Discrete Mathematics 312 (2012) 3544–3546

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A note on the adjacent vertex distinguishing total chromatic number of graphs

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ARTICLE INFO

Article history:
Received 9 June 2012
Received in revised form 7 August 2012
Accepted 8 August 2012
Available online 7 September 2012

Keywords:
Adjacent vertex distinguishing total coloring
Maximum degree

ABSTRACT

An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that any pair of adjacent vertices have different sets of colors. The minimum number of colors needed for such a total coloring of G is denoted by $\chi''_a(G)$. In this note, we show that $\chi''_a(G) \le 2\Delta$ for any graph G with maximum degree $\Delta \ge 3$.

1. Introduction

Only simple and finite graphs are considered in this paper. For a graph G, we use $V(G)$, $E(G)$, and $\Delta(G)$ (for short, Δ) to denote the set of vertices, the set of edges, and the maximum degree of G, respectively. A *total k-coloring* of a graph G is a mapping ϕ from $V(G) \cup E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \ne \phi(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The graph G is *total k-colorable* if it has a total k-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G is total k-colorable. Let ϕ be a total k-coloring of G. For a vertex $v \in V(G)$, we set $C_\phi(v) = \{\phi(uv) \mid uv \in E(G)\} \cup \{\phi(v)\}$. The coloring ϕ is called an *adjacent vertex distinguishing total coloring* or an *avd-total coloring* if $C_\phi(u) \ne C_\phi(v)$ for any pair of adjacent vertices u and v. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ of G is the smallest integer k such that G has a k-avd-total coloring.

It is evident that $\chi''_a(G) \ge \chi''(G) \ge \Delta + 1$ for any graph G. The well-known Total Coloring Conjecture (TCC) [1,6] asserts that $\chi''(G) \le \Delta + 2$ for any graph G. However, there exists many graphs such that $\chi''_a(G) > \Delta + 2$, for instance, a complete graph on odd order. Zhang et al. [8] first introduced and investigated the adjacent vertex distinguishing total coloring of graphs. In particular, they proposed the following challenging conjecture:

Conjecture 1. *If G is a connected graph with at least two vertices, then $\chi''_a(G) \le \Delta + 3$.*

Chen [2], and independently Wang [7], confirmed Conjecture 1 for graphs G with $\Delta \le 3$. Hulgan [5] presented a more concise proof for this result. Coker and Johannson [3] used a probabilistic method to establish an upper bound $\Delta + c$ for $\chi''_a(G)$, where $c > 0$ is a constant.

Let $\chi(G)$ and $\chi'(G)$ denote the chromatic number and chromatic index of a graph G, respectively. By the definitions, the following result is an easy observation:

Proposition 1. *For any graph G, $\chi''_a(G) \le \chi(G) + \chi'(G)$.*

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doi:10.1016/j.disc.2012.08.006

The celebrated Vizing's Theorem on the edge coloring says that every graph G has $\Delta \leq \chi'(G) \leq \Delta + 1$. G is of *Class* 1 if $\chi'(G) = \Delta$, and *Class* 2 if $\chi'(G) = \Delta + 1$. Suppose that G is neither a complete graph nor an odd cycle. Brooks' Theorem on the vertex coloring asserts that $\chi(G) \leq \Delta$. By Proposition 1, it is immediate to derive that $\chi''_a(G) \leq 2\Delta + 1$. For a planar graph G, by the Four-Color Theorem and Vizing's Theorem, we deduce that $\chi''_a(G) \leq \Delta + 5$. Moreover, if G is of Class 1, then $\chi''_a(G) \leq \Delta + 4$. More recently, Huang and Wang [4] verified Conjecture 1 for planar graphs G with $\Delta \geq 11$.

In this note, we show that if G is a graph with $\Delta \geq 3$, then $\chi''_a(G) \leq 2\Delta$.

2. Main result

Let G be a connected graph with $\chi(G) = k \geq 3$. Clearly, a proper (vertex) k-coloring of G admits a k-partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that $G[V_i]$, the subgraph of G induced by V_i, is edgeless. Let $\Lambda_k(G)$ denote the set of all such k-partitions $(V_1, V_2, \ldots, V_k)$ of $V(G)$. Given $\lambda_k(G) = (V_1, V_2, \ldots, V_k) \in \Lambda_k(G)$ and $i, j \in \{1, 2, \ldots, k\}$, let $E_{i,j}(\lambda)$ denote the set of edges of G joining a vertex in V_i to a vertex in V_j, and $e_{i,j}(\lambda) = |E_{i,j}(\lambda)|$. Further, we set $e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda))$, where

$$e_i(\lambda) = \sum_{j=1,\, j\neq i}^{k} e_{i,j}(\lambda) = \sum_{v\in V_i} d(v).$$

Suppose that $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are two distinct real sequences with $n \geq 1$. We say that A is *greater* than B in a lexicographical order if there is an index $1 \leq i \leq n$ such that $a_i > b_i$ and $a_j = b_j$ for all $j = 1, 2, \ldots, i-1$.

