + 1)g''(y(\tau_2))} \left[ -1 \pm \sqrt{1 + \frac{4(N + 1)}{N \beta^2} g''(y(\tau_2))} \right]
\]

where \( \beta = (g' - \frac{g''}{g'''})(y(\tau_2)) \). Clearly \( y'(\tau_2) \geq \alpha_+ \), which gives after Taylor expansion,

\[
y'(\tau_2) \geq \beta^{-1} + O(\beta^{-3}). \quad (8.31)
\]

Suppose \( y(\tau_2) \) is bounded for some subsequence of \( \gamma_k \), tending to infinity. Then \( \alpha_+ \) is bounded away from zero along this subsequence and so is \( y'(\tau_2) \). This is not possible due to the relation in (8.28). Hence \( y(\tau_2) \to \infty \) as \( \gamma_k \to \infty \) which implies \( \beta \sim (y(\tau_2))^{1/(N-1)} \) as \( \gamma_k \to \infty \).

In Eq. (8.28) taking logarithm on both sides, we get

\[
g(y(\tau_2)) - \tau_2 = \log \left( \frac{N - 1}{N} \right) + (N + 1) \log(y'(\tau_2)) + \log(g''(y(\tau_2))).
\]

Substituting the lower bound (8.31) for \( y'(\tau_2) \) in the above expression, we get for some \( c > 0 \)

\[
g(y(\tau_2)) - \tau_2 \geq -c \log(y(\tau_2)). \quad (8.32)
\]

Combining (8.8) and (8.32), we have

\[
g(y(\tau_2)) - \tau_2 = O(\log \tau_2). \quad (8.33)
\]

**Step 4.** We consider the case \( \tau_2 < \tau_1 \) for some subsequence of \( \gamma_k \). Then by (8.24),

\[
y'(\tau_2) \geq \frac{Ng(y(\tau_2))}{(N - 1)\tau_2 g'(y(\tau_2))}.
\]

Using (8.33) in the above inequality, we get

\[
y'(\tau_2) \geq \frac{N}{(N - 1)g'(y(\tau_2))} + O\left( \frac{\log(\tau_2)}{\tau_2^{1+1/N}} \right). \quad (8.34)
\]

Now, combining (8.7) and (8.34), we have the following asymptotic expression in the case \( \tau_2 < \tau_1 \) along a subsequence of \( \gamma_k \):

\[
y'(\tau_2) = \frac{N}{(N - 1)g'(y(\tau_2))} + O\left( \frac{\log(\tau_2)}{\tau_2^{1+1/N}} \right). \quad (8.35)
\]

This finishes Step 4.
Let $\tau = \min\{\tau_1, \tau_2\}$. From (8.23), (8.27), (8.33) and (8.35), we obtain

$$y'(\tau) = \frac{N}{(N - 1)g'(y(\tau))} + O\left(\frac{\log \tau}{\tau^{1 + \frac{1}{N}}\log \tau}\right).$$

From (8.36) and the fact $E(\tau) \geq 0$, we obtain

$$e^{g(y(\tau)) - t} \leq 1 - \frac{N - 1}{N} y'(\tau) g'(y(\tau)) = O\left(\frac{\log \tau}{\tau}\right).$$

The above equation and (8.36) imply that if we define $\alpha_0 = \frac{1}{N - 1}(1 - \frac{E(\tau)}{(-y'(\tau))N})$ then

$$\alpha_0 = \frac{1}{N - 1} + O\left(\frac{\log \tau}{\tau}\right).$$

Let $\delta = l \log(y(\tau))$ with $l$ as large (but fixed) as required in the subsequent arguments. Let

$$S_1 = \tau - \frac{\delta}{\alpha_0} + \frac{1}{\alpha_0} \log\left(\frac{g'(y(\tau))y'(\tau)}{N\alpha_0}\right).$$

Clearly we have $S_1 < \tau$.

Claim. $S_1 \geq t_0$ for all large $\gamma_k$.

Proof of the claim. Suppose for some subsequence $\gamma_{k_i} \to \infty$, we have $S_1 < t_0$. Then, $y(S_1) < s_0$ for large $\gamma_{k_i}$. Hence, by Taylor’s expansion, we have for some $\xi \in [S_1, \tau]$, and all large $\gamma_{k_i}$,

$$y(\tau) - y'(\tau) \frac{\delta}{\alpha_0} + y''(\xi) \frac{\delta^2}{2\alpha_0^2} < s_0.$$ 

Since $y'(\tau)\delta = O(\tau^{-1/(N-1)}\log \tau)$, we get for some positive constants $c_1$ and $c_2$,

$$y(\tau) \leq -c_1 y''(\xi)\delta^2 + c_2 \leq c_1 \delta^2 e^{g(y(\xi)) - \xi} + c_2.$$ 

