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Discrete Optimization 2 (2005) 123–134

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Ordered optimal solutions and parametric minimum cut problems☆

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Received 8 May 2002; received in revised form 22 September 2004; accepted 2 March 2005

This paper is dedicated to the memory of our friend, colleague and mentor, Shelby Brumelle (1939–2001)

Abstract

In this paper, we present an algebraic sufficient condition for the existence of a selection of optimal solutions in a parametric optimization problem that are totally ordered, but not necessarily monotone. Based on this result, we present necessary and sufficient conditions that ensure the existence of totally ordered selections of minimum cuts for some classes of parametric maximum flow problems. These classes subsume the class studied by Arai et al. [Discrete Appl. Math. 41 (1993) 69–74] as a special case.

MSC: 90C31; 90C27; 90C35; 06A05

Keywords: Parametric maximum flow; Minimum cut; Monotonicity; Ordered optimal solutions

1. Introduction

Our paper is concerned with parametric optimization problems and parametric maximum flow problems. It was motivated by a parametric maximum flow study carried out by Gallo, Grigoriadis and Tarjan (GGT) [8], and by a subsequent extension by Arai, Ueno and Kajitani (AUK) [1]. GGT considered a maximum network flow problem in which the capacities of arcs incident to the source or the sink may change as a function of a parameter. They showed that for $k = O(n)$ instances of the parameter, the maximum flows can be found in a time bound of one maximum flow, where n is the number of nodes, and that minimum cuts move monotonically with changes of the parameter. The monotonicity result was observed earlier by Eisner and Severance [6] for a restricted graph when the capacities of arcs incident to the source are parametrically increased, and by Stone [18] for a general graph when the capacities of arcs incident to the sink are parametrically decreased.[1] AUK extended the analysis of GGT and showed that when the capacities of arcs incident to a single node (other than the source or the sink) change, maximum flows for $k = O(n)$ instances of the parameter can be found in a time bound of two maximum flows. In their study, the minimum cuts are no longer monotone in the parameter. Rather, they may move "back and forth" as the parameter changes, but there always exists a selection of minimum cuts that are totally ordered.

☆ This paper is based on results in [4,5]. Research was partially supported by Natural Sciences and Engineering Research Council grants.

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[1] See also [17,13,2] for an analysis of the related parametric repair kit problem.

1572-5286/$ - see front matter
doi:10.1016/j.disopt.2005.03.002

The objective of this paper is to provide an algebraic sufficient condition for the existence of totally ordered optimal solutions for parametric optimization problems, and to demonstrate that this condition can, in turn, be used to provide necessary and sufficient conditions for the existence of a totally ordered selection of minimum cuts for some classes of parametric maximum flow problems.

Our results strengthen those derived by AUK by showing that, in their setting, every sub-selection of minimum cuts can be extended to a complete nested selection. Moreover, the classes for which a totally ordered selection of minimum cuts is shown to exist subsume the class of problems studied by AUK as a special case. We also present some other interesting special cases. For example, we show that one can further allow the capacity of the "center node" in the study of AUK to change and still maintain the existence of a totally ordered selection of minimum cuts. In another special case, we show that a totally ordered selection of minimum cuts exists when the capacities of arcs on a path are parametrically changed, given that no two arcs on the path are parallel.[2]

Finally, we note that our results are related to the various monotone selection theorems developed, e.g., by Topkis [19,20], Topkis and Veinott [22], Veinott [23], Granot and Veinott [11], Milgrom and Shannon [15], and Gautier et al. [9], for parametric optimization problems.[3] These theorems, which have numerous applications, present sufficient conditions under which it is possible to select an optimal solution for each instance in the parameter set such that the selected solutions are monotone in the parameters.

The rest of this article is organized as follows. Section 2 presents some preliminary results regarding lattices and a sufficient condition for the existence of a selection of totally ordered optimal solutions for parametric optimization problems. In Section 3 we develop some path closure properties, which are subsequently used in Section 4 to develop a characterization of classes of parametric capacity functions such that for every capacity function in these classes, there exists totally ordered selection of minimum cuts.

2. The totally ordered selection theorem

A *lattice* is a partially ordered set, say $(\mathscr{L}, \preceq)$, in which each pair of elements, x and y in $\mathscr{L}$, has a *supremum* or least upper bound and an *infimum* or greatest lower bound. The least upper bound (resp., greatest lower bound) of x and y in a lattice $\mathscr{L}$ is their *join* (resp., *meet*) and is denoted $x \vee y$ (resp., $x \wedge y$). A subset S of a lattice $\mathscr{L}$ is a *sublattice* if S contains the join and meet (with respect to $\mathscr{L}$) for each pair of elements of S.

Sublattice S is *lower* than sublattice T, or $S \preceq_V T$, if for every $x \in S$ and $y \in T$ we have $x \vee y \in T$ and $x \wedge y \in S$. On the set of all non-empty sublattices of a lattice, this order, which, according to Topkis [20], was introduced by Veinott, is reflexive, antisymmetric and transitive, and so partially orders sublattices of a lattice [20].

Suppose $(\Lambda, \leqslant)$ is a partially ordered set, $(\mathscr{L}, \preceq)$ is a lattice and $\{S_\lambda\}$ is a family of subsets of $\mathscr{L}$ indexed in Λ. If for any λ_1 and λ_2 in Λ, $\lambda_1 \leqslant \lambda_2$ implies that S_{λ_1} is lower than S_{λ_2} (respectively, S_{λ_2} is lower than S_{λ_1}), then $\{S_\lambda\}$ is *ascending* (respectively, *descending*) in λ on Λ. If $\lambda_1 \leqslant \lambda_2$ implies that $x \preceq y$ (respectively, $y \preceq x$) for each $x \in S_{\lambda_1}$ and $y \in S_{\lambda_2}$, then $\{S_\lambda\}$ is *strongly ascending* (respectively, *strongly descending)* in λ on Λ. See, e.g., [21] for a more detailed discussion.

