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The Chow group of zero-cycles on certain Châtelet surfaces over local fields

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1. INTRODUCTION

Let K be a finite extension of $\mathbb{Q}_p$ (p prime). By a Châtelet surface X over K we mean a smooth projective surface K-birational to a surface given by the equation:

$$(1) \qquad y^2 - dz^2 = f(x),$$

where $f(x)$ is a monic cubic separable polynomial in x with coefficients in K. Our main aim is to compute the Chow group $A_0(X)_0$ of 0-cycles of degree zero modulo rational equivalence on such surfaces. The case where $f(x)$ splits into three linear factors has been considered in [8] and [9]. In this paper we consider the remaining cases, in which $f(x)$ is either irreducible or of the form $x(x^2 - e)$, where $e \in K^*$ is not a square.

If $d \in K^{*2}$, then the Châtelet surface defined by the equation $y^2 - dz^2 = x(x^2 - e)$ is K-birational to $\mathbb{P}^2_K$. In fact, in this case the function field of this surface is $K(x, u)$, where $u = y + \sqrt{d}z$. By [2], Prop. 6.1, $A_0(\mathbb{P}^2_K)_0 = 0$. Since $A_0(X)_0$ is a birational invariant of a smooth projective geometrically integral surface [2], Prop. 6.3, we get that $A_0(X)_0$ is zero. Thus, we may assume that $d \notin K^{*2}$.

The main results of this paper are as follows.

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Theorem 1.1. *Let* X *be a Châtelet surface given by the equation* $y^2 - dz^2 = x(x^2 - e)$. *Let* $\mathrm{L} = \mathrm{K}(\sqrt{d})$ *and* $\mathrm{E} = \mathrm{K}(\sqrt{e})$. *If* L *and* E *are isomorphic extensions of* K, *then* $\mathrm{A}_0(\mathrm{X})_0 = \{0\}$.

Theorem 1.2. *Suppose that* $p \neq 2$. *Assume that the quadratic extensions* $\mathrm{L} = \mathrm{K}(\sqrt{d})$ *and* $\mathrm{E} = \mathrm{K}(\sqrt{e})$ *are not isomorphic. Then* $\mathrm{A}_0(\mathrm{X})_0$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.3. *Suppose that* $\mathrm{K} = \mathbb{Q}_2$. *Assume that* $\mathrm{L} = \mathrm{K}(\sqrt{d})$ *and* $\mathrm{E} = \mathrm{K}(\sqrt{e})$ *are non-isomorphic quadratic extensions of* K.

(1) *Suppose that* L/K *is unramified. Then the group* $\mathrm{A}_0(\mathrm{X})_0$ *is isomorphic to*
   (i) $\{0\}$ *if* $v_{\mathrm{K}}(e) \equiv 0 \pmod 4$;
   (ii) $\mathbb{Z}/2\mathbb{Z}$ *if* $v_{\mathrm{K}}(e) \not\equiv 0 \pmod 4$.
(2) *Suppose that* L/K *is a ramified extension. Then* $\mathrm{A}_0(\mathrm{X})_0$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.4. *Let* X *be a Châtelet surface* K*-birational to* $y^2 - dz^2 = f(x)$ *where* $f(x)$ *is an irreducible monic cubic polynomial in* x *with coefficients in* K. *Then* $\mathrm{A}_0(\mathrm{X})_0 = \{0\}$

2. THE METHOD OF COMPUTATION

Let $\mathrm{X} = \mathrm{X}_{d,e}$ denote the Châtelet surface corresponding to the equation

$$y^2 - dz^2 = x(x^2 - e).$$

Let π be a uniformiser of K. The change of variables $x = \pi^2 x'$, $y = \pi^3 y'$, $z = \pi^3 z'$ gives us

$$\mathrm{X}_{d,e'}\colon\ y'^2 - dz'^2 = x'(x'^2 - e'),$$

where $e' = \pi^{-4} e$. Thus it is enough to consider the cases $v_{\mathrm{K}}(e) = 0, 1, 2, 3$. Moreover, using yet another transformation $z \mapsto \lambda z$ for a suitable $\lambda \in \mathrm{K}^*$, it is clear that we need only to consider the cases $v_{\mathrm{K}}(d) = 0, 1$.

Let $\mathrm{CH}_0(\mathrm{X})$ = Chow group of zero cycles on X modulo rational equivalence. $\mathrm{A}_0(\mathrm{X})_0 = \mathrm{Ker}(\mathrm{H}_0(\mathrm{X}) \xrightarrow{\deg} \mathbb{Z})$.

We now describe a method due to Colliot-Thélène and Sansuc [4] which reduces the calculation of $\mathrm{A}_0(\mathrm{X})_0$ to a purely number-theoretic question.

