A note on the adjacent vertex distinguishing total chromatic number of graphs

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An adjacent vertex distinguishing total coloring of a graph $G$ is a proper total coloring of $G$ such that any pair of adjacent vertices have different sets of colors. The minimum number of colors needed for such a total coloring of $G$ is denoted by $\chi''_a(G)$. In this note, we show that $\chi''_a(G) \leq 2\Delta$ for any graph $G$ with maximum degree $\Delta \geq 3$.

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1. Introduction

Only simple and finite graphs are considered in this paper. For a graph $G$, we use $V(G)$, $E(G)$, and $\Delta(G)$ (for short, $\Delta$) to denote the set of vertices, the set of edges, and the maximum degree of $G$, respectively. A total $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G) \cup E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The graph $G$ is total $k$-colorable if it has a total $k$-coloring. The total chromatic number $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ is total $k$-colorable. Let $\phi$ be a total $k$-coloring of $G$. For a vertex $v \in V(G)$, we set $C_\phi(v) = \{\phi(\{uv\}) \mid u \in E(G) \cup \{\phi(v)\}\}$. The coloring $\phi$ is called an adjacent vertex distinguishing total coloring if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices $u$ and $v$. The adjacent vertex distinguishing total chromatic number $\chi''_a(G)$ of $G$ is the smallest integer $k$ such that $G$ has a $k$-avd-total coloring.

It is evident that $\chi''_a(G) \geq \chi''(G) \geq \Delta + 1$ for any graph $G$. The well-known Total Coloring Conjecture (TCC) [1,6] asserts that $\chi''(G) \leq \Delta + 2$ for any graph $G$. However, there exists many graphs such that $\chi''_a(G) > \Delta + 2$, for instance, a complete graph on odd order. Zhang et al. [8] first introduced and investigated the adjacent vertex distinguishing total coloring of graphs. In particular, they proposed the following challenging conjecture:

Conjecture 1. If $G$ is a connected graph with at least two vertices, then $\chi''_a(G) \leq \Delta + 3$.

Chen [2], and independently Wang [7], confirmed Conjecture 1 for graphs $G$ with $\Delta \leq 3$. Hulgan [5] presented a more concise proof for this result. Coker and Johannson [3] used a probabilistic method to establish an upper bound $\Delta + c$ for $\chi''_a(G)$, where $c > 0$ is a constant.

Let $\chi(G)$ and $\chi'(G)$ denote the chromatic number and chromatic index of a graph $G$, respectively. By the definitions, the following result is an easy observation:

Proposition 1. For any graph $G$, $\chi''_a(G) \leq \chi(G) + \chi'(G)$.
The celebrated Vizing’s Theorem on the edge coloring says that every graph $G$ has $\Delta \leq \chi'(G) \leq \Delta + 1$. $G$ is of Class 1 if $\chi'(G) = \Delta$, and Class 2 if $\chi'(G) = \Delta + 1$. Suppose that $G$ is neither a complete graph nor an odd cycle. Brooks’ Theorem on the vertex coloring asserts that $\chi(G) \leq \Delta$. By Proposition 1, it is immediate to derive that $\chi''(G) \leq 2\Delta + 1$. For a planar graph $G$, by the Four-Color Theorem and Vizing’s Theorem, we deduce that $\chi''(G) \leq 2\Delta + 5$. Moreover, if $G$ is of Class 1, then $\chi''(G) \leq \Delta + 4$. More recently, Huang and Wang [4] verified Conjecture 1 for planar graphs $G$ with $\Delta \geq 11$.

In this note, we show that if $G$ is a graph with $\Delta \geq 3$, then $\chi''(G) \leq 2\Delta$.

2. Main result

Let $G$ be a connected graph with $\chi(G) = k \geq 3$. Clearly, a proper (vertex) $k$-coloring of $G$ admits a $k$-partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that $G[V_i]$, the subgraph of $G$ induced by $V_i$, is edgeless. Let $A_k(G)$ denote the set of all such $k$-partitions $(V_1, V_2, \ldots, V_k)$ of $V(G)$. Given $\lambda_i(G) = (V_1, V_2, \ldots, V_k) \in A_k(G)$ and $i, j \in \{1, 2, \ldots, k\}$, let $E_{ij}(\lambda)$ denote the set of edges of $G$ joining a vertex in $V_i$ to a vertex in $V_j$, and $e_{ij}(\lambda) = |E_{ij}(\lambda)|$. Further, we set $e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda))$, where

$$e_i(\lambda) = \sum_{j=1, j \neq i}^k e_{ij}(\lambda) = \sum_{v \in V_i} d(v).$$

Suppose that $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are two distinct real sequences with $n \geq 1$. We say that $A$ is greater than $B$ in a lexicographical order if there is an index $1 \leq i \leq n$ such that $a_i > b_i$ and $a_j = b_j$ for all $j = 1, 2, \ldots, i - 1$.