Let $(\mathscr{L}, \preceq)$ be a lattice. Let $f(x, \lambda)$ be a function defined on $\mathscr{L} \times \Lambda$, where λ is the parameter. The objective is to minimize $f(\cdot, \lambda)$ for each λ. Let $F^*(\lambda)$ be the set of minimum solutions of $f(\cdot, \lambda)$ for a specific λ; i.e., $F^*(\lambda) = \mathrm{argmin}_x \{f(x, \lambda) : x \in \mathscr{L}\}$. For an arbitrary subset Λ_0 of Λ, a *monotone sub-selection of optimal solutions* is a mapping, $x(\cdot)$, from Λ_0 to $\mathscr{L}$, such that $x(\lambda) \in F^*(\lambda)$ for each $\lambda \in \Lambda_0$ and $x(\lambda)$ is monotone in λ. Similarly, a *totally ordered sub-selection of optimal solutions* is a mapping, $x(\cdot)$, from Λ_0 to $\mathscr{L}$, such that $x(\lambda) \in F^*(\lambda)$ for each $\lambda \in \Lambda_0$ and the collection $\{x(\lambda) | \lambda \in \Lambda_0\}$ is totally ordered by $\preceq$. A sub-selection is reffered to as a *selection*, if $\Lambda_0 = \Lambda$.

If $F^*(\lambda) = \emptyset$ for any $\lambda \in \Lambda$, then λ can be ignored in this study. Therefore, we assume in this section that $F^*(\lambda) \neq \emptyset$ for every $\lambda \in \Lambda$. For conditions which ensure non-emptiness of $F^*(\lambda)$ see, e.g., [20].

The following condition ensures the existence of a totally ordered optimal solutions:

Condition I. *For every $\lambda_1, \lambda_2 \in \Lambda$ and arbitrary $x \in F^*(\lambda_1)$, $y \in F^*(\lambda_2)$, either $x \wedge y \in F^*(\lambda_1)$ and $x \vee y \in F^*(\lambda_2)$ or $x \vee y \in F^*(\lambda_1)$ and $x \wedge y \in F^*(\lambda_2)$ holds.*

Theorem 1 (*Strong totally ordered selection theorem*). *Let $\mathscr{L}$ be a lattice, let $f(\cdot, \cdot)$ be defined on $\mathscr{L} \times \Lambda$, and suppose Condition* I *is satisfied.*

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[2] A definition of parallel arcs is given in Section 3.

[3] For related books with large bibliographies the reader is referred to, e.g., [24,21].

(i) *If Λ is a countable set, then a totally ordered sub-selection of optimal solutions* $\{x(\lambda) \in F^*(\lambda) | \lambda \in \Lambda_0\}$, *for a finite set* Λ_0, *can be extended to a totally ordered selection on* Λ.
(ii) *If* $\mathscr{L}$ *is finite, then* (i) *still holds without requiring* Λ *to be countable.*
(iii) *If a minimum element* $s(\lambda)$ (*respectively, maximum element* $S(\lambda)$) *exists in every* $F^*(\lambda)$, *then the collection* $\{s(\lambda)\}$ (*respectively,* $\{S(\lambda)\}$) *is totally ordered.*

Proof. (i) Consider the first claim. The mapping $x(\cdot)$ can be constructed by the following inductive procedure. Suppose that at a certain step, one has obtained a totally ordered sub-selection for Λ_k, which consists of k elements and contains Λ_0 as a subset. Since Λ_k is finite, the elements in Λ_k can be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_k$ so that $x_1 \preceq x_2 \preceq \cdots \preceq x_k$, where $x_i = x(\lambda_i)$ for $i = 1, 2, \ldots, k$. Choose an arbitrary λ_{k+1} in $\Lambda \backslash \Lambda_k$. We wish to find $x_{k+1} = x(\lambda_{k+1})$ such that $x_{k+1} \in F^*(\lambda_{k+1})$ and the collection $\mathscr{C}_{k+1} = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ is totally ordered.

We claim that $F^*(\lambda_{k+1})$ either contains an element m_1 such that $m_1 \preceq x_1$ or an element M_1 such that $x_1 \preceq M_1$. To show this, pick an arbitrary x in $F^*(\lambda_{k+1})$. Since Condition I holds, either $x \wedge x_1$ is in $F^*(\lambda_{k+1})$ or $x \vee x_1$ is in $F^*(\lambda_{k+1})$. If the former is true, we choose $m_1 = x \wedge x_1$, and if the latter is true, we choose $M_1 = x \vee x_1$.

If m_1 exists, let $x_{k+1} = m_1$ and $\mathscr{C}_{k+1} = \mathscr{C}_k \cup \{x_{k+1}\}$ is totally ordered. Otherwise, let l be the largest index such that there exists an element M_l in $F^*(\lambda_{k+1})$ for which $x_l \preceq M_l$. If $l = k$, let $x_{k+1} = M_l$ and $\mathscr{C}_{k+1} = \mathscr{C}_k \cup \{x_{k+1}\}$ is again totally ordered. If $l < k$, then there does not exist an element M_{l+1} in $F^*(\lambda_{k+1})$ such that $x_{l+1} \preceq M_{l+1}$. Consider $x_{l+1} \vee M_l$ and $x_{l+1} \wedge M_l$. By Condition I, one of them must be in $F^*(\lambda_{k+1})$. The former cannot be in $F^*(\lambda_{k+1})$, since otherwise, there exists an element $M_{l+1} = x_{l+1} \vee M_l$ in $F^*(\lambda_{k+1})$ satisfying $x_{l+1} \preceq M_{l+1}$, contradicting the assumption that no such M_{l+1} exists. So $x_{l+1} \wedge M_l$ is in $F^*(\lambda_{k+1})$. Let $x_{k+1} = x_{l+1} \wedge M_l$. Clearly, $x_{k+1} \preceq x_{l+1}$. Since $x_l \preceq M_l$ and $x_l \preceq x_{l+1}$, it follows from the definition of $\wedge$ that $x_l \preceq x_{l+1} \wedge M_l = x_{k+1}$. Thus the collection $\mathscr{C}_{k+1} = \mathscr{C}_k \cup \{x_{k+1}\}$ is totally ordered. This completes the proof of the first claim.

(ii) Observe that if for two values of the parameter, λ_1 and λ_2, $F^*(\lambda_1) = F^*(\lambda_2)$, then one only has to select an optimal solution $x(\lambda_1)$ for λ_1 and let $x(\lambda_2) = x(\lambda_1)$. Let $n = |\mathscr{L}|$ be the cardinality of $\mathscr{L}$. For each $\lambda \in \Lambda$, $F^*(\lambda)$ must be one of the 2^n subsets of $\mathscr{L}$. Therefore, one has to consider at most 2^n representative instances in Λ. Since this set of representative instances is finite and thus countable, claim (i) applies and the proof of (ii) is complete.