Lemma 2. *Let G be a connected graph with $k = \chi(G)$. Let $\lambda^* = (V_1^*, V_2^*, \ldots, V_k^*)$ be a lexicographically maximal sequence in $\Lambda_k(G)$ according to $e(\lambda^*) = (e_1(\lambda^*), e_2(\lambda^*), \ldots, e_k(\lambda^*))$. Assume that $x \in V_i^*$ with $2 \leq i \leq k$. Then for each $1 \leq j \leq i-1$, there exists a vertex $y \in V_j^*$ such that $xy \in E(G)$.*

Lemma 2 holds obviously.

Lemma 3 ([8,5]). $\chi''_a(K_n) = \begin{cases} n+1, & \text{if } n \text{ is even,} \\ n+2, & \text{if } n \text{ is odd.} \end{cases}$

Theorem 4. *For any graph G with $\Delta \geq 3$, we have $\chi''_a(G) \leq 2\Delta$.*

Proof. Let $\Delta = k$. The theorem holds automatically for complete graphs by Lemma 3. So assume that G is not a complete graph. By Brooks' Theorem, $\chi(G) \leq k$. If $\chi(G) \leq k-1$, it follows from Proposition 1 that $\chi''_a(G) \leq \chi(G) + \chi'(G) \leq k-1+k+1 = 2k$. Thus, assume that $\chi(G) = k$. Let $\lambda = (V_1, V_2, \ldots, V_k) \in \Lambda_k(G)$ be a lexicographically maximal sequence in $\Lambda_k(G)$ according to $e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda))$. By Lemma 2, if $x \in V_i$ with $2 \leq i \leq k$, then for each $1 \leq j \leq i-1$, there exists a vertex $y \in V_j$ such that $xy \in E(G)$.

For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, we use $G[X, Y]$ to denote the subgraph of G induced by all the edges with an endpoint in X and the other endpoint in Y. Clearly, $G[X, Y]$ is a bipartite graph.

To give our coloring scheme, we need to define the following bipartite subgraphs:

$$H_i = G\left[V_i, \bigcup_{j=1}^{i-1} V_j\right] \quad \text{for } i = k, k-1, \ldots, 2.$$

Note that H_i is of Class 1 by König's Theorem. Now let us construct a proper total $2k$-coloring ϕ of G in the following ways.

Step 1. For $i = 1, 2, \ldots, k$, color all the vertices in V_i with i.

Step 2. Color $E(H_k)$ using the color set $C_0 = \{k+1, k+2, \ldots, 2k\}$.

Step 3. Color $E(G) \setminus E(H_k)$ by the following procedure:

(a) Let $i = 1$.
(b) Let $C_i = \{k-i+1, k-i+2, \ldots, 2k\}$. Color $\bigcup_{j=1}^{k-i-1} E_{k-i,j}(\lambda)$ with C_i in the following ways:
  (b1) Let $j = 1$.
  (b2) Color $E_{k-i,j}(\lambda)$ properly with C_i. When an edge $e \in E_{k-i,j}(\lambda)$ cannot be colored, we leave e uncolored and continue.
  (b3) If $j = k-i-1$, go to (c). Otherwise, set $j := j+1$, go to (b2).
(c) If $i = k-1$, stop. Otherwise, set $i := i+1$, go to (b).

First, we show that $E(G)$ can be properly colored by the above procedure. To do this, it suffices to show that, for each fixed $1 \leq i \leq k-1$, all edges in $E_{k-i,j}(\lambda)$ can be colored properly using the colors in C_i for each $j = 1, 2, \ldots, k-i-1$. Assume to the contrary that there exists an edge $v_{k-i}v_j \in E_{k-i,j}(\lambda)$, $1 \leq j \leq k-i-1$, which cannot be colored properly. Suppose that $d(v_{k-i}) = s$ and $d(v_j) = t$. Let $w_1, w_2, \ldots, w_{s-1}$ be the neighbors of v_{k-i} other than v_j, and $u_1, u_2, \ldots, u_{t-1}$ be the neighbors of v_j other than v_{k-i}. By Lemma 2, for each $1 \leq l \leq j-1$, there exists a vertex $u_l \in V_l$ such that $v_ju_l \in E(G)$. Note that v_ju_l remains uncolored at the current step by (c). Similarly, for each $j+1 \leq q \leq k-i-1$, there exists an uncolored edge $v_{k-i}w_q$ in G where $w_q \in V_q$.

