

Note

The cubicity of hypercube graphs

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Abstract

For a graph G , its *cubicity* $\text{cub}(G)$ is the minimum dimension k such that G is representable as the intersection graph of (axis-parallel) cubes in k -dimensional space. (A k -dimensional cube is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where R_i is a closed interval of the form $[a_i, a_i + 1]$ on the real line.) Chandran et al. [L.S. Chandran, C. Mannino, G. Oriolo, On the cubicity of certain graphs, *Information Processing Letters* 94 (2005) 113–118] showed that for a d -dimensional hypercube H_d , $\frac{d-1}{\log d} \leq \text{cub}(H_d) \leq 2d$. In this paper, we use the probabilistic method to show that $\text{cub}(H_d) = \Theta\left(\frac{d}{\log d}\right)$. The parameter *boxicity* generalizes cubicity: the boxicity $\text{box}(G)$ of a graph G is defined as the minimum dimension k such that G is representable as the intersection graph of axis-parallel boxes in k -dimensional space. Since $\text{box}(G) \leq \text{cub}(G)$ for any graph G , our result implies that $\text{box}(H_d) = O\left(\frac{d}{\log d}\right)$. The problem of determining a non-trivial lower bound for $\text{box}(H_d)$ is left open.

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1. Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe U , where V is an index set. The intersection graph $\Omega(\mathcal{F})$ of \mathcal{F} has V as a vertex set, and two distinct vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometrical objects are a well-studied topic in graph theory. Two well-known concepts in this area of graph theory are the cubicity and the boxicity. These concepts were introduced by Roberts in 1969 [12] and they find applications in niche overlap in ecology and to problems of fleet maintenance in operations research. (See [7].)

A k -dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where R_i (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line. A k -dimensional cube is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where R_i is a closed interval of the form $[a_i, a_i + 1]$ on the real line. For a graph G , its *boxicity* is the minimum dimension k , such that G is representable as the intersection graph of (axis-parallel) boxes in k -dimensional space. We denote the boxicity of a graph G by $\text{box}(G)$. When the boxes are restricted to be (axis-parallel) k -dimensional cubes, the minimum dimension k required to represent G is called the *cubicity* of G and is denoted by $\text{cub}(G)$. It is easy to see that for any graph G , $\text{box}(G) \leq \text{cub}(G)$.

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The d -dimensional hypercube H_d on 2^d vertices is defined as follows. The vertices of H_d correspond to the 2^d binary strings each of length d , two of the vertices being adjacent *if and only if* the corresponding binary strings differ in exactly one-bit position. Hypercubes are a well-studied class of graphs, which arise in the context of parallel computing, coding theory, algebraic graph theory and many other areas. Hypercubes are popular among graph theorists because of their symmetry, small diameter and many other interesting graph-theoretic properties.

It was shown by Chandran, Mannino and Oriolo [2] that $\frac{d-1}{\log d} \leq \text{cub}(H_d) \leq 2d$. In this paper, we show the following:

$$\text{cub}(H_d) = \Theta\left(\frac{d}{\log d}\right).$$

Since $\text{cub}(G)$ is an upper bound for $\text{box}(G)$, clearly the above result also implies that $\text{box}(H_d) \leq \frac{cd}{\log d}$, where c is a constant. Such an upper bound for $\text{box}(H_d)$ was not known before. We leave open the question of determining a non-trivial lower bound for $\text{box}(H_d)$.

1.1. A brief literature survey on cubicity and boxicity

It was shown by Cozzens [6] that computing the boxicity of a graph is NP-hard. This was later improved by Yannakakis [17], and finally by Kratochvíl [11] who showed that even deciding whether boxicity of a graph is at most 2 is NP-complete. The complexity of finding the maximum independent set in graphs whose boxicity is at most a given constant was considered by [10,9].

There have also been attempts to estimate or bound the boxicity of graph classes with special structure. Scheinerman [13] showed that the boxicity of outer planar graphs is at most 2. Thomassen [15] proved that the boxicity of planar graphs is bounded above by 3. Upper bounds for the boxicity of many other graph classes such as chordal graphs, AT-free graphs, permutation graphs etc., were shown in [3] by relating the boxicity of a graph with its treewidth.

Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [16], the rectangle number [5], grid dimension [1], circular dimension [8,14] and the boxicity of digraphs [4] are some examples.

2. Definitions and notation

Let G be a undirected simple graph. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of G , respectively.