Now, we consider two cases: (i) $\xi \geq t_0$ and (ii) $\xi < t_0$. In case (i), we have $y(\xi) \geq s_0$ and using (8.39) we get, for some positive constant $c$ and all large $\gamma_{k_i}$,

$$\delta^2 e^{g(y(\xi)) - \xi} \leq \frac{c\delta^2}{g'(y(\xi))^N} \leq \frac{c\delta^2}{g'(s_0)^N}.$$ 

If (ii) holds, we have $y(\xi) \leq s_0$ and hence,
In either case, (8.39) implies \( y(\tau) = O(\delta^2) \) which is a contradiction to the fact that \( y(\tau) \sim \tau^{(N-1)/N} \). This proves the claim. \( \square \)

Hence for all large \( \gamma_k \), we have \( S_1 \in [t_0, \tau] \). Since \( \theta_j(t) \not\equiv 0 \) in \([t_0, \tau] \) and \( y' \) is a decreasing function in \([t_0, \tau] \), \( \frac{E}{(y')^{\gamma_k}} \) is a decreasing function in \([t_0, \tau] \). Thus, Lemmas 8.4 and 8.5 apply and the relations (8.10)–(8.18) hold in \([t_0, \tau] \). Using (8.14) we get \( \forall t \in [t_0, \tau] \),

\[
g(y(t)) - t \leq g(y(t)) - \tau + \frac{1}{\alpha_0N}g'(y(t))y(\tau) - y(t) + \frac{1}{\alpha_0} \log\left(\frac{\alpha_0N}{g'(y(\tau))y'(\tau)}\right)
+ \frac{1}{\alpha_0} \log\left[1 + e^{-\frac{g'(y(t))y'(\tau)}{N}(y(\tau)-y(t))}\left(\frac{1}{\alpha_0}g'(y(\tau))y'(\tau) - 1\right)\right]
\equiv \psi(y(t)) + O(1),
\]

where \( \psi(s) = g(s) - \tau + \frac{1}{\alpha_0N}g'(y(t))(y(\tau) - s) \). Since \( \psi''(s) \geq 0 \) for \( s \geq s_0 \), given any \( S_2 \in [t_0, S_1] \), we have \( \forall t \in [S_2, S_1] \),

\[
g(y(t)) - t \leq \max\{\psi(y(S_1)), \psi(y(S_2))\} + O(1). \tag{8.40}
\]

We want to choose \( S_2 \) so that \( g(y(t)) - t \leq -\delta + O(1) \) \( \forall t \in [S_2, S_1] \). First we evaluate \( \psi(y(S_1)) \). From (8.8), since \( y(\tau) \to \infty \) as \( \gamma_k \to \infty \), we get for all large \( \gamma_k \),

\[
g(y(\tau)) - \tau \leq -N\log(g'(y(\tau))) + \log c_N < 0.
\]

Hence, using (8.16), (8.18), (8.37) and (8.38) we get

\[
\psi(y(S_1)) = g(y(\tau)) - \tau - N\delta + \frac{\delta}{\alpha_0} + O\left(\frac{\delta^2}{g'(y(\tau))}\right) \leq -\delta + O(1).
\]

From Step 4, Eqs. (8.37), (8.38) we obtain,

\[
\psi(y(S_2)) = g(y(S_2)) - g(y(\tau)) + \frac{1}{\alpha_0N}g'(y(\tau))y(\tau) - \frac{1}{\alpha_0N}g'(y(\tau))y(S_2) + O(\log \tau)
= y(S_2)^{N/(N-1)} + g_1(y(S_2)) - \frac{1}{\alpha_0N}g'(y(\tau))y(S_2) + \frac{1}{\alpha_0N}g'_1(y(\tau))y(\tau)
- g_1(y(\tau)) + O(\log \tau) + O(\log y(S_2)). \tag{8.41}
\]

Let \( \rho(x) = |x|^{N/(N-1)} + g_1(x) - \frac{1}{\alpha_0N}g'(y(\tau))x + \frac{1}{\alpha_0N}g'_1(y(\tau))y(\tau) - g_1(y(\tau)) + m \log(y(\tau)) \).