For $u \in V(G)$, let $B(u)$ denote the sets of colors assigned to the edges incident to u under the coloring ϕ. Since $v_{k-i}v_j$ is uncolored, we have the following:

$$|B(v_{k-i})| \leq d(v_{k-i}) - 1 - (k - i - 1 - (j + 1) + 1) = s - k + i + j,$$
$$|B(v_j)| \leq d(v_j) - 1 - (j - 1) = t - j.$$

Thus

$$|B(v_{k-i}) \cup B(v_j)| \leq |B(v_{k-i})| + |B(v_j)| \leq s + t - k + i.$$

If $s < k$ or $t < k$, then it is easy to derive that $|B(v_{k-i}) \cup B(v_j)| \leq k + i - 1$. Since $|C_i| = 2k - (k - i + 1) + 1 = k + i$, then $v_{k-i}v_j$ can be colored with a color in $C_i \setminus (B(v_{k-i}) \cup B(v_j))$, contradicting the hypothesis. So suppose that $s = t = k$, and all the colors in C_i occur on the edges incident to v_{k-i} or v_j. This implies that $B(v_{k-i}) \cup B(v_j) = C_i$. Since $|B(v_{k-i})| \leq s - k + i + j = i + j$ and $|B(v_j)| \leq k - j$, we have that $k + i = |C_i| = |B(v_{k-i}) \cup B(v_j)| \leq |B(v_{k-i})| + |B(v_j)| \leq (i + j) + (k - j) = k + i$. It follows that $|B(v_{k-i})| = i + j$, $|B(v_j)| = k - j$, and $B(v_{k-i}) \cap B(v_j) = \emptyset$.

The proof is split into the following two cases:

Case 1. $k - i + 1 \in B(v_{k-i})$.

Let $\phi(w_r v_{k-i}) = k - i + 1$, where $r \in \{1, 2, \ldots, s - 1\}$. Since $|B(v_{k-i})| \leq d(v_{k-i}) - 1 = k - 1$, $C_0 \subseteq C_i = B(v_{k-i}) \cup B(v_j)$, and $|C_0| = k$, we have $B(v_j) \cap C_0 \neq \emptyset$. If there is a color $\alpha \in (B(v_j) \cap C_0) \setminus B(w_r)$, then we can recolor $w_r v_{k-i}$ with α, and color $v_{k-i}v_j$ with $k - i + 1$, deriving a contradiction. Hence $B(v_j) \cap C_0 \subseteq B(w_r)$. Since $k - i + 1 \in B(w_r) \setminus C_0$, $|C_0| = k$, and $|B(w_r)| \leq k$, there is a color $\beta \in C_0 \setminus B(w_r) \subseteq C_0 \setminus B(v_j) \subseteq C_0 \cap B(v_{k-i})$. Further, let $\gamma \in C_0 \setminus B(v_{k-i}) \subseteq B(v_j) \cap C_0 \subseteq B(w_r)$. So $\beta, \gamma \in C_0$, $\beta \in B(v_{k-i})$, and $\gamma \in B(v_j)$. This implies that $\beta \neq \gamma$.

Let P be the longest (β, γ)-alternating path originating from v_{k-i}, i.e., $E(P)$ is colored alternately with the colors β and γ. Switch the colors of the edges on P. If P does not terminate at v_j, then we may color $v_{k-i}v_j$ with β, a contradiction. Otherwise, P is a (γ, β)-alternating path from v_{k-i} to v_j. Clearly, P cannot arrive at w_r. Let Q denote the longest (γ, β)-alternating path originating from w_r. Then Q cannot terminate at v_{k-i}, for otherwise, there is some vertex $x \in V(P)$ that is incident to two edges colored the same color β or γ. This contradicts the fact that ϕ is a proper partial total coloring of G. Thus, we switch the colors of the edges on Q, and color $w_r v_{k-i},\ v_{k-i}v_j$ with $\gamma,\ k - i + 1$, also a contradiction.

Case 2. $k - i + 1 \in B(v_j)$.

With a similar discussion as in Case 1, we can color $v_{k-i}v_j$ properly.

Next, we need to prove that ϕ is an avd-total coloring of G. Let $uv \in E(G)$ with $d(u) = d(v)$, and assume that $u \in V_i$, $v \in V_j$, and $i < j$. Note that $C_\phi(u) \subseteq \{i, i+1, i+2, \ldots, 2k\}$ and $i \in C_\phi(u)$, $C_\phi(v) \subseteq \{j, j+1, j+2, \ldots, 2k\}$ and $j \in C_\phi(v)$. It follows that $i \notin C_\phi(v)$ and hence $C_\phi(u) \neq C_\phi(v)$. Thus, ϕ is a 2Δ-avd-total coloring of G. □

When $\Delta = 3$, our Theorem 4 asserts that $\chi''_a(G) \leq 6$, which implies the result of [2,7,5].

Acknowledgments

The first author's research was supported by the Foundation of Zhejiang Educational Committee (No. Y201226078). The second author's research was supported by the NSFC (No. 11071223), the ZJNSFC (No. Z6090150), the ZJIP (No. T200905), the ZSDZZZZXK13, and the IP-OCNS-ZJNU.

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