The surface X comes equipped with a morphism $f\colon \mathrm{X} \to \mathbb{P}_1$ whose fibres are conics. We denote by O the singular point of the fibre above ∞. By [2], Théorème C, the map

$$\gamma\colon \mathrm{X}(\mathrm{K}) \to \mathrm{A}_0(\mathrm{X})_0, \qquad \gamma(\mathrm{Q}) = \mathrm{Q} - \mathrm{O},$$

is surjective. We also have a natural injection (see [4])

$$\phi\colon \mathrm{A}_0(\mathrm{X})_0 \to \mathrm{H}^1\big(\mathrm{K}, \mathrm{S}(\overline{\mathrm{K}})\big),$$

where $\overline{K}$ is an algebraic closure of K and S is the K-torus whose character group is the $Gal(\overline{K}/K)$-module $Pic(\overline{X})$ where $\overline{X} = X \times_K \overline{K}$. Thus with the following identifications (see [4])

$$H^1(K, S(\overline{K})) \to K^*/N_{L/K}L^* \times E^*/N_{LE/E}LE^* \to (\mathbb{Z}/2\mathbb{Z})^2$$

the calculation of $A_0(X)_0$ reduces to computing the image of the composite map

$$X(K) \to A_0(X)_0 \to H^1(K, S(\overline{K})) \xrightarrow{\cong} (\mathbb{Z}/2\mathbb{Z})^2.$$

As all the points in the same fibre of the map $f : X(K) \to \mathbb{P}^1(K)$ are mutually equivalent 0-cycles, what we have to compute is the image of the induced map $\chi : f(X(K)) \to (\mathbb{Z}/2\mathbb{Z})^2$. The subset $f(X(K)) \subset \mathbb{P}^1(K)$ is clearly equal to

$$M = \{x \in K^* \mid x(x^2 - e) \in N_{L/K}L^*\} \cup \{0\}.$$

The exact description of the map $\chi : M \to (\mathbb{Z}/2\mathbb{Z})^2$ is given by (see [2,14])

$$\chi(x) = \begin{cases} (\bar{x}, (x - \sqrt{e})^-) & \text{if } x \neq 0, \\ (-\bar{e}, (-\sqrt{e})^-) & \text{if } x = 0, \end{cases}$$

where the bar denotes the image in $K^*/N_{L/K}L^*$ and $E^*/N_{LE/E}LE^*$ respectively, both these quotients being identified with $\mathbb{Z}/2\mathbb{Z}$. By using this map χ we will now prove Theorem 1.1.

Proof of Theorem 1.1. By the above method, to show that $A_0(X)_0 = \{0\}$, it is enough to show that $\chi(M) = \{0\}$. As L and E are isomorphic, the extension LE/E is trivial. Thus the group $E^*/N_{LE/E}LE^*$ is trivial. Therefore for any $x \in M$ we get $\chi(x) = (\bar{x}, 0)$. Since $N_{L/K}L^* = N_{E/K}E^*$, $-e \in N_{L/K}L^*$. Thus $\chi(0) = (0, 0)$. Now let $x \in M\backslash\{0\}$. Note that $x^2 - e \in N_{E/K}E^* = N_{L/K}L^*$. This, together with the fact that $x(x^2 - e) \in N_{L/K}L^*$, implies that $x \in N_{L/K}L^*$ and $\chi(x) = (0, 0)$. □

Before proving Theorem 1.2 we observe that $\chi(M)$ is contained in the diagonal subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$ when L and E are non-isomorphic.

Lemma 2.1. *Let $L/K = K(\sqrt{d})$ and $E/K = K(\sqrt{e})$ be non-isomorphic quadratic extensions. Then $\chi(M)$ is contained in the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, $A_0(X)_0$ is either $\{0\}$ or $\mathbb{Z}/2\mathbb{Z}$.*

Proof. By class field theory (see [16], p. 212) we have the commutative diagram

$$\begin{array}{ccc} E^*/N_{LE/E}LE^* & \xrightarrow{N_{E/K}} & K^*/N_{L/K}L^* \\ \downarrow \text{rec} & & \downarrow \text{rec} \\ Gal(LE/E) & \xrightarrow{\cong} & Gal(L/K), \end{array}$$

where the vertical maps are isomorphisms. The map $\mathrm{Gal}(\mathrm{LE}/\mathrm{E}) \to \mathrm{Gal}(\mathrm{L}/\mathrm{K})$ is an isomorphism since L and E are linearly disjoint. Thus $\mathrm{E}^*/\mathrm{N}_{\mathrm{LE}/\mathrm{E}}\mathrm{LE}^* \to \mathrm{K}^*/\mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$ is an isomorphism, i.e. an element $t \in \mathrm{E}^*$ belongs to $\mathrm{N}_{\mathrm{LE}/\mathrm{E}}\mathrm{LE}^*$ if and only if $\mathrm{N}_{\mathrm{E}/\mathrm{K}}(t)$ belongs to $\mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$. Therefore, for any $x \in \mathrm{M}\backslash\{0\}$, $x \in \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$ if and only if $x - \sqrt{e} \in \mathrm{N}_{\mathrm{LE}/\mathrm{E}}\mathrm{LE}^*$. Thus $\chi(x) = (0,0)$ or $(1,1)$. Similarly, $\chi(0) = (-\bar{e}, (-\sqrt{e})^-) = (0,0)$ or $(1,1)$ depending upon whether $-e \in \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$ or $-e \notin \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$. This shows that $\chi(\mathrm{M})$ is contained in the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Corollary 2.2. *The group* $\mathrm{A}_0(\mathrm{X})_0 \cong \mathbb{Z}/2\mathbb{Z}$ *if and only if at least one of the following condition holds.*

(i) $-e \notin \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$.
(ii) *There exists* $x \in \mathrm{K}^*$, *such that* $x \notin \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$ *and* $x^2 - e \notin \mathrm{N}_{\mathrm{L}/\mathrm{K}}\mathrm{L}^*$.

3. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. Throughout this section let p denote an odd prime and let K denote a finite extension of $\mathbb{Q}_p$.

Lemma 3.1. *Let* F/K *be a quadratic extension of* K.

(i) *If* F/K *is unramified, then an element* $x \in \mathrm{K}^*$ *belongs to* $\mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$ *if and only if* $v_{\mathrm{K}}(x)$ *is even.*
(ii) *If* F/K *is ramified,* π_{F} *is a uniformiser of* F *and* $\pi_{\mathrm{K}} = \mathrm{N}_{\mathrm{F}/\mathrm{K}}(\pi_{\mathrm{F}})$, *then* $x \in \mathrm{K}^*$ *belongs to* $\mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$ *if and only if* $x/\pi_{\mathrm{K}}^{v_{\mathrm{K}}(x)}$ *is a square.*

Proof. (i) It is easy to see that $\mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$ is contained in the subgroup of elements of even valuation. Since both these subgroups are of index two in K^*, they are equal.

(ii) Let N' be the subgroup of all elements $x \in \mathrm{K}^*$ such that $x/\pi_{\mathrm{K}}^{v_{\mathrm{K}}(x)}$ is a square. Using the fact that π_{K} belongs to $\mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$, it is clear that we have $\mathrm{N}' \subset \mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$. Since N' and $\mathrm{N}_{\mathrm{F}/\mathrm{K}}\mathrm{F}^*$ are index-two subgroups of K^*, they must be equal. □

Lemma 3.2. *Let* K *be a finite extension of* $\mathbb{Q}_p$ *where* p *is an odd prime. Suppose that* $e \in \mathrm{K}^*$ *is not a square. Then* $\mathrm{E} = \mathrm{K}(\sqrt{e})$ *is a ramified extension of* K *if and only if* $v_{\mathrm{K}}(e)$ *is odd.*

Proof. If $v_{\mathrm{K}}(e)$ is odd, then we make the reduction to the case where $v_{\mathrm{K}}(e) = 1$ by multiplying e by a square. It is clear that E is ramified when $v_{\mathrm{K}}(e) = 1$. Now suppose $v_{\mathrm{K}}(e)$ is even. We may assume that e is a unit by modifying e by a square. Since p is odd, the polynomial $T^2 - e$ is separable over the residue field, and hence irreducible in the residue field. Thus E/K is unramified in this case. □

Proof of Theorem 1.2. We split the proof into following two cases. To show that $\mathrm{A}_0(\mathrm{X})_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it suffices to verify that one of the conditions of Corollary 2.2 holds.

Case (1): E is unramified over K. In this case L/K, being non-isomorphic to E/K, is a ramified extension. If $-e \notin N_{L/K}L^*$ then $\chi(0) = (1, 1)$ and we are done. Suppose that $-e \in N_{L/K}L^*$. Let us first show that $e \notin N_{L/K}L^*$. Indeed, as $E = K(\sqrt{e})$ is the unramified quadratic extension, $v_K(e)$ is even by Lemma 3.2 and the unit $e\pi^{-v_K(e)}$ is not a square for any uniformiser π of K. Lemma 3.1(ii) now implies that $e \notin N_{L/K}L^*$. Hence $-1 \notin N_{L/K}L^*$; in particular, -1 is not a square.

Write $-1 = x^2 - ey^2$ for some $x, y \in K$. This is possible because $-1 \in N_{E/K}E^*$ by Lemma 3.1(i). Clearly $x \neq 0$ and $y \neq 0$, because neither -1 nor e is a square.

Put $\alpha = x/y$. Replacing α by $-\alpha$ if necessary, we may assume that $\alpha \notin N_{L/K}L^*$. Moreover, $\alpha^2 - e \notin N_{L/K}L^*$ because $\alpha^2 - e = -1/y^2$ and $-1 \notin N_{L/K}L^*$. It follows that $\alpha(\alpha^2 - e) \in N_{L/K}L^*$. Thus $\alpha \in M$ and as $\chi(\alpha) = (1, 1)$ we are done.

Case (2): E is ramified over K. We will show that $\chi(0) = (1, 1)$. As E/K and L/K are quadratic extensions, by local class field theory, their norm subgroups $N_{E/K}E^*$ and $N_{L/K}L^*$ are two index-two subgroups of K^*. These two subgroups are not equal as E and L are not isomorphic. Then, their intersection must be an index-four subgroup of K^*. Since $(K^*)^2 \subset N_{E/K}E^* \cap N_{L/K}L^*$ is also an index four subgroup of K^*, we get $(K^*)^2 = N_{E/K}E^* \cap N_{L/K}L^*$. We know that $-e = N_{E/K}(\sqrt{e}) \in N_{E/K}E^*$. If $-e \in N_{L/K}L^*$ then $-e$ would be a square. This contradicts the fact (by Lemma 3.2) that $v_K(e) = v_K(-e)$ is odd. Thus $-e \notin N_{L/K}L^*$ and $\chi(0) = (1, 1)$. □

4. PRELIMINARIES ON HILBERT SYMBOLS

In this section we review the notion of Hilbert symbol as given in [15].