**Lemma 2.** Let $G$ be a connected graph with $k = \chi(G)$. Let $\lambda^* = (V_1^*, V_2^*, \ldots, V_k^*)$ be a lexicographically maximal sequence in $A_k(G)$ according to $e(\lambda) = (e_1(\lambda^*), e_2(\lambda^*), \ldots, e_k(\lambda^*))$. Assume that $x \in V_i^*$ with $2 \leq i \leq k$. Then for each $1 \leq j \leq i - 1$, there exists a vertex $y \in V_j^*$ such that $xy \in E(G)$.

**Lemma 2** holds obviously.

**Theorem 4.** For any graph $G$ with $\Delta \geq 3$, we have $\chi''(G) \leq 2\Delta$.

**Proof.** Let $\Delta = k$. The theorem holds automatically for complete graphs by **Lemma 3**. So assume that $G$ is not a complete graph. By Brooks’ Theorem, $\chi(G) \leq k$. If $\chi(G) \leq k - 1$, it follows from Proposition 1 that $\chi''(G) \leq \chi(G) + \chi'(G) \leq k - 1 + 1 + 1 = 2k$. Thus, assume that $\chi(G) = k$. Let $\lambda = (V_1, V_2, \ldots, V_k) \in A_k(G)$ be a lexicographically maximal sequence in $A_k(G)$ according to $e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda))$. By **Lemma 2**, if $x \in V_i$ with $2 \leq i \leq k$, then for each $1 \leq j \leq i - 1$, there exists a vertex $y \in V_j$ such that $xy \in E(G)$.

For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, we use $G[X, Y]$ to denote the subgraph of $G$ induced by all the edges with an endpoint in $X$ and the other endpoint in $Y$. Clearly, $G[X, Y]$ is a bipartite graph.

To give our coloring scheme, we need to define the following bipartite subgraphs:

$$H_i = \left[ V_i, \bigcup_{j=1}^{i-1} V_j \right]$$

for $i = k, k - 1, \ldots, 2$.

Note that $H_i$ is of Class 1 by König’s Theorem. Now let us construct a proper total $2k$-coloring $\phi$ of $G$ in the following ways.

**Step 1.** For $i = 1, 2, \ldots, k$, color all the vertices in $V_i$ with $i$.

**Step 2.** Color $E(H_0)$ using the color set $C_0 = \{k + 1, k + 2, \ldots, 2k\}$.

**Step 3.** Color $E(G) \setminus E(H_k)$ by the following procedure:

(a) Let $i = 1$.

(b) Let $C_i = \{k + i + 1, k + i + 2, \ldots, 2k\}$. Color $\bigcup_{j=1}^{k-i+1} E_{k-i,j}(\lambda)$ with $C_i$ in the following ways:

(b1) Let $j = 1$.

(b2) Color $E_{k-1,j}(\lambda)$ properly with $C_i$. When an edge $e \in E_{k-1,j}(\lambda)$ cannot be colored, we leave $e$ uncolored and continue.

(b3) If $j = k - i - 1$, go to (c). Otherwise, set $j := j + 1$, go to (b2).

(c) If $i = k - 1$, stop. Otherwise, set $i := i + 1$, go to (b).

First, we show that $E(G)$ can be properly colored by the above procedure. To do this, it suffices to show that, for each fixed $1 \leq i \leq k - 1$, all edges in $E_{k-i,j}(\lambda)$ can be colored properly using the colors in $C_i$ for each $j = 1, 2, \ldots, k - 1$. Assume to the contrary that there exists an edge $v_{k-i,j}u_j \in E_{k-i,j}(\lambda)$, $1 \leq j \leq k - i - 1$, which cannot be colored properly. Suppose that $d(u_{k-i}) = s$ and $d(u_j) = t$. Let $u_1, u_2, \ldots, u_{s-1}$ be the neighbors of $v_{k-i}$ other than $u_j$, and $u_1, u_2, \ldots, u_{t-1}$ be the neighbors of $u_j$ other than $v_{k-i}$. By **Lemma 2**, for each $1 \leq i \leq j - 1$, there exists a vertex $u_i \in V_i$ such that $v_{k-i}u_i \in E(G)$. Note that $u_{k-i}$ remains uncolored at the current step by (c). Similarly, for each $j + 1 \leq q \leq k - i - 1$, there exists an uncolored edge $v_{k-i}w_q$ in $G$ where $w_q \in V_q$. 