(iii) To prove the third part of the theorem, consider two arbitrary instances of the parameter λ_1 and λ_2. Since $s(\lambda_1) \wedge s(\lambda_2)$ is either in $F^*(\lambda_1)$ or in $F^*(\lambda_2)$, $s(\lambda_1) \wedge s(\lambda_2)$ must be equal to either $s(\lambda_1)$ or $s(\lambda_2)$. In the former case, $s(\lambda_1) \preceq s(\lambda_2)$ and in the latter case, $s(\lambda_2) \preceq s(\lambda_1)$. Similarly, $S(\lambda_1)$ and $S(\lambda_2)$ must be ordered by $\preceq$. □

The reader is referred to [12] for a weaker sufficient condition than Condition I, for the existence of a totally ordered selection of optimal solutions in a parametric optimization problem. However, this condition does not necessarily ensure that an arbitrary sub-selection of ordered optimal solutions can be extended to a complete one.

The strong totally ordered selection theorem can be easily used to strengthen the AUK result by demonstrating that every totally ordered sub-selection of minimum cuts in their parametric network flow problem can be extended to a totally ordered selection of minimum cuts. To show it, we first need to recall some basic definitions in graph theory.

Let $G(N, A)$ denote a directed network, with nodes N and arcs A. Let s and t be the source node and the sink node, respectively. A *cut* is a bi-partition $(X, \overline{X})$ of the node set N, where $s \in X$ and $t \in \overline{X}$. A partial order can be defined on the set of cuts in a network. Namely, $(X, \overline{X}) \preceq_c (Y, \overline{Y})$ if $X \subseteq Y$. Under this partial order, the set of cuts in a network is a lattice.

Let c_{ij} denote the capacity associated with arc (i, j). In this study, we assume that arc capacities are strictly positive, in order to avoid some degenerate cases. For two subsets of nodes X and Y, (X, Y) can be interpreted as the set of arcs $\{(i, j) | i \in X, j \in Y\}$. Let $c(X, Y) = \sum_{i \in X, j \in Y} c_{ij}$ be the capacity of (X, Y). In particular, the capacity of cut $(X, \overline{X})$ is defined as $c(X, \overline{X}) = \sum_{i \in X, j \in \overline{X}} c_{ij}$. A *minimum cut* in a network is one whose capacity is minimum among all cuts in the network.

Finally, recall that in the AUK model, arcs incident to a node v, $v \neq s$, $v \neq t$ have linearly increasing capacities in a real-valued parameter λ, and arcs elsewhere have fixed capacities.

Lemma 2. *In the AUK setting, let* $(Y, N \backslash Y)$ *be a minimum cut at* λ_Y *and let* $(W, N \backslash W)$ *be a minimum cut at* λ_W. *Then, either* $(Y \cap W, N \backslash (Y \cap W))$ *is a minimum cut at* λ_Y *and* $(Y \cup W, N \backslash (Y \cup W))$ *is a minimum cut at* λ_W *or* $(Y \cup W, N \backslash (Y \cup W))$ *is a minimum cut at* λ_Y *and* $(Y \cap W, N \backslash (Y \cap W))$ *is a minimum cut at* λ_W.

Proof. The proof is similar to the proof of Lemma 2 in AUK and thus omitted. □

Lemma 2 can now be used to extend the AUK parametric result.

Theorem 3. *For a given set of parameter values* $\Lambda = \{\lambda_1, \ldots, \lambda_k \ldots\}$, *in the AUK setting, a totally ordered sub-selection of minimum cuts* $\{(X(\lambda), N \backslash X(\lambda)) : \lambda \in \Lambda_0, \Lambda_0 \subseteq \Lambda\}$ *for a finite set* Λ_0, *can be extended to a totally ordered selection on* Λ.

Proof. By Lemma 2, Condition I holds. Further, the set of all cuts in a network is a finite lattice. The proof then follows by Theorem 1 (ii). □

3. Parallel arcs and path closure properties

This section introduces the notion of parallel arcs, which is independent of arc capacities and only depends on the network topology. It also develops some path closure properties along lines motivated by Picard and Queyranne [16] and Granot et al. [10]. Our study of ordered minimum cuts in the next subsection will depend on these concepts.

A *path* in G is a sequence, $< n_1, (n_1, n_2), n_2, (n_2, n_3), n_3, \ldots, n_{k-1}, (n_{k-1}, n_k), n_k >$, of nodes and arcs. When there is no ambiguity, the above path will also be denoted as $< n_1, n_2, \ldots, n_k >$. A *simple path* is one in which the nodes do not repeat. An *s-t path* is a simple path from the source node s to the sink node t. For simplicity of presentation, we will write $p_1 \cap p_2 = \emptyset$ to denote the fact that paths p_1 and p_2 are vertex disjoint; $e \in p_1$ to denote the fact that arc e is in path p_1; and $u \in p_1$ to denote the fact that node u is in path p_1.

Definition 4. Two arcs (i, j) and (k, l) are *parallel* if there exist simple paths p_1 from s to i, p_2 from s to k, p_3 from j to t, and p_4 from l to t such that $p_1 \cap p_4 = \emptyset$, $p_2 \cap p_3 = \emptyset$, $p_1 \cap p_3 = \emptyset$ and $p_2 \cap p_4 = \emptyset$. In this case, the subgraph of G spanned by (i, j), (k, l), p_1, p_2, p_3 and p_4 *is called a bypass* between (i, j) and (k, l).

Intuitively, if two arcs (i, j) and (k, l) are parallel in G, then there exists a *simple s–t* path $p_a = < p_1, (i, j), p_3 >$ from s to t that bypasses (k, l) and another *simple s–t* path $p_b = < p_2, (k, l), p_4 >$ from s to t that bypasses (i, j).

Parallel arcs are illustrated in Fig. 1, where arcs (i, j) and (k, l) (shown in bold lines) are parallel.

For an arbitrary directed network, verifying whether a pair of arcs is parallel is NP-hard. This is due to the fact that finding two vertex-disjoint paths between two pairs of nodes in directed graphs is NP-hard. See [7].

Definition 5. Let (i, j) and (k, l) be two parallel arcs in a directed network G. Arc (i, j) is called a *front arc* with respect to (k, l) if in every bypass between (i, j) and (k, l), i is the last common node in p_1 and p_2. Arc (k, l) is called a *back arc* with respect to (i, j) if in every bypass between (i, j) and (k, l), l is the first common node in p_3 and p_4.

Fig. 2 illustrates the notion of front and back parallel arcs. In Diagram (a), (i, j) is a front arc with respect to (k, l) and (k, l) is a back arc with respect to (i, j). In Diagram (b), (i, j) is a front arc with respect to (k, l), but (k, l) is not a back arc with respect to (i, j), since in a bypass spanned by the two arcs and p_1, p_2, p_3' and p_4, l is not on p_3'. In Diagram (c), arc (i, j) is both a front arc and a back arc with respect to (k, l).