Let K be a field. Let $a, b \in K^*$. We put $(a, b)_K = 1$ if $ax^2 + by^2 = 1$ has a solution $(x, y) \in K^2$ and $(a, b)_K = -1$ otherwise. The number $(a, b)_K$ is called the *Hilbert symbol* of a and b relative to the field K. It can be shown that the Hilbert symbol has the following properties.

1. $(a, b)_K = (b, a)_K$ (Symmetry).
2. $(ab, c)_K = (a, c)_K \cdot (b, c)_K$ (Bilinearity).
3. $(a^2, b)_K = 1$.
4. $(a, 1 - a)_K = 1$ for $a \neq 1$.

Thus the Hilbert symbol can be thought of as a symmetric bilinear form on the $\mathbb{F}_2$-vector space K^*/K^{*2} with values in the group $\{1, -1\}$.

The following proposition gives an equivalent definition of the Hilbert symbol.

Proposition 4.1 [15], p. 19. *Let $a, b \in K^*$ and let $K_b = K(\sqrt{b})$. Then $(a, b)_K = 1$ if and only if $a \in N_{K_b/K}(K_b^*)$.*

When $K = \mathbb{Q}_p$ we denote the Hilbert symbol by $(a, b)_p$ instead of $(a, b)_{\mathbb{Q}_p}$.

We now describe a formula for calculating the Hilbert symbol when $K = \mathbb{Q}_2$. Let $\mathbb{Z}_2^*$ be the group of units of $\mathbb{Z}_2$ and $U_3 = 1 + 8\mathbb{Z}_2$. Let $\varepsilon, \omega : \mathbb{Z}_2^* \to \mathbb{Z}/2\mathbb{Z}$ be the homomorphisms given by

$$\varepsilon(z) = \frac{z-1}{2} \pmod 2, \qquad \omega(z) = \frac{z^2-1}{8} \pmod 2.$$

Put $a = 2^{\alpha} u, b = 2^{\beta} v$ where u and v are units. Then, according to [15], p. 20,

$$(a, b)_2 = (-1)^{\varepsilon(u)\varepsilon(v) + \omega(v)\alpha + \omega(u)\beta}.$$

Using the above formula for the 2-adic Hilbert symbol, we prove the following lemma which will be used in the proof of Theorem 1.3.

Lemma 4.2. *Let* $K = \mathbb{Q}_2$. *Let* e, d, L, E *be as in Theorem* 1.3. *Assume that* $v_K(d) = 1$. *Then,*

(i) *If* $v_K(e) = 1$, *then at least one of the elements* $-1, 1-e, e$ *does not belong to* $N_{L/K}L^*$
(ii) *If* $v_K(e) = 3$, *then at least one of the elements* $-1, (1-e/4), e$ *does not belong to* $N_{L/K}L^*$.

Proof. (i) Assume that all three elements $-1, 1-e, e$ belong to $N_{L/K}L^*$. Then by Proposition 4.1, $(-1, d)_2 = (e, d)_2 = (1-e, d)_2 = 1$. Let $e = 2u$ and $d = 2v$ where $u, v \in \mathbb{Z}_2^*$. We have,

$$(-1, d)_2 = (-1, 2v)_2 = (-1, 2)_2(-1, v)_2 = (-1)^{\varepsilon(v)}.$$

As $(-1, d)_2 = 1$ by assumption, we have

(2) $\qquad \varepsilon(v) = 0 \quad \text{and thus} \quad v \equiv 1 \pmod 4.$

Also,

$$(1-e, d)_2 = (1-2u, 2v)_2 = (-1)^{\varepsilon(1-2u)\varepsilon(v) + \omega(1-2u)}.$$

As $\varepsilon(v) = 0$ and by assumption $(1-e, d)_2 = 1$, we get the following

(3) $\qquad \omega(1-2u) = 0 \quad \text{and} \quad u \equiv 1 \pmod 4.$

Now,

$$(e, d)_2 = (2u, 2v)_2 = (-1)^{\varepsilon(u)\varepsilon(v) + \omega(v) + \omega(u)}.$$

As $\varepsilon(v) = 0$ and $(e, d)_2 = 1$ by assumption, we get $\omega(u) = \omega(v)$. Since both u, v are congruent to 1 modulo 4 by (2) and (3), one can check that $u \equiv v \pmod 8$. As any element of U_3 is a square, we get that u and v differ by a square unit. Thus the extensions $L = \mathbb{Q}_2(\sqrt{2v})$ and $E = \mathbb{Q}_2(\sqrt{2u})$ are isomorphic. This contradicts the hypothesis that L and E are non-isomorphic extensions of K.