We note that \( \rho \) is a strictly convex function on \( \mathbb{R} \). Then for a fixed \( \gamma_k \), \( \rho(0) = \frac{1}{\alpha_0N}g'_1(y(\tau))y(\tau) - g_1(y(\tau)) + m \log(y(\tau)) \) > 0 (from (H2) and (H3)). Let \( a > 1 \). Since \( g'(y(\tau)) \sim y(\tau)^{1/(N-1)} \) and \( g_1(s) < 0 \) for all large \( s > 0 \), we have \( \forall \) large \( \gamma_k \),
\[
\rho(a|g_1(y(\tau))|^{N-1}) \leq O\left(|g'_1(y(\tau))|^{N} - \frac{a}{\alpha_0N}g'_1(y(\tau))\right)^{N-1}y(\tau)^{1/(N-1)} + \frac{1}{\alpha_0N}g'_1(y(\tau))y(\tau) - g_1(y(\tau)) + m\log(y(\tau)).
\]

Using \((H_2)\) and \((H_3)\) we get
\[
\lim_{s \to +\infty} \frac{|g'_1(s)|^N}{as^{1/(N-1)}|g'_1(s)|^{N-1} + g_1(s)\left[\frac{g'_1(s)}{g_1(s)} - \alpha_0N\right]} = 0,
\]
\[
\lim_{s \to +\infty} \frac{\log s}{as^{1/(N-1)}|g'_1(s)|^{N-1} + g_1(s)\left[\frac{g'_1(s)}{g_1(s)} - \alpha_0N\right]} = 0.
\]

In particular we get \(\rho(a|g'_1(y(\tau))|^{N-1}) < 0\) for all \(\gamma_k\) large enough. Hence there exists a zero \(\theta\) of \(\rho\) in \([0, a|g'_1(y(\tau))|^{N-1}]\) with \(\rho'(\theta) < 0\). Clearly \(\theta\) is the first of the two zeroes of \(\rho\). Let
\[
\rho_1(x) = |x|^{N/(N-1)} - \frac{1}{\alpha_0N}g'(y(\tau))x + \mu \quad \text{where} \quad \mu = g_1(\theta) - g_1(y(\tau)) + \frac{1}{\alpha_0N}g'_1(y(\tau))y(\tau) + m\log(y(\tau)).
\]

Note that \(\rho_1(\theta) = 0\). This implies \(\theta \to \infty\) as \(\gamma_k \to \infty\). Otherwise, \(\theta\) is bounded along some subsequence of \(\gamma_k\) and explicitly writing out the equation \(\rho_1(\theta) = 0\) we arrive at the conclusion that \(g(y(\tau))\) and \(g'_1(y(\tau))\) have the same sign for all large \(\gamma_k\) which is impossible in view of assumption \((H_2)\). Now, fix \(k_1\) a large integer. Let \(\eta := 4k_1\log(y(\tau))/g'_1(y(\tau))\). Then, for some \(\xi \in (\theta, \theta + \eta)\),
\[
\rho(\theta + \eta) = \rho'(\theta)\eta + \rho''(\xi)\eta^2
\]
and using \(g_1(\theta) < 0\), we get
\[
\rho'(\theta)\eta \leq \left(\frac{4k_1N}{N-1}\right)\left[\frac{a^{1/(N-1)}|g'_1(y(\tau))|}{g'(y(\tau))} - 1\right]\log(y(\tau)).
\]

By \((H_4)\) we have that \(\rho''\) is bounded in \([\theta, \theta + \eta]\). Hence \(\rho(\theta + \eta) \leq -k_1\log(y(\tau))\). We now choose \(S_2\) so that \(y(S_2) = \theta + \eta\). By choosing \(k_1 > l\), Eq. (8.41) immediately gives \(y(S_2) \leq -\delta + O(1)\). Hence, from (8.40), \(g(y(t)) - t \leq -\delta + O(1)\) \(\forall t \in [S_2, S_1]\). Moreover,
\[
y'(S_2)^{N-1} = y'(S_1)^{N-1} + \int_{S_2}^{S_1} e^{g(y(s)) - s} ds = y'(S_1)^{N-1} + (S_1 - S_2)O(e^{-\delta}).
\]

Hence, from (8.17), we get \(\forall t \in [S_2, S_1], \forall \text{ large } \gamma_k\),
\[
y'(t) = \frac{N\alpha_0}{g'(y(\tau))}\left[1 + O\left(\frac{\delta^2}{g^{2/N}(y(\tau))}\right)\right].
\]

We now evaluate \(y(S_2) = \theta + \eta\). Using \((H_3)\) we get that \(\forall \text{ large } \gamma_k\), \(|g'_1(y(\tau))|^{N-1} = o(1)(y(\tau) + o(1))\). Hence, using the fact that \(g_1\) is a decreasing function, we get for all large \(\gamma_k\):
\[
|g_1(\theta)| \leq |g_1(a|g'_1(y(\tau))|^{N-1})| = |g_1(o(1)(y(\tau) + o(1)))| \leq |g_1(y(\tau))|.
\]
Therefore, \( \mu = O(g_1(y(\tau))) \). Since \( \rho_1(\theta) = 0 \), we get using (8.38) for all large \( \gamma_k \):

\[
-\frac{N-1}{N} g'(y(\tau)) \theta + \mu \leq 0
\]  