The following definitions and results help identify sets of arcs related to minimum cuts, across which each pair of arcs is parallel.

For each set of nodes, X, define the *s-kernel* by

$$_s[X] = \{i : \text{there exists a path from } s \text{ to } i \text{ which is contained in X}\};$$

define the *t-kernel* by

$$[\overline{X}]_t = \{i : \text{there exists a path from } i \text{ to } t \text{ which is contained in } \overline{X}\};$$

define the set S_X by

$$S_X = \{i : \text{there exists a path from } s \text{ to } i \text{ which does not meet } [\bar{X}]_t\};$$

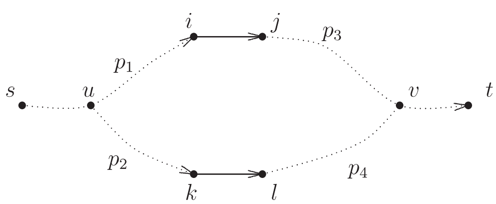

Fig. 1. Parallel arcs.

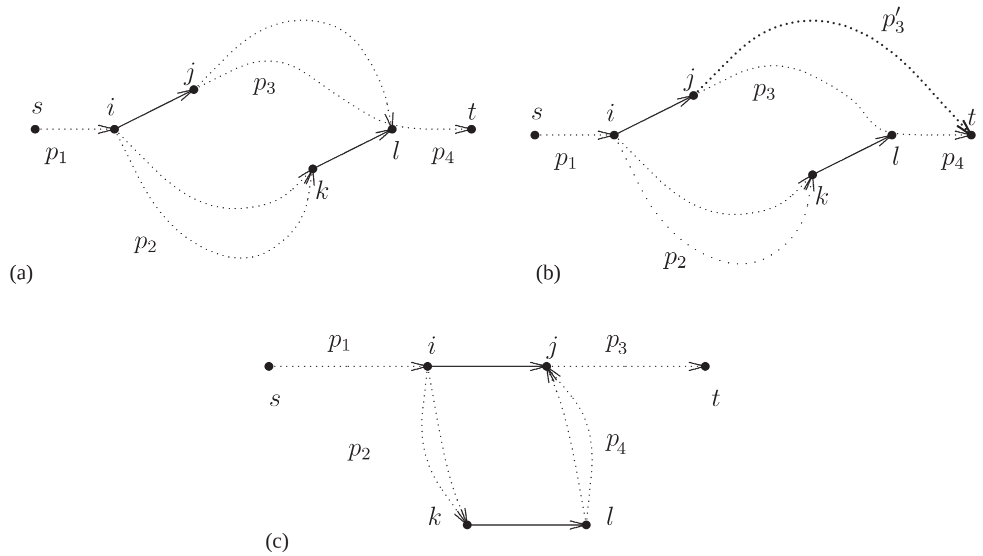

Fig. 2. Front and back arcs.

define the set T_X by

$$T_X = \{i : \text{there exists a path from } i \text{ to } t \text{ which does not meet } S_X\};$$

and define the set Z_X to consist of all other nodes neither in S_X nor in T_X.

Define a set, X, to be *s-closed* if for each node i in X there is a path from s to i which does not meet $\bar{X}$; and define it to be *t-closed* if for each node i in X there is a path from i to t which does not meet $\bar{X}$.

Lemma 6. *For each cut* $(X, \overline{X})$,

(i) *The sets* ${}_s[X]$ *and* S_X *are s-closed, and the sets* $[\bar{X}]_t$ *and* T_X *are t-closed*;
(ii) ${}_s[S_X] = S_X$ *and* $[T_X]_t = T_X$;
(iii) $S_X \cap T_X = \emptyset$;
(iv) $(S_X, \overline{S_X}) = (\overline{T_X}, T_X) = (S_X, [\bar{X}]_t) = (S_X, T_X) \subseteq (X, \overline{X})$;
(v) *Each pair of arcs in* $(S_X, \overline{S_X})$ *is parallel.*

Proof. Statements (i), (ii) and (iii) essentially follow from the associated definitions. Thus, for brevity, we will only prove (iv) and (v).

If (i, j) is in $(S_X, \overline{S_X})$, then j must belong to $[\bar{X}]_t$; otherwise by the definition of S_X there would be a path from s to j (via i) which does not meet $[\bar{X}]_t$. Hence, $(S_X, \overline{S_X}) \subseteq (S_X, [\bar{X}]_t)$. However, since $[\bar{X}]_t \subseteq \overline{S_X}$ it follows that $(S_X, \overline{S_X}) = (S_X, [\bar{X}]_t)$. The last equality in (iv) now follows since $[\bar{X}]_t \subseteq T_X \subseteq \overline{S_X}$. Further, for $(i, j) \in (S_X, \overline{S_X})$, we must have that $i \in X$. Indeed, if i is not in X, then the path consisting of the arc (i, j), followed by the path from j to t which is contained in $[\bar{X}]_t$,would lie in $\bar{X}$ and i would be in $[\bar{X}]_t$, contradicting the fact that it is in S_X. Hence $(S_X, \overline{S_X}) \subseteq (X, \overline{X})$.

To finish the proof of (iv), it remains to show that $(S_X, [\bar{X}]_t) = (\overline{T_X}, T_X)$. Since $S_X \subseteq \overline{T_X}$ and $[\bar{X}]_t \subseteq T_X$, it is clear that $(S_X, [\bar{X}]_t) \subseteq (\overline{T_X}, T_X)$. On the other hand, suppose that (i, j) is in $(\overline{T_X}, T_X)$. Then i is in S_X; otherwise, the path consisting of arc (i, j) followed by the path from j to t which is contained in T_X would not meet S_X and i would be in T_X. The previous paragraph established that if i is in S_X, then j must be in $[\bar{X}]_t$. Hence $(S_X, [\bar{X}]_t) \supseteq (\overline{T_X}, T_X)$, and we conclude that $(S_X, [\bar{X}]_t) = (\overline{T_X}, T_X)$.

To prove (v), suppose that (i, j) and (k, l) are in the cut $(S_X, \overline{S_X})$. Then by (iv) they are also contained in the set of arcs $(S_X, [\bar{X}]_t)$. Since i and k are in S_X, there are paths p_1 and p_2 contained in S_X connecting s to i and to k, respectively. Since j and l are in $[\bar{X}]_t$, there are paths p_3 and p_4 contained in $[\bar{X}]_t$ connecting j and l to t, respectively. The paths p_1 and p_2 must be disjoint from p_3 and p_4 since they are in disjoint sets. Therefore, by definition, arcs (i, j) and (k, l) are parallel. $\square$

Lemma 7. *The following are equivalent*:

(i) $(X, \overline{X})$ *is a minimum cut for some capacity function* c;
(ii) $(S_X, T_X) = (X, \overline{X})$;
(iii) $(S_X, \overline{S_X}) = (X, \overline{X})$.