(ii) Since the proof of this case is similar to the one above, we only give a sketch. Assume that all three elements $-1, (1 - e/4), e$ belong to $N_{L/K}L^*$. Let $e = 2^3 u$ and $d = 2v$ where $u, v \in \mathbb{Z}_2^*$. As in (i), using $(-1, d)_2 = 1$ we get $\varepsilon(v) = 0$ and thus $v \equiv 1$ mod 4. Similarly $(1 - e/4, d)_2 = 1$ gives $\omega(1 - 2u) = 1$ and thus $u \equiv 1$ mod 4. By properties 2 and 3 of the Hilbert symbol mentioned earlier, we deduce that

$$(e, d)_2 = \left(2^3 u, 2v\right)_2 = (2u, 2v)_2$$

Thus $(e, d)_2 = 1$ gives $\omega(u) = \omega(v) = 1$. As in (i), we arrive at a contradiction by showing that L and E are isomorphic extension of $\mathbb{Q}_2$. □

5 SOME RESULTS ON RAMIFIED QUADRATIC EXTENSION

Throughout this section K will denote a finite extension of $\mathbb{Q}_2$. Let L/K be a ramified quadratic extension. Let k be the residue field of K. Since L/K is totally ramified, k is also the residue field of L. For $i \geqslant 0$, let $U_{i,L} = \{x \in U_L \mid v_L(1 - x) \geqslant i\}$. The subgroups $\{U_{i,L}\}_{i \geqslant 0}$ define a decreasing filtration of U_L. Similarly we define $U_{i,K}$.

Theorem 5.1 [10], III.1.4. *Let π_L be a uniformiser of L. Let σ be the generator of* Gal(L/K). *Then $\frac{\sigma(\pi_L)}{\pi_L} \in U_{1,L}$. Further, if s is the largest integer such that $\frac{\sigma(\pi_L)}{\pi_L} \in U_{s,L}$, then s is independent of the uniformiser π_L.*

Thus the integer s defined above depends only on the extension L/K. Therefore we will denote it by $s(L/K)$.

Theorem 5.2 [10], III.2.3. *Let e_K be the ramification index of K over $\mathbb{Q}_2$. Then $s(L/K) \leqslant 2e_K$.*

Remark 5.3. When $K = \mathbb{Q}_2$, K has six ramified quadratic extensions, namely $K(\sqrt{-1}), K(\sqrt{-5}), K(\sqrt{\pm 2}), K(\sqrt{\pm 10})$. For the first two extensions $s = 1$ and for the remaining extensions $s = 2$.

Theorem 5.4 [10], III.1.5. *Let L, K be as above. Assume that $k = \mathbb{F}_2$. Let $s = s(L/K)$. Choose a uniformiser π_L of L. Let $\pi_K = N_{L/K}(\pi_L)$. Define*

$$\lambda_{i,L} : U_{i,L} \to \mathbb{F}_2, \quad \lambda_{i,L}\left(1 + \theta \pi_L^i\right) = \bar{\theta}.$$

Similarly define $\lambda_{i,K}$ using the uniformiser π_K. Then

1. *The following diagrams commute:*

$$\begin{array}{ccc} U_{i,L} & \xrightarrow{\lambda_{i,L}} & \mathbb{F}_2 \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle Id} \\ U_{i,K} & \xrightarrow{\lambda_{i,K}} & \mathbb{F}_2 \end{array} \qquad \textit{if } 1 \leqslant i < s,$$

$$\begin{array}{ccc} U_{s,L} & \xrightarrow{\lambda_{s,L}} & \mathbb{F}_2 \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle 0} \\ U_{s,K} & \xrightarrow{\lambda_{s,K}} & \mathbb{F}_2, \end{array}$$

$$\begin{array}{ccc} U_{s+2i,L} & \xrightarrow{\lambda_{s+2i,L}} & \mathbb{F}_2 \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle Id} \\ U_{s+i,K} & \xrightarrow{\lambda_{s+i,K}} & \mathbb{F}_2 \end{array} \qquad \textit{if } i > 0.$$

2. $N_{L/K}(U_{s+i,L}) = N_{L/K}(U_{s+i+1,L})$ *for* $i > 0$, $p \nmid i$.
3. $N_{L/K}(U_{s+1,L}) = U_{s+1,K}$.

The following corollary is an easy consequence of the above theorem. However part (iii) of the corollary will play an important role in the proof of Theorem 1.3.

Corollary 5.5. *With the notation as above,* $N_{L/K}(U_{i,L}) \subset U_{i,K}$ *for* $1 \leqslant i \leqslant s+1$. *Thus for* $i \leqslant s$, *we have induced maps*

$$N^i_{L/K} : \frac{U_{i,L}}{U_{i+1,L}} \to \frac{U_{i,K}}{U_{i+1,K}}.$$

Further,

(i) $N^i_{L/K}$ *is an isomorphism for* $1 \leqslant i < s$.
(ii) $N^s_{L/K}$ *is the zero map.*
(iii) $\gamma \in U_{s,K} \backslash U_{s+1,K} \Longrightarrow \gamma \notin N_{L/K}L^*$.

Proof. (1) and (2) are immediate consequences of the first two commutative diagrams in Theorem 5.4 and the fact that $U_{i+1,L}$ is the kernel of $\lambda_{i,L}$. Suppose that $\gamma \in U_{s,K} \backslash U_{s+1,K}$. This gives that $\lambda_{s,K}(\gamma) \neq 0$ and thus the second commutative diagram in Theorem 5.4 tells us that $\gamma \notin N_{L/K}(U_{s,L})$. Now we want to show that for $i \neq s$ and for any $x \in U_{i,L} \backslash U_{i+1,L}$, $N_{L/K}(x) \neq \gamma$. For $i < s$ this is easy to see by (i) above. For $i > s$, Theorem 5.4(3) implies that $N_{L/K}(x) \in U_{s+1,K}$. Thus $\gamma \neq N_{L/K}(x)$. □

6 THE PROOF OF THEOREM 1.3

We first state the following lemma from [16].

