A note on the adjacent vertex distinguishing total chromatic number of graphs

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An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that any pair of adjacent vertices have different sets of colors. The minimum number of colors needed for such a total coloring of G is denoted by $\chi''_a(G)$. In this note, we show that $\chi''_a(G) \leq 2\Delta$ for any graph G with maximum degree $\Delta \geq 3$.

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1. Introduction

Only simple and finite graphs are considered in this paper. For a graph $G$, we use $V(G)$, $E(G)$, and $\Delta(G)$ (for short, $\Delta$) to denote the set of vertices, the set of edges, and the maximum degree of $G$, respectively. A total k-coloring of a graph $G$ is a mapping $\phi$ from $V(G) \cup E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The graph $G$ is total k-colorable if it has a total k-coloring. The total chromatic number $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ is total $k$-colorable. Let $\phi$ be a total $k$-coloring of $G$. For a vertex $v \in V(G)$, we set $C_\phi(v) = \{\phi(uv) | uv \in E(G)\} \cup \{\phi(v)\}$. The coloring $\phi$ is called an adjacent vertex distinguishing total coloring or an avd-total coloring if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices $u$ and $v$. The adjacent vertex distinguishing total chromatic number $\chi'_a(G)$ of $G$ is the smallest integer $k$ such that $G$ has a $k$-avd-total coloring.

It is evident that $\chi''(G) \geq \chi''_a(G) \geq \Delta + 1$ for any graph $G$. The well-known Total Coloring Conjecture (TCC) [1,6] asserts that $\chi''(G) \leq \Delta + 2$ for any graph $G$. However, there exists many graphs such that $\chi''_a(G) > \Delta + 2$, for instance, a complete graph on odd order. Zhang et al. [8] first introduced and investigated the adjacent vertex distinguishing total coloring of graphs. In particular, they proposed the following challenging conjecture:

**Conjecture 1.** If $G$ is a connected graph with at least two vertices, then $\chi''_a(G) \leq \Delta + 3$.

Chen [2], and independently Wang [7], confirmed Conjecture 1 for graphs $G$ with $\Delta \leq 3$. Hulgan [5] presented a more concise proof for this result. Coker and Johannson [3] used a probabilistic method to establish an upper bound $\Delta + c$ for $\chi''_a(G)$, where $c > 0$ is a constant.

Let $\chi(G)$ and $\chi'(G)$ denote the chromatic number and chromatic index of a graph $G$, respectively. By the definitions, the following result is an easy observation:

**Proposition 1.** For any graph $G$, $\chi''_a(G) \leq \chi(G) + \chi'(G)$.

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The celebrated Vizing’s Theorem on the edge coloring says that every graph \( G \) has \( \Delta \leq \chi'(G) \leq \Delta + 1 \). \( G \) is of Class 1 if \( \chi'(G) = \Delta \), and Class 2 if \( \chi'(G) = \Delta + 1 \). Suppose that \( G \) is neither a complete graph nor an odd cycle. Brooks’ Theorem on the vertex coloring asserts that \( \chi(G) \leq \Delta \). By Proposition 1, it is immediate to derive that \( \chi''_{\Delta}(G) \leq 2\Delta + 1 \). For a planar graph \( G \), by the Four-Color Theorem and Vizing’s Theorem, we deduce that \( \chi''_{\Delta}(G) \leq 5 \). Moreover, if \( G \) is of Class 1, then \( \chi''_{\Delta}(G) \leq \Delta + 4 \). More recently, Huang and Wang [4] verified Conjecture 1 for planar graphs \( G \) with \( \Delta \geq 11 \).

In this note, we show that if \( G \) is a graph with \( \Delta \geq 3 \), then \( \chi''_{\Delta}(G) \leq 2\Delta \).

2. Main result

Let \( G \) be a connected graph with \( \chi(G) = k \geq 3 \). Clearly, a proper (vertex) \( k \)-coloring of \( G \) admits a \( k \)-partition \( (V_1, V_2, \ldots, V_k) \) of \( V(G) \) such that \( G[V_i] \), the subgraph of \( G \) induced by \( V_i \), is edgeless. Let \( \Lambda_k(G) \) denote the set of all such \( k \)-partitions \( (V_1, V_2, \ldots, V_k) \) of \( V(G) \). Given \( \lambda_k(G) = (V_1, V_2, \ldots, V_k) \in \Lambda_k(G) \) and \( i, j \in \{1, 2, \ldots, k\}, \) let \( E_{ij}(\lambda) \) denote the set of edges of \( G \) joining a vertex in \( V_i \) to a vertex in \( V_j \), and \( e_{ij}(\lambda) = |E_{ij}(\lambda)| \). Further, we set \( e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda)) \), where

\[
e_i(\lambda) = \sum_{j=1, j \neq i}^k e_{ij}(\lambda) = \sum_{v \in V_i} d(v).
\]

Suppose that \( A = (a_1, a_2, \ldots, a_k) \) and \( B = (b_1, b_2, \ldots, b_k) \) are two distinct real sequences with \( n \geq 1 \). We say that \( A \) is greater than \( B \) in a lexicographical order if there is an index \( 1 \leq i \leq n \) such that \( a_i > b_i \) and \( a_j = b_j \) for all \( j = 1, 2, \ldots, i - 1 \).