Each of the above implies the following:

(iv) $S_X \subseteq X$ *and* $T_X \subseteq \bar{X}$;
(v) $S_X = {}_s[X]$ *and* $T_X = [\bar{X}]_t$.

Moreover, $S_X \subseteq X$ *if and only if* $S_X = {}_s[X]$, *and* $T_X \subseteq \bar{X}$ *if and only if* $T_X = [\bar{X}]_t$.

Proof. From Lemma 6(iv) we have that

$$(S_X, T_X) = (S_X, \overline{S_X}) \subseteq (X, \overline{X}). \tag{1}$$

This, together with the standing hypothesis that capacities are positive, shows that (i) implies (ii) and that (ii) implies (iii).

Suppose that (iii) holds. Define a capacity function, c, by $c(e) = 1$ if $e \in (S_X, \overline{S_X})$ and $c(e) = \infty$ otherwise. Then $(S_X, \overline{S_X}) = (X, \overline{X})$ is a minimum cut for c, and so (iii) implies (i). From the previous paragraph it follows that (i), (ii), and (iii) are equivalent.

From the definitions of the s and t kernels, it is clear that ${}_s[X] \subseteq X$ and that $[\bar{X}]_t \subseteq \bar{X}$. Hence each part of (v) implies the corresponding part of (iv).

Also from the definitions of the s and t kernels, it is clear that the kernel operators are monotone, so that if (iv) holds it follows that ${}_s[S_X] \subseteq {}_s[X]$. Since by Lemma 6(ii) we have $S_X = {}_s[S_X]$, it follows that $S_X \subseteq {}_s[X]$. By the definitions of ${}_s[X]$ and S_X, it follows that ${}_s[X] \subseteq S_X$. Hence $S_X = {}_s[X]$. A similar argument shows that $T_X \subseteq \bar{X}$ implies that $T_X = [\bar{X}]_t$. Hence each part of (iv) implies the corresponding part of (v).

To finish the proof it is sufficient to show that (ii) implies (iv). Suppose that $(S_X, T_X) = (X, \overline{X})$. If $i \in S_X \backslash X$, then because S_X is s-closed, there is a path, p, from s to i in S_X. Since $s \in S_X$ and $i \notin X$, there is at least one arc in p which is in $(X, \overline{X})$ but not in (S_X, T_X), which contradicts the hypothesis that $(S_X, T_X) = (X, \overline{X})$. Hence $S_X \subseteq X$.

A similar argument shows that $T_X \subseteq \bar{X}$, so that (ii) implies (iv). □

Lemma 8. *Let* X *and* Y *be two subsets of nodes such that* $s \in X \cap Y$ *and* $t \notin X \cup Y$.

(i) *If* $X \subseteq Y$, *then* $S_X \subseteq S_Y$ *and* $T_Y \subseteq T_X$.
(ii) $S_{X \cap Y} \subseteq S_X \cap S_Y$ *and* $S_{X \cup Y} \supseteq S_X \cup S_Y$.
(iii) $T_{X \cap Y} \supseteq T_X \cup T_Y$ *and* $T_{X \cup Y} \subseteq T_X \cap T_Y$.

Proof. The hypothesis in (i) implies $\bar{Y} \subseteq \bar{X}$, so that $[\bar{Y}]_t \subseteq [\bar{X}]_t$. Suppose node i is in S_X. Then there exists a path from s to i which does not meet $[\bar{X}]_t$. But this path cannot meet $[\bar{Y}]_t$ since it is contained in $[\bar{X}]_t$. Consequently, $i \in S_Y$ and $S_X \subseteq S_Y$. Suppose i is in T_Y. Then there exists a path from i to t which does not meet S_Y. This path cannot meet S_X since it is contained in S_Y. Consequently, $i \in T_X$ and $T_Y \subseteq T_X$ completing the proof of (i). (ii) and (iii) follow from (i). □

Define two cuts, say $(X, \overline{X})$ and $(Y, \overline{Y})$, to be *equivalent* if $S_X = S_Y$. If two cuts are each minimum (not necessarily for the same capacity function), then the next lemma shows that they are equivalent if and only if they have the same set of arcs, although the node sets might not be identical.

Lemma 9. *If* $(X, \overline{X})$ *is a minimum cut for some capacity function and* $S_X = S_Y$, *then* $(X, \overline{X}) \subseteq (Y, \overline{Y})$. *If, in addition*, $(Y, \overline{Y})$ *is also a minimum cut for some capacity function then* $(X, \overline{X}) = (Y, \overline{Y})$.

Proof. Since $(X, \overline{X})$ is assumed to be a minimum cut, by Lemma 7(iii) it follows that $(S_X, \overline{S_X}) = (X, \overline{X})$. By hypothesis, $S_X = S_Y$. Thus, $(X, \overline{X}) = (S_X, \overline{S_X}) = (S_Y, \overline{S_Y}) \subseteq (Y, \overline{Y})$, where the set inclusion follows from Lemma 6(iv). If, in addition, $(Y, \overline{Y})$ is a minimum cut for some capacity function c, then an identical argument applied to $(Y, \overline{Y})$ shows that $(Y, \overline{Y}) \subseteq (X, \overline{X})$. So in this case $(X, \overline{X}) = (Y, \overline{Y})$. □

4. Ordered selection of minimum cuts

In this section, we use the totally ordered selection theorem to derive a necessary and sufficient condition for the existence of a totally ordered selection of minimum cuts in a parametric maximum flow problem.

A *parametric maximum flow* problem is a maximum flow problem in which the capacities of the arcs may change as functions of a parameter λ. In such a problem, the notation is modified to include the parameter, so the capacity of an arc (i, j) is denoted by $c(\lambda; i, j)$ and the capacity of (X, Y) is written as $c(\lambda; X, Y)$. Again, we assume that the capacities on the arcs are positive to avoid degenerate cases.

A parametric maximum flow problem was studied by GGT (1989). Therein, they have shown that the lattices of minimum cuts are ascending in a parametric maximum flow problem in which the capacities of arcs incident to the source are increasing functions of a parameter λ and the capacities of arcs incident to the sink are decreasing functions of λ. Notice that this result does not pertain to any specific parametric capacity function $c(\cdot; \cdot)$ on G. Rather, a class of parametric capacity functions is specified such that the aforementioned property holds for every parametric capacity function in this class.