Lemma 6.1 [16], p. 212. *Let* $K/\mathbb{Q}_2$ *be a finite extension. Let* e_K *be the ramification index of* $K/\mathbb{Q}_2$. *For every* $n > e_K$ *the map* $U_{n,K} \stackrel{x\mapsto x^2}{\longrightarrow} U_{n+e_K,K}$ *is an isomorphism of* $\mathbb{Z}_2$*-modules.*

Now the proof of Theorem 1.3 will occupy the rest of this section. Henceforth $K = \mathbb{Q}_2$. For simplicity of notation we write $\mathbb{Z}_2^*$ instead of $U_{\mathbb{Q}_2}$, U_n instead of $U_{n,\mathbb{Q}_2}$ and v instead of $v_{\mathbb{Q}_2}$. For any field extension $F/\mathbb{Q}_2$, we will write $N(F^*)$ instead of $N_{F/\mathbb{Q}_2}(F^*)$.

Proof of Theorem 1.3. Recall from Section 2 that it is enough to consider the cases $v(d) = 0, 1$ and $v(e) = 0, 1, 2$ or 3.

(1) L *is unramified over* $\mathbb{Q}_2$.

Case 1 $v(e) = 0$. To show that $A_0(X)_0$ is zero, we have to show that $\chi(x) = 0$ for every $x \in M$. If $x \in M\backslash\{0\}$ such that $v(x) > 0$ then

$$v\big(x^2 - e\big) = \min\{2v(x), v(e)\} = 0.$$

If $x \in M\backslash\{0\}$ such that $v(x) < 0$ then

$$v\big(x^2 - e\big) = \min\{2v(x), v(e)\} = 2v(x).$$

Thus whenever $v(x) \neq 0$, $v(x^2 - e)$ is even and hence by Lemma 3.1, $x^2 - e \in N(L^*)$. Thus for any $x \in M\backslash\{0\}$ with $v(x) \neq 0$, $\chi(x) = (0, 0)$. If $x \in M$ such that $v(x) = 0$ then $x \in N(L^*)$ and thus $\chi(x) = (0, 0)$. We now claim that $\chi(0) = (0, 0)$. For this we need to show that $-e \in N(L^*)$. But this is clear since $v(-e) = 0$ and L is the unramified extension of $\mathbb{Q}_2$. This proves Theorem 1.3(1)(i).

Now we prove Theorem 1.3(1)(ii).

Case $v(e) = 1$ or 3. Choose $x \in \mathbb{Q}_2^*$ such that $v(x)$ is odd and $2v(x) > v(e)$. Then by Lemma 3.1, $x \notin N(L^*)$ and $x^2 - e \notin N(L^*)$. Therefore $x \in M$ and $\chi(x) = (1, 1)$ i.e $A_0(X)_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Case $v(e) = 2$. We claim that $\chi(2) = (1, 1)$. Let $\beta = e/4$. Since e is not a square in $\mathbb{Q}_2$, β cannot be a square and thus $\beta \notin U_3$. We now show that $\beta \notin U_2$. Note that, an element $\alpha \in U_2\backslash U_3$, can be written as $\alpha = 5\gamma^2$ for some $\gamma \in U_1$. Thus, $\mathbb{Q}_2(\sqrt{5}) = \mathbb{Q}_2(\sqrt{\alpha})$, where $\mathbb{Q}_2(\sqrt{5})$ is the unramified quadratic extension of $\mathbb{Q}_2$. This implies that $\beta \notin U_2$, since $\mathbb{Q}_2(\sqrt{\beta}) = E$ is a ramified quadratic extension of $\mathbb{Q}_2$. Thus $\beta \in U_1\backslash U_2$ and $v(2^2 - e) = v(2^2(1 - \beta)) = 3$. By Lemma 3.1(i), $2^2 - e \notin N(L^*)$ and similarly $2 \notin N(L^*)$. Therefore $2 \in M$ and $\chi(2) = (1, 1)$ i.e. $A_0(X)_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The proof of this case completes the proof of the Theorem 1.3(1).

(2) L *is ramified over* $\mathbb{Q}_2$.

We first prove the following lemma which will be crucially used in the proof.

Lemma 6.2. *Suppose that* $L/\mathbb{Q}_2$ *is a ramified extension and either* -1 *or* e *does not belong to* $N_{L/K}L^*$. *Then* $A_0(X)_0 \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Suppose that $-1 \notin N(L^*)$. Since L and E are non-isomorphic quadratic extensions, by class field theory we can choose $\alpha \in N(E^*)$ such that $\alpha \notin N(L^*)$. Thus $\alpha = x^2 - ey^2$ for some $x, y \in \mathbb{Q}_2$. Since $\alpha \notin N(L^*)$, $y \neq 0$. If $x = 0$ then $-e \notin N(L^*) \Rightarrow \chi(0) = (1, 1) \Rightarrow A_0(X)_0 \cong \mathbb{Z}/2\mathbb{Z}$. Thus we assume that $x \neq 0$. Now

$$\left(\frac{x}{y}\right)^2 - e = \frac{\alpha}{y^2} \notin N(L^*).$$

If $\frac{x}{y} \notin N(L^*)$ then $\frac{x}{y} \in M$ and $\chi(\frac{x}{y}) = (1, 1)$. Otherwise since $-1 \notin N(L^*)$, $-\frac{x}{y} \in M$ and $\chi(-\frac{x}{y}) = (1, 1)$.