**Lemma 2.** Let \( G \) be a connected graph with \( k = \chi(G) \). Let \( \lambda^* = (V_1^*, V_2^*, \ldots, V_k^*) \) be a lexicographically maximal sequence in \( \Lambda_k(G) \) according to \( e(\lambda) = (e_1(\lambda)^*, e_2(\lambda)^*, \ldots, e_k(\lambda)^*) \). Assume that \( x \in V_i^* \) with \( 2 \leq i \leq k \). Then for each \( 1 \leq j \leq i - 1 \), there exists a vertex \( y \in V_j^* \) such that \( xy \in E(G) \). Theorem 2 holds obviously.

**Lemma 3** ([8,5]), \( \chi''_{\Delta}(K_n) = \begin{cases} n + 1, & \text{if } n \text{ is even,} \\ n + 2, & \text{if } n \text{ is odd.} \end{cases} \)

**Theorem 4.** For any graph \( G \) with \( \Delta \geq 3 \), we have \( \chi''_{\Delta}(G) \leq 2\Delta \).

**Proof.** Let \( \Delta = k \). The theorem holds automatically for complete graphs by Lemma 3. So assume that \( G \) is not a complete graph. By Brooks’ Theorem, \( \chi(G) \leq k \). If \( \chi(G) \leq k - 1 \), it follows from Proposition 1 that \( \chi''_{\Delta}(G) \leq \chi(G) + \chi'(G) \leq k - 1 + k + 1 = 2k \). Thus, assume that \( \chi(G) = k \). Let \( \lambda = (V_1, V_2, \ldots, V_k) \in \Lambda_k(G) \) be a lexicographically maximal sequence in \( \Lambda_k(G) \) according to \( e(\lambda) = (e_1(\lambda), e_2(\lambda), \ldots, e_k(\lambda)) \). By Lemma 2, if \( x \in V_i \) with \( 2 \leq i \leq k \), then for each \( 1 \leq j \leq i - 1 \), there exists a vertex \( y \in V_j \) such that \( xy \in E(G) \).

For \( X, Y \subseteq V(G) \) with \( X \cap Y = \emptyset \), we use \( G[X, Y] \) to denote the subgraph of \( G \) induced by all the edges with an endpoint in \( X \) and the other endpoint in \( Y \). Clearly, \( G[X, Y] \) is a bipartite graph.

To give our coloring scheme, we need to define the following bipartite subgraphs:

\[
H_i = G \begin{bmatrix} \vdots \vdots \vdots \end{bmatrix} V_i, \quad \text{for } i = k, k - 1, \ldots, 2.
\]

Note that \( H_i \) is of Class 1 by König’s Theorem. Now let us construct a proper total \( 2k \)-coloring \( \phi \) of \( G \) in the following ways.

**Step 1.** For \( i = 1, 2, \ldots, k \), color all the vertices in \( V_i \) with \( i \).

**Step 2.** Color \( E(H_k) \) using the color set \( C_0 = \{ k + 1, k + 2, \ldots, 2k \} \).

**Step 3.** Color \( E(G) \setminus E(H_k) \) by the following procedure:

(a) Let \( i = 1 \).

(b) Let \( C_i = \{ k - i + 1, k - i + 2, \ldots, 2k \} \). Color \( \bigcup_{j=1}^{k-i} E_{k-i}(\lambda) \) with \( C_i \) in the following ways:

(b1) Let \( j = 1 \).

(b2) Color \( E_{k-i}(\lambda) \) properly with \( C_i \). When an edge \( e \in E_{k-i}(\lambda) \) cannot be colored, we leave \( e \) uncolored and continue.

(b3) If \( j = k - i - 1 \), go to (c). Otherwise, set \( j := j + 1 \), go to (b2).

(c) If \( i = k - 1 \), stop. Otherwise, set \( i := i + 1 \), go to (b).