The main concern in this section is to characterize classes of parametric capacity functions such that for *every* parametric capacity function in the specified class a totally ordered selection of minimum cuts exists. To that end, we introduce the following terminology. A parametric capacity function, $c(\cdot; \cdot)$, is *constant* on a set of arcs $A_0 \subseteq A$ if for each arc e in A_0, $c(\lambda_1; e) = c(\lambda_2; e)$ for all λ_1 and λ_2 in Λ; it *changes in the same direction* on A_0 if for each λ_1 and λ_2 in Λ, $c(\lambda_1; \tilde{e}) - c(\lambda_2; \tilde{e}) > 0$ for some $\tilde{e} \in A_0$ implies that $c(\lambda_1; e) - c(\lambda_2; e) \geqslant 0$ for all $e \in A_0$; it *changes in opposite directions* between two sets of arcs A^+ and A^- if for each λ_1 and λ_2 in Λ, $c(\lambda_1; \tilde{e}) - c(\lambda_2; \tilde{e}) > 0$ for some $\tilde{e} \in A^+$ implies that $c(\lambda_1; e) - c(\lambda_2; e) \leqslant 0$ for all $e \in A^-$ and $c(\lambda_1; \tilde{e}) - c(\lambda_2; \tilde{e}) > 0$ for some $\tilde{e} \in A^-$ implies that $c(\lambda_1; e) - c(\lambda_2; e) \leqslant 0$ for all $e \in A^+$.

Next, we introduce a mechanism for specifying classes of parametric capacity functions, which we refer to as Generalized GGT (G^3T) classes. A G^3T class of parametric capacity functions is obtained by specifying a set of arcs, A_v, on which the capacities can change. In addition, two subsets of A_v, say A_v^+ and A_v^-, are specified which restrict the direction of change. Given A_v, A_v^+ and A_v^-, the G^3T class of parametric capacity functions, $\mathscr{F}(A_v, A_v^+, A_v^-)$, are those capacity functions which are constant on $A \backslash A_v$, change in the same direction on A_v^+ and on A_v^-, and change in opposite directions between A_v^+ and A_v^-.

Examples of G^3T classes have been used in the literature. GGT have introduced a class of parametric capacity functions which are included in $\mathscr{F}(A_v, A_v^+, A_v^-)$, where $A_v^+ = \{(s, i) | (s, i) \in A, \forall i \in N\}$, $A_v^- = \{(i, t) | (i, t) \in A, \forall i \in N\}$ and $A_v = A_v^+ \cup A_v^-$ (assume that arc (s, t) does not exist in the network). The GGT class imposes some explicit monotonicity conditions on the capacity functions which are slightly more restrictive than the directional restrictions which we use in G^3T classes. AUK examined the class of capacity functions $\mathscr{F}(A_v, A_v^+, A_v^-)$, where $A_v^+ = \{(v, i) | (v, i) \in A, \forall i \in N\} \cup \{(i, v) | (i, v) \in A, \forall i \in N\}$, $A_v^- = \emptyset$ and $A_v = A_v^+$ for a specific node v called the "center node". They showed that for every parametric capacity function in $\mathscr{F}(A_v, A_v^+, A_v^-)$, a totally ordered selection of minimum cuts exists. Our Corollary 14 provides an alternative proof.

The G^3T specification of parametric capacity functions is fairly broad. As just mentioned, it subsumes the classes of parametric capacity functions studied by GGT and AUK as special cases. However, there are other possible ways of defining classes of parametric functions which are not included in our framework. For example, McCormick [14] restricts the magnitude of the parametric capacity change.

Our goal is to characterize the sets A_v, A_v^+ and A_v^- which will ensure the existence of a totally ordered selection of minimum cuts. Indeed, Theorem 12 provides a necessary and sufficient condition for the existence of a totally ordered selection of minimum cuts for every capacity function in the class $\mathscr{F}(A_v, A_v^+, A_v^-)$.

Condition II. *Let $G(N, A)$ be a directed network and $\mathscr{F}(A_v, A_v^+, A_v^-)$ be a G^3T class of parametric capacity functions. For each pair of parallel arcs e_1 and e_2 in A_v at least one of the following conditions is valid*:

(a) *e_1 and e_2 are both front arcs or are both back arcs, and e_1 and e_2 are either both in A_v^+ or both in A_v^-;*
(b) *e_1 or e_2 is a front arc and the other is a back arc, and one of them is in A_v^+ and the other is in A_v^-;*
(c) *e_1 or e_2 is both a front arc and a back arc.*

Lemma 10. *Suppose that Condition* II *holds, that $(X, \overline{X})$ is a minimum cut for the capacity function $c_1(\cdot) = c(\lambda_1; \cdot)$, and that $(Y, \overline{Y})$ is a minimum cut for the capacity function $c_2(\cdot) = c(\lambda_2; \cdot)$.*

Then either

$$c_1(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) + c_2(X\backslash Y, T_{X\cup Y}) \leqslant c_1(X\backslash Y, T_{X\cup Y}) + c_2(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) \tag{2}$$

or

$$c_2(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) + c_1(Y\backslash X, T_{X\cup Y}) \leqslant c_2(Y\backslash X, T_{X\cup Y}) + c_1(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) \tag{3}$$

holds.

Proof. Suppose that neither (2) nor (3) holds. Then

$$c_1(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) - c_2(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) + c_2(X\backslash Y, T_{X\cup Y}) - c_1(X\backslash Y, T_{X\cup Y}) > 0 \tag{4}$$

and

$$c_2(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) - c_1(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) + c_1(Y\backslash X, T_{X\cup Y}) - c_2(Y\backslash X, T_{X\cup Y}) > 0. \tag{5}$$

From (4), either

$$c_1(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) - c_2(S_{X\cap Y}, \bar{Y}\backslash\bar{X}) > 0, \tag{6}$$

or

$$c_2(X\backslash Y, T_{X\cup Y}) - c_1(X\backslash Y, T_{X\cup Y}) > 0. \tag{7}$$

From (5), either

$$c_2(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) - c_1(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) > 0, \tag{8}$$

or

$$c_1(Y\backslash X, T_{X\cup Y}) - c_2(Y\backslash X, T_{X\cup Y}) > 0. \tag{9}$$

Selecting one inequality from each set gives four cases. The proof will be completed when we show that none of the four cases can hold.