Now suppose $e \notin N(L^*)$. We may assume $-1 \in N(L^*)$ since otherwise we are done by the above case. Thus $-e \notin N(L^*)$. Hence $\chi(0) = (1, 1)$. □

Now we prove the Theorem 1.3(2). The proof splits into two parts depending upon the invariant $s(L/\mathbb{Q}_2)$ of the field extension. Since $L/\mathbb{Q}_2$ is a ramified quadratic extension, $s(L/\mathbb{Q}_2) = 1$ or 2, by Theorem 5.2. We want to show that $A_0(X)_0 \cong \mathbb{Z}/2\mathbb{Z}$. If $s(L/\mathbb{Q}_2) = 1$, we will appeal to Lemma 6.2. If $s(L/\mathbb{Q}_2) = 2$, we will appeal to Corollary 2.2.

Case $s(L/K) = 1$. It is clear that $-1 \in U_1 \backslash U_2$. Thus by Corollary 5.5(iii), $-1 \notin N(L^*)$. Thus $A_0(X)_0 \cong \mathbb{Z}/2\mathbb{Z}$ by Lemma 6.2.

Case $s(L/K) = 2$. We may assume that -1 and e belong to $N(L^*)$, since otherwise we are done by Lemma 6.2. The proof of this case consists of following subcases.

Case $v(e) = 0$. Choose a uniformiser π of $\mathbb{Q}_2$ which does not belong to $N(L^*)$. Put $x = \pi e$. Since we have assumed that $e \in N(L^*)$, we get $x \notin N(L^*)$. Write $x^2 - e = -e(1 - \pi^2 e)$. Then $1 - \pi^2 e \in U_2 \backslash U_3$ and by Corollary 5.5(iii) we get $1 - \pi^2 e \notin N(L^*)$. Since $-e \in N(L^*)$, $x^2 - e \notin N(L^*)$. Therefore $x \in M$ and $\chi(x) = (1, 1)$.

Case $v(e) = 1$. Since $s(L/\mathbb{Q}_2) = 2$, by Remark 5.3 we know that $v(d) = 1$. In this case, by Lemma 4.2(i), at least one of the elements $-1, e, 1 - e$ does not belong to $N(L^*)$. By our assumption, $-1, e \in N(L^*)$. Thus $-e \in N(L^*)$ and by Lemma 4.2(i), $1 - e \notin N(L^*)$.

Put $x = eu$ where u is any unit which is not in $N(L^*)$. Thus $x \notin N(L^*)$. Then $x^2 - e = -e(1 - eu^2)$. Since $u^2 \in U_3$, $1 - eu^2 \equiv 1 - e \pmod 8$. This implies that $1 - eu^2 \notin N(L^*)$ and thus $x^2 - e \notin N(L^*)$. Therefore $x \in M$ and $\chi(x) = (1, 1)$.

Case $v(e) = 2$. Suppose $x \in U_2 \backslash U_3$. Then $x \notin N(L^*)$ by Corollary 5.5(iii). Now, $x^2 \in U_3$ and $v(e) = 2$ implies that $x^2 - e \in U_2 \backslash U_3$. Therefore $x^2 - e \notin N(L^*)$ by Corollary 5.5(iii). Thus, $\chi(x) = (1, 1)$.

Case $v(e) = 3$. Choose an element $x = 2u$ such that $x \notin N(L^*)$. Then $x^2 - e = 4(u^2 - e/4)$. We have, $u^2 - e/4 \equiv (1 - e/4) \pmod 8$. Since $s(L/\mathbb{Q}_2) = 2$, by Remark 5.3, we get that $v(d) = 1$. Thus by Lemma 4.2(ii), at least one of the elements $-1, e, (1 - e/4)$ does not belong to $N(L^*)$. By our assumption, $-1, e \in N(L^*)$. Thus $1 - e/4 \notin N(L^*)$ which implies that $x^2 - e \notin N(L^*)$. Therefore $x \in M$ and $\chi(x) = (1, 1)$.

This completes the proof of Theorem 1.3. □

7 THE IRREDUCIBLE CUBIC CASE

Let K be any finite extension of $\mathbb{Q}_p$. In this section X will denote a smooth projective surface K-birational to the surface defined by the equation

$$y^2 - dz^2 = f(x),$$

where $f(x) = x^3 + ax^2 + bx + c$ is an irreducible monic cubic polynomial with coefficients in K. In this section we prove Theorem 1.4 which says that $A_0(X)_0 = \{0\}$. As mentioned in Section 1, $d \in K^{*2}$ implies that X is K-birational to $\mathbb{P}^2$ and hence $A_0(X)_0 = \{0\}$. Thus we may assume $d \notin K^{*2}$. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $f(x)$ in an algebraic closure of K. Let $E_i = K(\alpha_i)$ and $L = K(\sqrt{d})$.