As mentioned in the introduction, a string of length d consisting only of 0's and 1's (i.e., a *binary string*) can be associated (in one-to-one correspondence) with each vertex of a d -dimensional hypercube H_d , such that two vertices u and v are adjacent if and only if their corresponding binary strings differ in exactly one position. Let $f(v)$ denote the binary string associated with the vertex v . The value of the binary digit (i.e., *bit*) at the i th position of the binary string $f(v)$ will be denoted by $f_i(v)$.

Given two vertices u and v , let $D(u, v) = \{i : 1 \leq i \leq d \text{ and } f_i(v) \neq f_i(u)\}$. That is, $D(u, v)$ is the set of “positions” where the bit values of $f(u)$ and $f(v)$ differ from each other. The Hamming distance between $f(u)$ and $f(v)$ is defined to be $|D(u, v)|$. It is easy to observe that the shortest distance $\delta(u, v)$ between two vertices u and v equals the Hamming distance between $f(u)$ and $f(v)$. That is, $\delta(u, v) = |D(u, v)|$.

Unit Interval graphs: A graph G is a unit interval graph if and only if each vertex of G can be mapped to a (closed) interval of unit length on the real line such that two distinct vertices are adjacent in G if and only if the corresponding (unit) intervals intersect.

The following characterization of cubicity is easy to prove. (See [12].)

Lemma 1. *Let G be a simple graph. Let t be the minimum integer such that there exist t unit interval graphs G_1, G_2, \dots, G_t on the same vertex set as G (i.e., $V(G_i) = V(G)$ for each i , $1 \leq i \leq t$), such that $E(G) = E(G_1) \cap E(G_2) \cap \dots \cap E(G_t)$. Then $\text{cub}(G) = t$.*

3. Upper bound for $\text{cub}(H_d)$

In this section we will show that there exists a constant c , such that $\text{cub}(H_d) \leq \frac{cd}{\log d}$. By Lemma 1, it is sufficient to demonstrate that there exist $\frac{c \cdot d}{\log d}$ unit interval graphs (where c is a constant) on the same vertex set as H_d , such that the edge set of H_d is the intersection of the edge sets of these unit interval graphs. With this in mind, corresponding to each vertex $x \in V(H_d)$, we define below a special unit interval graph I_x .

Construction of the unit interval graph I_x : We map x to the interval $[0, 1]$. Let $u \in V(H_d) - \{x\}$. We map u to the interval $[\delta(x, u), \delta(x, u) + 1]$. (Recall that $\delta(x, u)$ denotes the shortest distance between u and x in H_d .) Let I_x be the resulting interval graph (with vertex set $V(H_d)$).

Lemma 2. For any $x \in V(H_d)$, $E(I_x) \supseteq E(H_d)$.

Proof. Let $(u, v) \in E(H_d)$. Without loss of generality, assume that $\delta(x, v) \geq \delta(x, u)$. Note that H_d is a bipartite graph. Hence, if (u, v) is an edge of H_d then $\delta(x, u) \neq \delta(x, v)$. (Otherwise, there will be an odd cycle in H_d .) Moreover, $\delta(x, v) \leq \delta(x, u) + 1$, since $(u, v) \in E(H_d)$. It follows that $\delta(x, v) = \delta(x, u) + 1$. Thus the intervals associated with u and v in I_x intersect. (They touch each other at $\delta(x, v) = \delta(x, u) + 1$.) Therefore the Lemma is true. ■

Our plan is to show that there exists a subset $S \subset V(H_d)$ with $|S| \leq \frac{cd}{\log d}$ (where c is a constant) such that $\bigcap_{x \in S} E(I_x) = E(H_d)$. The reader may note that in view of Lemma 2, it is sufficient to show that such a set S has the following Property, which we refer to as Property P .

Definition 1 (Property P). A subset $S \subset V(H_d)$ is said to have Property P if and only if for each $(u, v) \notin E(H_d)$, there exists a vertex $x \in S$, such that $(u, v) \notin E(I_x)$.

Choosing the subset S randomly: We select a random subset S of $V(H_d)$ by conducting the following experiment: We select a binary string x such that the bit at position i is set to 1 with probability $\frac{1}{2}$. That is, for any i , $1 \leq i \leq d$, $Pr(f_i(x) = 0) = \frac{1}{2}$ and $Pr(f_i(x) = 1) = \frac{1}{2}$. We do this experiment $\frac{cd}{\log d}$ times, thus selecting $\frac{cd}{\log d}$ binary strings. Let S be the multi-set of vertices which correspond to the strings so selected. Clearly, $|S| = \frac{cd}{\log d}$.