First, we show that \( E(G) \) can be properly colored by the above procedure. To do this, it suffices to show that, for each fixed \( 1 \leq i \leq k - 1 \), all edges in \( E_{k-i}(\lambda) \) can be colored properly using the colors in \( C_i \) for each \( j = 1, 2, \ldots, k - i - 1 \). Assume to the contrary that there exists an edge \( v_{k-i}u_j \in E_{k-i-j}(\lambda) \), \( 1 \leq j \leq k - i - 1 \), which cannot be colored properly. Suppose that \( d(u_{k-j}) = s \) and \( d(u_j) = t \). Let \( u_1, u_2, \ldots, u_{s-1} \) be the neighbors of \( u_{k-j} \), other than \( u_j \), and \( u_1, u_2, \ldots, u_{t-1} \) be the neighbors of \( u_j \), other than \( u_{k-j} \). By Lemma 2, for each \( 1 \leq l \leq j - 1 \), there exists a vertex \( u_l \in V_l \) such that \( u_lu_j \in E(G) \). Note that \( u_lu_j \) remains uncolored at the current step by (c). Similarly, for each \( j + 1 \leq q \leq k - i - 1 \), there exists an uncolored edge \( v_{k-j}w_q \) in \( G \) where \( w_q \in V_q \).
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If \( s < k \) or \( t < k \), then it is easy to derive that \( |B(v_{k-i}) \cup B(v_j)| \leq k + i - 1 \). Since \( |C_{i}| = 2k - (k - i + 1) + 1 = k + i \), then \( v_{k-i}v_j \) can be colored with a color in \( C_{i} \setminus (B(v_{k-i}) \cup B(v_j)) \), contradicting the hypothesis. So suppose that \( s = t = k \), and all the colors in \( C_{i} \) occur on the edges incident to \( v_{k-i} \) or \( v_j \). This implies that \( B(v_{k-i}) \cup B(v_j) = C_{i} \). Since \( |B(v_{k-i})| \leq s - k + i + j = i + j \) and \( |B(v_j)| \leq k - j \), we have that \( k + i = |C_{i}| = |B(v_{k-i}) \cup B(v_j)| \leq |B(v_{k-i})| + |B(v_j)| \leq (i + j) + (k - j) = k + i \). It follows that \( |B(v_{k-i})| = i + j, |B(v_j)| = k - j, \) and \( B(v_{k-i}) \cap B(v_j) = \emptyset \). The proof is split into the following two cases:

Case 1. \( k - i + 1 \in B(v_{k-i}) \).

Let \( \phi(u, v_{k-i}) = k - i + 1 \), where \( r \in \{1, 2, \ldots, s - 1\} \). Since \( |B(v_{k-i})| \leq d(v_{k-i}) - 1 = k - 1 \), \( C_{0} \subseteq C_{i} = B(v_{k-i}) \cup B(v_j) \), and \( |C_{0}| = k \), we have \( B(v_j) \cap C_{0} \neq \emptyset \). If there is a color \( \alpha \in (B(v_j) \cap C_{0}) \setminus B(w_r) \), then we can recolor \( w_r, v_{k-i} \) with \( \alpha \), and color \( v_{k-i}v_j \) with \( k - i + 1 \), deriving a contradiction. Hence \( B(v_j) \cap C_{0} \subseteq B(w_r) \). Since \( k - i + 1 \in B(w_r) \setminus C_{0}, |C_{0}| = k \), and \( |B(v_j)| \leq k \), there is a color \( \beta \in C_{0} \setminus B(v_j) \subseteq C_{0} \setminus B(v_j) \subseteq C_{0} \setminus B(v_{k-i}) \). Further, let \( \gamma \in C_{0} \setminus B(v_{k-i}) \subseteq B(v_j) \cap C_{0} \subseteq B(w_r) \). So \( \beta, \gamma \in C_{0}, \beta \in B(v_{k-i}), \) and \( \gamma \in B(v_j) \). This implies that \( \beta \neq \gamma \).

Let \( P \) be the longest (\( \beta, \gamma \))-alternating path originating from \( v_{k-i} \), i.e., \( E(P) \) is colored alternately with the colors \( \beta \) and \( \gamma \). Switch the colors of the edges on \( P \). If \( P \) does not terminate at \( v_j \), then we may color \( v_{k-i}v_j \) with \( \beta \), a contradiction. Otherwise, \( P \) is a (\( \beta, \gamma \))-alternating path from \( v_{k-i} \) to \( v_j \). Clearly, \( P \) cannot arrive at \( w_r \). Let \( Q \) denote the longest (\( \gamma, \beta \))-alternating path originating from \( w_r \). Then \( Q \) cannot terminate at \( v_{k-i} \), for otherwise, there is some vertex \( x \in V(P) \) that is incident to two edges colored the same color \( \beta \) or \( \gamma \). This contradicts the fact that \( \phi \) is a proper partial total coloring of \( G \). Thus, we switch the colors of the edges on \( Q \), and color \( v_{k-i}v_j, v_{k-i}v_j \) with \( \gamma, k - i + 1 \), also a contradiction.

Case 2. \( k - i + 1 \in B(v_j) \).

With a similar discussion as in Case 1, we can color \( v_{k-i}v_j \) properly.

Next, we need to prove that \( \phi \) is an avd-total coloring of \( G \). Let \( uv \in E(G) \) with \( d(u) = d(v) \), and assume that \( u \in V_r, v \in V_j \), and \( i < j \). Note that \( C\phi(u) \subseteq \{i, i + 1, i + 2, \ldots, 2k\} \) and \( i \in C\phi(u), C\phi(v) \subseteq \{i, j + 1, j + 2, \ldots, 2k\} \) and \( j \in C\phi(v) \). It follows that \( i \not\in C\phi(v) \) and hence \( C\phi(u) \neq C\phi(v) \). Thus, \( \phi \) is a 2-avd-total coloring of \( G \).

When \( \Delta = 3 \), our Theorem 4 asserts that \( \chi''_a(G) \leq 6 \), which implies the result of [2,7,5].

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