7.1. The method of computation of $A_0(X)_0$

Let $M = \{x \in K \mid x^3 + ax^2 + bx + c \in N_{L/K}L^*\}$. As in Section 2, results of [4] reduce the problem of computing the Chow group to the determination of the image of M under the map,

$$\chi : M \to E_1^*/N_{LE_1/E_1}(LE_1^*) \times E_2^*/N_{LE_2/E_2}(LE_2^*) \times E_3^*/N_{LE_3/E_3}(LE_3^*),$$
$$x \mapsto (x - \alpha_1, x - \alpha_2, x - \alpha_3).$$

Lemma 7.1. $x - \alpha_i \in N_{LE_i/E_i}(LE_i^*)$ *if and only if* $x^3 + ax^2 + bx + c \in N_{L/K}L^*$.

Proof. As in the proof of Lemma 2.1, class field theory implies that $x - \alpha_i \in N_{LE_i/E_i}(LE_i^*)$ if and only if $N_{E_i/K}(x - \alpha_i) \in N_{L/K}L^*$. Since $N_{E_i/K}(x - \alpha_i) = x^3 + ax^2 + bx + c$ the lemma follows. □

Proof of Theorem 1.4. By definition of M, $x \in M$ implies $x^3 + ax^2 + bx + c \in N_{L/K}L^*$. Therefore by Lemma 7.1 we get $x - \alpha_i \in N_{LE_i/E_i}(LE_i^*)$ for $i = 1, 2, 3$. Thus $\chi(x) = (0, 0, 0)$. Thus $A_0(X)_0 \cong \{0\}$.

8. THE GLOBAL CASE

Let K be a number field. Let X be any smooth projective surface K-birational to the surface given by the equation

$$y^2 - dz^2 = x(x^2 - e),$$

where $d \notin K^{*2}$. Let K_v be the completion of K at v and $X_v = X \times_K K_v$.

By the result of Bloch [1], Theorem (0.4), $A_0(X_v)_0 = 0$ for almost all places v of K. We can also observe this directly as follows. At almost all places of K, $K_v(\sqrt{d})$ and $K_v(\sqrt{e})$ are unramified extensions of K_v. If $d \in K_v^{*2}$, then we have already seen in the section 1 that $A_0(X_v)_0 = 0$. If $e \in K_v^{*2}$ and v is not a place lying above the prime ideal (2) then, $\sqrt{e}, -\sqrt{e}, 2\sqrt{e}$ are units. Thus by [2], Proposition 4.7, $A_0(X_v)_0 = 0$. If neither e, nor d is in K_v^{*2}, then $K_v(\sqrt{d})$ and $K_v(\sqrt{e})$ are both unramified quadratic extensions and thus isomorphic extensions of K_v. In this case, by Theorem 1.1 of this paper we get that $A_0(X_v)_0 = 0$.

Let us recall briefly how the results of Colliot-Thélène, Sansuc and Swinnerton-Dyer [5], Colliot-Thélène and Sansuc [4] (see Salberger [13] for more general statement valid for all conic bundles over $\mathbb{P}^1_K$), which allow one to compute $A_0(X)_0$ from the knowledge of $A_0(X_v)_0$. For each place v of K, we have a map $A_0(X)_0 \to A_0(X_v)_0$ and hence, a diagonal map

$$\delta : A_0(X)_0 \to \prod_v A_0(X_v)_0.$$

As we have seen above, $A_0(X_v)_0 = 0$ for almost all places v, the target of δ is the same as $\bigoplus_v A_0(X_v)_0$. The exactness of the following sequence is proved in [6], Section 8

$$0 \to Ш^1(K, S) \to A_0(X)_0 \xrightarrow{\delta} \bigoplus_v A_0(X_v)_0 \to \mathrm{Hom}\big(H^1(K, \hat{S}), \mathbb{Q}/\mathbb{Z}\big)$$

in which $\hat{S} = \mathrm{Pic}(\overline{X})$, a free $\mathbb{Z}$-module of finite rank with a $\mathrm{Gal}(\overline{K}/K)$-action, and S is the K-torus dual to $\hat{S}$. The exactness of this sequence reduces the computation of $A_0(X)_0$ to the local problem of computing $A_0(X_v)_0$.

Let us indicate how the results of this paper contribute to the solution of this local problem, at least when $K = \mathbb{Q}$. Let X be a Châtelet surface as above. When v is a finite place of $\mathbb{Q}$, $A_0(X_v)_0$ can be calculated using results of this paper when $x^2 - e$ remains irreducible and [8,9] when $x^2 - e$ splits into linear factors over v. One now needs to do these calculations when v is the real place of $\mathbb{Q}$, i.e. $K_v = \mathbb{R}$. If $x^2 - e$ remains irreducible over $\mathbb{R}$, then either $L/\mathbb{R}$ is the trivial extension or the extensions L and E become isomorphic. Thus in both these cases, $A_0(X_v)_0 = \{0\}$. When $x^2 - e$ is reducible, we use results of [3] to calculate $A_0(X_v)_0$.

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