We show that if subset S is constructed randomly as explained above, then $Pr(S \text{ does not satisfy Property } P) < 1$. As a consequence, it follows that there exists a subset S of $V(H_d)$, where $|S| \leq \frac{cd}{\log d}$ (c being a constant), such that S satisfies Property P .

The following lemma is an easy consequence of the construction of I_x .

Lemma 3. For any vertex $x \in V(H_d)$, $(u, v) \in E(I_x)$ if and only if $|\delta(x, u) - \delta(x, v)| \leq 1$.

Given three vertices $x, u, v \in V(H_d)$, we partition the bits of $f(x)$ into three categories:

1. If $i \notin D(u, v)$, then $f_i(x)$ is defined to be a *neutral* bit of $f(x)$ (with respect to u and v). (The reason why we name $f_i(x)$ a neutral bit is the following: If $i \notin D(u, v)$, then it is the case that either $i \in D(u, x)$ and $i \in D(v, x)$, or $i \notin D(u, x)$ and $i \notin D(v, x)$.)
2. If $i \in D(u, v)$, then $f_i(x)$ is called a *u-bit* if and only if $f_i(x) \neq f_i(u)$. Clearly in that case $f_i(x) = f_i(v)$. The reader may note that if $f_i(x)$ is a u-bit then $i \in D(x, u)$, whereas $i \notin D(x, v)$. The number of u-bits of $f(x)$ will be denoted by $n_u(x)$.
3. If $i \in D(u, v)$, then $f_i(x)$ is called a *v-bit* if and only if $f_i(x) \neq f_i(v)$. Clearly if $i \in D(u, v)$, $f_i(x)$ is a v-bit if and only if it is not a u-bit. It may be noted that if $f_i(x)$ is a v-bit then $i \in D(x, v)$, whereas $i \notin D(x, u)$. The number of v-bits of $f(x)$ will be denoted by $n_v(x)$.

The next lemma follows immediately, from the discussion above.

Lemma 4. Let $x, u, v \in V(H_d)$. Then $|D(u, v)| = n_u(x) + n_v(x)$ and $|\delta(x, u) - \delta(x, v)| = |n_u(x) - n_v(x)|$.

Let x be a vertex corresponding to a randomly chosen binary string: i.e., $Pr(f_i(x) = 1) = \frac{1}{2}$. We now bound $Pr((u, v) \in E(I_x))$ for each pair of non-adjacent vertices (u, v) . By Lemmas 3 and 4, $Pr((u, v) \in E(I_x)) = Pr(|n_u(x) - n_v(x)| \leq 1)$. We consider two cases:

Case 1: $\delta(u, v) = r$ is even. Since u and v are non-adjacent, $r \geq 2$. Since $r = \delta(u, v) = |D(u, v)| = n_u(x) + n_v(x)$ by Lemma 4, clearly $n_u(x) - n_v(x)$ is also even. Thus we have $Pr((u, v) \in E(I_x)) = Pr(|n_u(x) - n_v(x)| = 0)$. Noting that for any $i \in D(u, v)$, $f_i(x)$ is a u -bit with probability $\frac{1}{2}$ and it is a v -bit with probability $\frac{1}{2}$, we have:

$$Pr((u, v) \in E(I_x)) = \binom{r}{r/2} 2^{-r}. \tag{1}$$

Because r is even and $r \geq 2$, we have $\binom{r}{r/2} 2^{-r} > \binom{r+2}{(r+2)/2} 2^{-(r+2)}$. It follows that $Pr((u, v) \in E(I_x))$ is maximized at $r = 2$ and thus $Pr((u, v) \in E(I_x)) \leq \frac{1}{2}$.

Case 2: $\delta(u, v) = r$ is odd. Since u and v are non-adjacent, $r \geq 3$. Clearly, $n_u(x) - n_v(x)$ is odd. Thus we have $Pr((u, v) \in E(I_x)) = Pr(|n_u(x) - n_v(x)| = 1)$. It follows that:

$$\begin{aligned} Pr((u, v) \in E(I_x)) &= Pr(n_u(x) - n_v(x) = 1) + Pr(n_u(x) - n_v(x) = -1) \\ &= \left(\binom{r}{(r+1)/2} + \binom{r}{(r-1)/2} \right) 2^{-r} \\ &= \binom{r}{(r+1)/2} 2^{-(r-1)}. \end{aligned} \tag{2}$$

Because r is odd and $r \geq 3$, we have $\binom{r}{(r+1)/2} 2^{-(r-1)} > \binom{r+2}{(r+3)/2} 2^{-(r+1)}$. It follows that $Pr((u, v) \in E(I_x))$ is maximized at $r = 3$ and thus $Pr((u, v) \in E(I_x)) \leq \frac{3}{4}$.