Suppose, on the contrary, that inequalities (6) and (8) hold. Then, because the capacities are positive, there exist arcs $(i, j) \in A_v \cap (S_{X\cap Y}, \bar{Y}\backslash\bar{X})$ and $(k, l) \in A_v \cap (S_{X\cap Y}, \bar{X}\backslash\bar{Y})$ such that $c_1(i, j) - c_2(i, j) > 0$ and $c_2(k, l) - c_1(k, l) > 0$. These two arcs are parallel by Lemma 6(v) since they belong to the cut $(S_{X\cap Y}, \overline{S_{X\cap Y}})$ and so must satisfy one of the criteria in Condition II.

Condition II(a) cannot hold since the capacities change in opposite directions.

Arc (i, j) cannot be a back arc with respect to (k, l). Indeed, since $l \in T_X$ by Lemma 7(ii), it follows from Lemma 6(i) that there exists a path from l to t in T_X. By Lemma 7(iv), $T_X \subseteq \bar{X}$; and since $j \in \bar{Y}\backslash\bar{X}$, this path cannot pass through j. Similarly, (k, l) cannot be a back arc with respect to (i, j). Hence neither II(b) nor II(c) can hold. Consequently, there is a contradiction and inequalities (6) and (8) cannot hold.

Next suppose, on the contrary, that inequalities (6) and (9) hold. Then there exist arcs $(i, j) \in A_v \cap (S_{X\cap Y}, \bar{Y}\backslash\bar{X})$ and $(k, l) \in A_v \cap (Y\backslash X, T_{X\cup Y})$ such that $c_1(i, j) - c_2(i, j) > 0$ and $c_1(k, l) - c_2(k, l) > 0$. By Lemma 8, $S_{X\cap Y} \subseteq S_Y$ and $T_{X\cup Y} \subseteq T_Y$; and by Lemma 7(iv), $S_Y \subseteq Y$ and $T_Y \subseteq \bar{Y}$. Thus, both $(S_{X\cap Y}, \bar{Y}\backslash\bar{X})$ and $(Y\backslash X, T_{X\cup Y})$ are included in the cut $(Y, \overline{Y})$. Therefore the arcs (i, j) and (k, l) are parallel by Lemma 6(v), since they belong to the cut $(Y, \overline{Y}) = (S_Y, \overline{S_Y})$ and so must satisfy one of the criteria in Condition II.

Condition II(b) cannot hold since the capacity functions change in the same direction. Condition II(a) cannot hold since the nodes i, j, k and l are in four disjoint sets, namely $X \cap Y$, $\bar{Y}\backslash\bar{X}$, $Y\backslash X$ and $\bar{X} \cap \bar{Y}$.

Arc (i, j) cannot be a back arc with respect to (k, l). Indeed, by Lemma 8(iii), $T_{X\cup Y} \subseteq T_X$, and by Lemma 7(iv), $T_X \subseteq \bar{X}$. Thus if $l \in T_{X\cup Y} \subseteq T_X$, then by Lemma 6(i) there exists a path from l to t in T_X which cannot pass through $j \in \bar{Y}\backslash\bar{X}$. Similarly (k, l) cannot be a front arc with respect to (i, j). So Condition II(c) cannot be satisfied. Consequently, there is a contradiction and inequalities (6) and (9) cannot hold.

The last two cases are proved similarly. □

The next theorem shows that the set of cuts which are minimum for some parametric capacity function in a G^{3}T class forms a lattice with the order $\preceq_c$ defined in Section 2.

Theorem 11. *Let $G(N, A)$ be a directed network and $\mathscr{F} = \mathscr{F}(A_v, A_v^+, A_v^-)$ be a G^3T class of parametric capacity functions satisfying Condition* II. *Let $\mathscr{C}$ be the set of cuts which are minimum for some parametric capacity function $c(\cdot; \cdot) \in \mathscr{F}$. Then $\mathscr{C}$ ordered by $\preceq_c$ is a sublattice of the lattice of all cuts. Moreover, if $(X, \overline{X})$ and $(Y, \overline{Y})$ are minimum cuts for $c_1(\cdot) = c(\lambda_1; \cdot)$ and $c_2(\cdot) = c(\lambda_2; \cdot)$, respectively, then $(X \cap Y, \overline{X \cap Y})$ is a minimum cut for one of the capacity functions c_1 or c_2 and $(X \cup Y, \overline{X \cup Y})$ is a minimum cut for the other.*

Proof. Let $R = (\bar{X} \cap \bar{Y}\backslash T_{X\cup Y})$ and let $L = (X \cap Y\backslash S_{X\cap Y})$.

By Lemma 10, Condition II ensures that either (2) or (3) holds. Suppose that (2) holds. The following set of inequalities (11)–(18) are chosen to make up the difference between (2) and

$$c_1(S_{X\cap Y}, \bar{X}\cup\bar{Y}) + c_2(X\cup Y, T_{X\cup Y}) \leqslant c_1(X,\bar{X}) + c_2(Y,\bar{Y}). \tag{10}$$

Inequalities (11)–(14) follow from the inequalities in Lemma 8 and since $S_X \subseteq X$, $T_X \subseteq \bar{X}$, $S_Y \subseteq Y$, and $T_Y \subseteq \bar{Y}$ hold by Lemma 7(iv). The last four inequalities hold since the capacity functions are assumed to be positive.

$$c_1(S_{X\cap Y}, \bar{X}\backslash\bar{Y}) \leqslant c_1(X\cap Y, \bar{X}\backslash\bar{Y}), \tag{11}$$
$$c_2(Y\backslash X, T_{X\cup Y}) \leqslant c_2(Y\backslash X, \bar{X}\cap\bar{Y}), \tag{12}$$
$$c_1(S_{X\cap Y}, \bar{X}\cap\bar{Y}) \leqslant c_1(X\cap Y, \bar{X}\cap\bar{Y}), \tag{13}$$
$$c_2(X\cap Y, T_{X\cup Y}) \leqslant c_2(X\cap Y, \bar{X}\cap\bar{Y}), \tag{14}$$
$$0 \leqslant c_1(X\backslash Y, R), \tag{15}$$
$$0 \leqslant c_2(L, \bar{Y}\backslash\bar{X}), \tag{16}$$
$$0 \leqslant c_1(X\backslash Y, \bar{X}\backslash\bar{Y}), \tag{17}$$
$$0 \leqslant c_2(Y\backslash X, \bar{Y}\backslash\bar{X}). \tag{18}$$

Adding the above inequalities to (2), yields (10).