(8.43)

and hence

\[
\theta \geq \frac{N \mu}{(N-1)g'(y(\tau))} = O\left(\frac{g_1(y(\tau))}{g'(y(\tau))}\right).  
\]  

(8.44)

Noting that \( \frac{g_1(y(\tau))}{g'(y(\tau))} = o(1) \), we also get that \( \theta \leq \frac{g_1(y(\tau))}{g'(y(\tau))} \) and hence \( \theta \sim \frac{g_1(y(\tau))}{g'(y(\tau))} \).

In particular, the above equation together with (H3) gives that \( \mu \sim |g_1(y(\tau))| \) as \( \gamma_k \to \infty \). Hence, we may rewrite \( y(S_2) = \theta + \eta \) as:

\[
y(S_2) = \frac{N \mu}{(N-1)g'(y(\tau))} + O\left(\frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{1+1/(N-1)}}\right) + O\left(\frac{\log \tau}{g'(y(\tau))}\right).  
\]  

(8.45)

It remains to evaluate \( S_2 \). Using (8.16), (8.42) and (8.45), we have \( \forall \) large \( \gamma_k \) and for some \( \xi \in [S_2, S_1] \)

\[
S_2 = S_1 + \frac{y(S_2) - y(S_1)}{y'(\xi)} = \tau - \frac{\delta}{\alpha_0} + \frac{1}{\alpha_0} \log \left(\frac{g'(y(\tau))y'(\tau)}{N\alpha_0}\right) + \frac{g'(y(\tau))}{N\alpha_0} \left(1 + O\left(\frac{\delta^2}{(g(y(\tau)))^{2/N}}\right)\right) K_{N,\delta,\tau}
\]

where

\[
K_{N,\delta,\tau} = \frac{\mu N}{(N-1)g'(y(\tau))} + O\left(\frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{(2N-1)/(N-1)}}\right) + O\left(\frac{\log \tau}{g'(y(\tau))}\right) - y(\tau) + \frac{N \delta}{g'(y(\tau))} + O\left(\frac{\delta^2}{(g(y(\tau)))^{3/N}}\right).
\]

Using (8.37) and (8.38) the above equation gives \( \forall \) large \( \gamma_k \)

\[
S_2 = g(y(\tau)) + \mu - \left(\frac{N-1}{N}\right) y(\tau) g'(y(\tau)) + O(\log \tau) + O\left(\frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{1+1/(N-1)}}\right)
\]

Substituting in the above equation the expression for \( y(S_2) \) from (8.41) we get
\[ S_2 = \left( g\left(y(S_2)\right) - \psi\left(y(S_2)\right) \right) - \frac{(N - 1)}{N} g'(y(\tau))y(S_2) + \mu + O(\log \tau) \]
\[ + O\left( \frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{N/(N-1)}} \right). \]

Substituting now for \( y(S_2) \) from (8.45), using \( \psi(y(S_2)) = o(\log \tau) \), \( g(y(S_2)) = O\left( \frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{N/(N-1)}} \right) \), we get

\[ S_2 = O\left( \frac{|g_1(y(\tau))|^{N/(N-1)}}{(g'(y(\tau)))^{N/(N-1)}} \right) + O(\log \tau). \]

**Step 6.** Deriving the contradiction:

Since \( \theta'_1(S_2) \geq 0 \), from (8.3) we obtain that

\[ \frac{N}{N - 1} \leq \frac{S_2 g'(y(S_2)) y'(S_2)}{g(y(S_2))} = O\left( \frac{S_2 y'(S_2)}{y(S_2)} \right). \]  

(8.46)

Using (8.42)–(8.46) we get from the above inequality

\[ \frac{N}{N - 1} \leq O\left( \frac{S_2}{\mu} \right) = O\left( \frac{|g_1(y(\tau))|^{1/(N-1)}}{(g'(y(\tau)))^{N/(N-1)}} \right). \]

Since (H3) implies that \( \frac{|g_1(y(\tau))|}{(g'(y(\tau)))^{N}} \to 0 \) as \( \gamma_k \to \infty \) we get a contradiction from the last inequality. This completes the proof of the Lemma 8.2 and hence that of Proposition 8.3 thereby proving Theorem 1.2. \( \square \)

**Acknowledgment**

The authors thank the anonymous referee for suggesting a number of valuable improvements and corrections.

**References**