From the above two cases, it follows that

$$Pr((u, v) \in E(I_x)) \leq \frac{3}{4}. \tag{3}$$

Since each $x \in S$ is chosen independently and uniformly at random,

$$Pr((u, v) \in E(I_x), \forall x \in S) \leq \left(\frac{3}{4}\right)^{|S|} \leq \left(\frac{3}{4}\right)^{\frac{cd}{\log d}}. \tag{4}$$

The obvious next step in order to derive an upper bound for $Pr(S \text{ does not satisfy Property } P)$ would be to use the union bound, that is, $Pr(S \text{ does not satisfy Property } P) \leq \sum_{(u,v) \notin E(H_d)} Pr((u, v) \in E(I_x), \forall x \in S)$. Unfortunately, there are $\binom{2^d}{2} - 2^{d-1}d = O(2^{2d})$ non-adjacent pairs of vertices in H_d , and a straightforward application of the union bound as above would not suffice: the bound given by Inequality (4) is too weak. But, by examining the Inequalities (1) and (2) more carefully, the reader can easily see that as r becomes larger, the probability that a non-adjacent pair (u, v) with $\delta(u, v) = r$ being adjacent in I_x decreases, and for sufficiently large r , this probability can be much smaller than what is guaranteed by Inequality (3). In fact, by applying Stirling’s approximation (i.e., $n! \sim (n/e)^n \sqrt{2\pi n}$) on Inequalities (1) and (2), it is easy to verify that, there exists a constant c_1 , such that for a pair of non-adjacent vertices (u, v) ,

$$Pr((u, v) \in E(I_x)) \leq \frac{c_1}{\sqrt{\delta(u, v)}}. \tag{5}$$

Based on this observation, we partition the non-adjacent pairs of vertices in H_d into two subsets A and B as follows:

$$\begin{aligned} A &= \left\{ (u, v) : u, v \text{ are non-adjacent in } H_d \text{ and } \delta(u, v) > \frac{d}{\log^2 d} \right\} \\ B &= \left\{ (u, v) : u, v \text{ are non-adjacent in } H_d \text{ and } \delta(u, v) \leq \frac{d}{\log^2 d} \right\}. \end{aligned}$$

Definition 2. A subset S of $V(H_d)$ is said to satisfy Property P_A (respectively P_B) if and only if for each non-adjacent pair $(u, v) \in A$ (respectively in B), there exists a vertex $x \in S$, such that $(u, v) \notin E(I_x)$.

It is easy to see the following:

$$Pr(S \text{ does not satisfy } P) \leq Pr(S \text{ does not satisfy } P_A) + Pr(S \text{ does not satisfy } P_B). \tag{6}$$

We will show that each of the two terms in the right-hand side is strictly less than $\frac{1}{2}$, so that the left-hand side is strictly less than 1, as required.

Since $|A| \leq 2^{2d}$, and recalling that for any pair $(u, v) \in A$, we have $\delta(u, v) > \frac{d}{\log^2 d}$, we can apply union bound to show that,

$$\begin{aligned} Pr(S \text{ does not satisfy } P_A) &\leq \sum_{(u,v) \in A} \left(\frac{c_1}{\sqrt{\delta(u, v)}} \right)^{\frac{cd}{\log d}} \\ &\leq \left(\frac{c_1 \log d}{\sqrt{d}} \right)^{\frac{cd}{\log d}} 2^{2d} \\ &\leq 2^{2d + (\log c_1 + \log \log d - \frac{\log d}{2}) \frac{cd}{\log d}} \\ &\leq 2^{-\frac{c}{8}d} < \frac{1}{2}, \end{aligned} \tag{7}$$

when c is a suitably large constant and when d is sufficiently large. (This is because, for sufficiently large d , we can assume $\frac{\log c_1 + \log \log d}{\log d} \leq \frac{1}{4}$. Also, for a suitably large constant c , we can assume $2d \leq \frac{cd}{8}$.)