For $u \in V(G)$, let $B(u)$ denote the sets of colors assigned to the edges incident to $u$ under the coloring $\phi$. Since $v_{k-i}v_j$ is uncolored, we have the following:

\[ |B(v_{k-i})| \leq d(v_{k-i}) - 1 - (k-i-1-(j+1)+1) = s-k+i+j, \]

\[ |B(v_j)| \leq d(v_j) - 1 - (j-1) = t-j. \]

Thus

\[ |B(v_{k-i}) \cup B(v_j)| \leq |B(v_{k-i})| + |B(v_j)| \leq s + t - k + i. \]

If $s < k$ or $t < k$, then it is easy to derive that $|B(v_{k-i}) \cup B(v_j)| \leq k+i-1$. Since $|C_i| = 2k - (k-i+1) + 1 = k + i$, then $v_{k-i}v_j$ can be colored with a color in $C_i \setminus (B(v_{k-i}) \cup B(v_j))$, contradicting the hypothesis. So suppose that $s = t = k$, and all the colors in $C_i$ occur on the edges incident to $v_{k-i}$ or $v_j$. This implies that $B(v_{k-i}) \cup B(v_j) = C_i$. Since $|B(v_{k-i})| \leq s-k+i+j = i+j$ and $|B(v_j)| \leq k-j$, we have that $k + i = |C_i| = |B(v_{k-i}) \cup B(v_j)| \leq |B(v_{k-i})| + |B(v_j)| \leq (i+j) + (k-j) = k + i$. It follows that $|B(v_{k-i})| = i+j$, $|B(v_j)| = k-j$, and $B(v_{k-i}) \cap B(v_j) = \emptyset$.

The proof is split into the following two cases:

Case 1. $k-i+1 \in B(v_{k-i})$.

Let $\phi(w_rv_{k-i}) = k-i+1$, where $r \in \{1,2,\ldots,s-1\}$. Since $|B(v_{k-i})| \leq d(v_{k-i}) - 1 = k - 1$, $C_0 \subseteq C_i = B(v_{k-i}) \cup B(v_j)$, and $|C_0| = k$, we have $B(v_j) \cap C_0 \neq \emptyset$. If there is a color $\alpha \in (B(v_j) \cap C_0) \setminus B(w_r)$, then we can recolor $v_{k-i}v_j$ with $\alpha$, and color $v_{k-i}v_j$ with $k - i + 1$, deriving a contradiction. Hence $B(v_j) \subseteq B(w_r)$. Since $k - i + 1 \in B(v_r) \setminus C_0$, $|C_0| = k$, and $|B(w_r)| \leq k$, there is a color $\beta \in C_0 \setminus B(w_r) \subseteq C_0 \setminus B(v_j) \subseteq C_0 \cap B(v_{k-i})$. Further, let $\gamma \in C_0 \setminus B(v_{k-i}) \subseteq B(v_j) \cap C_0 \subseteq B(w_r)$. So $\beta, \gamma \in C_0$, $\beta \neq \gamma$. Let $P$ be the $(\beta, \gamma)$-alternating path originating from $v_{k-i}$, i.e., $E(P)$ is colored alternately with the colors $\beta$ and $\gamma$. Switch the colors of the edges on $P$. If $P$ does not terminate at $v_j$, then we may color $v_{k-i}v_j$ with $\beta$, a contradiction. Otherwise, $P$ is a $(\gamma, \beta)$-alternating path from $v_{k-i}$ to $v_j$. Clearly, $P$ cannot arrive at $w_r$. Let $Q$ denote the longest $(\gamma, \beta)$-alternating path originating from $w_r$. Then $Q$ cannot terminate at $v_{k-i}$, for otherwise, there is some vertex $x \in V(P)$ that is incident to two edges colored the same color $\beta$ or $\gamma$. This contradicts the fact that $\phi$ is a proper partial total coloring of $G$. Thus, we switch the colors of the edges on $Q$, and color $v_{k-i}v_j, v_{k-i}v_r$ with $\gamma$. If $k-i+1$ also a contradiction.

Case 2. $k-i+1 \notin B(v_j)$.

With a similar discussion as in Case 1, we can color $v_{k-i}v_j$ properly.

Next, we need to prove that $\phi$ is a $\alpha$-total coloring of $G$. Let $w \in E(G)$ with $d(w) = d(v)$, and assume that $u \in V_i, v \in V_j$, and $i < j$. Note that $C_\phi(u) \subseteq \{i, i+1, i+2, \ldots, 2k\}$ and $i \in C_\phi(u), C_\phi(v) \subseteq \{i, j+1, j+2, \ldots, 2k\}$ and $j \in C_\phi(v)$. It follows that $i \notin C_\phi(v)$ and hence $C_\phi(v) \neq C_\phi(u)$. Thus, $\phi$ is a 2.3-\alpha-total coloring of $G$.

When $\Delta = 3$, our Theorem 4 asserts that $\chi''_\alpha(G) \leq 6$, which implies the result of [2,7,5].

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