Now we deal with the pairs in B . Recall that an upper bound for $Pr((u, v) \in E(I_x) \forall x \in S)$ is given by Inequality (4). But, unfortunately $|B|$ is too big to infer that $Pr(S \text{ does not satisfy } P_B) < \frac{1}{2}$, by a simple application of union bound. To overcome this difficulty, we define an equivalence relation \mathcal{R} on B such that the pairs in the same equivalence class behave identically, i.e. if (u_1, v_1) and (u_2, v_2) belong to the same equivalence class then for any $x \in V(H_d)$, $(u_1, v_1) \in E(I_x)$ if and only if $(u_2, v_2) \in E(I_x)$.

Recall that $f(u)$ denotes the binary string associated with u . Let $\mathcal{P} = \{k_1, k_2, \dots, k_i\}$ where $1 \leq k_1 < k_2 < \dots < k_i \leq d$. We denote by $f_{\mathcal{P}}(u)$ the binary string obtained by concatenating the bits $f_{k_1}(u), f_{k_2}(u), \dots, f_{k_i}(u)$ in that order. We call $f_{\mathcal{P}}(u)$ the *bit pattern* of u at the set of positions \mathcal{P} .

From now on, for any pair of vertices u and v , we choose to represent it by the ordered pair (u, v) if $f_{D(u,v)}(u)$ is less than $f_{D(u,v)}(v)$ in the lexicographic order; else we represent it by (v, u) . (The reader may observe that the bit pattern $f_{D(u,v)}(u)$ is the complement of the bit pattern $f_{D(u,v)}(v)$.)

We define the equivalence relation \mathcal{R} as follows: Consider two pairs (u_1, v_1) and (u_2, v_2) .

$$(u_1, v_1) \mathcal{R} (u_2, v_2) \Leftrightarrow D(u_1, v_1) = D(u_2, v_2) \quad \text{and} \quad f_{D(u_1, v_1)}(u_1) = f_{D(u_1, v_1)}(u_2).$$

That is, (u_1, v_1) and (u_2, v_2) are related by \mathcal{R} if and only if: (1) the set of bit positions where u_1 differs from v_1 is identical to the set of positions where u_2 differs from v_2 and (2) the bit patterns of u_1 and u_2 at those bit positions are identical.

Let B_1, \dots, B_α be the equivalence classes of B under \mathcal{R} . Note that each equivalence class B_k corresponds to a unique pair (\mathcal{P}, s) , where \mathcal{P} is a set of i distinct bit positions, where $2 \leq i \leq \frac{d}{\log^2 d}$, and s is a binary string of length i . It is easy to see that the number of equivalence classes α has the following upper bound. Let $t = \lfloor \frac{d}{\log^2 d} \rfloor$. Then,

$$\alpha \leq \sum_{i=2}^{t} \binom{d}{i} 2^i \leq t \binom{d}{t} 2^t \leq t(2d)^t. \tag{8}$$

Now, from the definition of the relation \mathcal{R} , it is easy to see that if (u_1, v_1) and (u_2, v_2) are in the same equivalence class B_k then for any $x \in V(H_d)$, $|n_{u_1}(x) - n_{v_1}(x)| = |n_{u_2}(x) - n_{v_2}(x)|$ and therefore $(u_1, v_1) \in E(I_x)$ if and only if $(u_2, v_2) \in E(I_x)$.

Thus applying the union bound using (4) and using the inequality (8) for α , we get:

$$Pr(S \text{ does not satisfy } P_B) = Pr(\exists (u, v) \in B : \text{ such that } (u, v) \in E(I_x), \forall x \in S)$$

$$\leq \alpha \left(\frac{3}{4}\right)^{\frac{cd}{\log d}} \leq t(2d)^t \left(\frac{3}{4}\right)^{\frac{cd}{\log d}} < \frac{1}{2} \quad (9)$$

for a suitably large constant c .

Thus recalling inequality (6), we have $Pr(S \text{ does not satisfy Property } P) < 1$. It follows that there exists a subset $S \subseteq V(H_d)$, with $|S| \leq \frac{cd}{\log d}$, such that S satisfies Property P . In other words:

Theorem 1. $\text{cub}(H_d) \leq \frac{cd}{\log d}$, where c is a constant.

The following lower bound for the cubicity of H_d was shown in [2].

Theorem 2 (Chandran et al. [2]). $\text{cub}(H_d) \geq \frac{d-1}{\log d}$.

Finally combining the upper bound of Theorem 1 with the lower bound of Theorem 2, we have:

Theorem 3. $\text{cub}(H_d) = \Theta\left(\frac{d}{\log d}\right)$.

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