By the definition of the operator S, $(S_{X\cap Y}, L) = \emptyset$. The set of arcs $(R, T_{X\cup Y})$ is also empty, since the hypothesis that $(X, \overline{X})$ and $(Y, \overline{Y})$ are minimum cuts implies by Lemmas 7(iv) and 8(iii) that $T_{X\cup Y} \subseteq \overline{X\cup Y}$. Then by Lemma 7, $T_{X\cup Y} = [\overline{X\cup Y}]_t$, from which the emptiness of $(R, T_{X\cup Y})$ follows.

It follows from $(S_{X\cap Y}, L) = (R, T_{X\cup Y}) = \emptyset$ that $(S_{X\cap Y}, \overline{S_{X\cap Y}}) = (S_{X\cap Y}, \bar{X}\cup\bar{Y}) \cup (S_{X\cap Y}, L) = (S_{X\cap Y}, \bar{X}\cup\bar{Y})$ and $(\overline{T_{X\cup Y}}, T_{X\cup Y}) = (X\cup Y, T_{X\cup Y}) \cup (R, T_{X\cup Y}) = (X\cup Y, T_{X\cup Y})$. Hence (10) and the assumptions that $(X, \overline{X})$ and $(Y, \overline{Y})$ are minimum cuts for c_1 and c_2, respectively, imply that $(S_{X\cap Y}, \overline{S_{X\cap Y}})$ and $(\overline{T_{X\cup Y}}, T_{X\cup Y})$ are also minimum cuts for c_1 and c_2, respectively. Moreover, inequalities (2), (10) and (11)–(18) are satisfied as equalities.

Next, we show that $(S_{X\cap Y}, \bar{X}\cup\bar{Y}) = (X\cap Y, \bar{X}\cup\bar{Y})$ and $(X\cup Y, T_{X\cup Y}) = (X\cup Y, \bar{X}\cap\bar{Y})$. The sets of arcs in the arguments of the capacity functions c_1 and c_2 on the left-hand sides of (11)–(14) are subsets of the corresponding sets on the right-hand sides. Since capacities are assumed to be positive, these sets of arcs must be equal. Similarly, the sets of arcs involved in (15)–(18) must be empty. So from (11), $(L, \bar{X}\backslash\bar{Y}) = \emptyset$; from (13), $(L, \bar{X}\cap\bar{Y}) = \emptyset$; and from (16), $(L, \bar{Y}\backslash\bar{X}) = \emptyset$. Also, note that $(X\cap Y, \bar{X}\cup\bar{Y}) = (S_{X\cap Y}, \bar{X}\cup\bar{Y}) \cup (L, \bar{X}\cup\bar{Y})$ and that $(L, \bar{X}\cup\bar{Y}) = (L, \bar{X}\backslash\bar{Y}) \cup (L, \bar{X}\cap\bar{Y}) \cup (L, \bar{Y}\backslash\bar{X})$. Hence, $(S_{X\cap Y}, \bar{X}\cup\bar{Y}) = (X\cap Y, \bar{X}\cup\bar{Y})$.

An analogous argument shows that $(X\cup Y, T_{X\cup Y}) = (X\cup Y, \bar{X}\cap\bar{Y})$ follows from (12), (14) and (15).

Consequently, if (2) holds, then

$$c_1(X\cap Y, \bar{X}\cup\bar{Y}) + c_2(X\cup Y, \bar{X}\cap\bar{Y}) = c_1(X,\overline{X}) + c_2(Y,\overline{Y}). \tag{19}$$

A similar argument shows that if (3) holds, then

$$c_2(X\cap Y, \bar{X}\cup\bar{Y}) + c_1(X\cup Y, \bar{X}\cap\bar{Y}) = c_1(X,\overline{X}) + c_2(Y,\overline{Y}). \tag{20}$$

In either case, both $X\cup Y$ and $X\cap Y$ are minimum cuts and so belong to $\mathscr{C}$. □

Theorem 12. *Let $G(N, A)$ be a directed network and $\mathscr{F}(A_v, A_v^+, A_v^-)$ be a G^3T class of parametric capacity functions. A totally ordered selection of minimum cuts exists for every capacity function $c(\cdot;\cdot)$ in $\mathscr{F}(A_v, A_v^+, A_v^-)$ if and only if Condition* II *holds. Furthermore, if Condition* II *is satisfied, then each totally ordered sub-selection of minimum cuts can be extended to a totally ordered selection.*

Proof. If Condition II holds, then by Theorem 11, $\mathscr{C}$ is a sublattice and Condition I is satisfied as equalities on $\mathscr{C}$. The claimed selection and extension properties follow from Theorem 1.

To prove the "only if" part, suppose there exists a pair of parallel arcs $e_1 = (i, j)$ and $e_2 = (k, l)$ in A_v such that none of the three criteria in Condition II is satisfied. It suffices to construct a capacity function $c(\cdot;\cdot) \in \mathscr{F}(A_v, A_v^+, A_v^-)$ for which a totally ordered selection of minimum cuts does not exist.

Since e_1 and e_2 do not satisfy criterion II(c), neither of them is both a front arc and a back arc. Since neither criterion II(a) nor criterion II(b) holds, at least one of the following three cases must be valid.

Case 1: At least one of the arcs e_1 and e_2 is neither a front arc nor a back arc.

Case 2: e_1 and e_2 are either both front arcs or are both back arcs (i.e. $i = j$ or $k = l$), and the capacities of e_1 and e_2 are allowed to change in opposite directions.

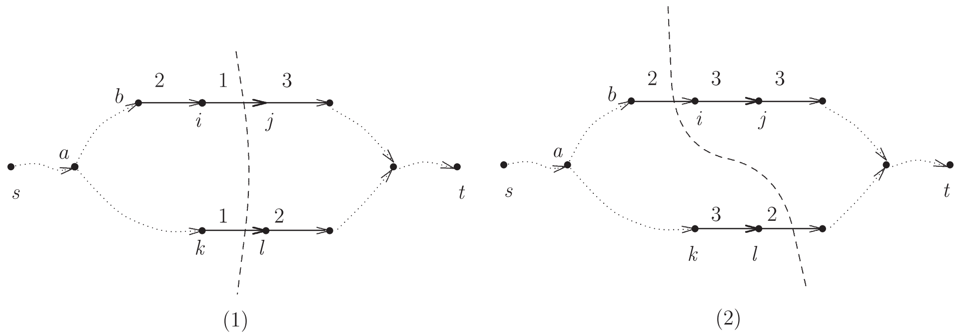

Fig. 3. G' for Case 1.

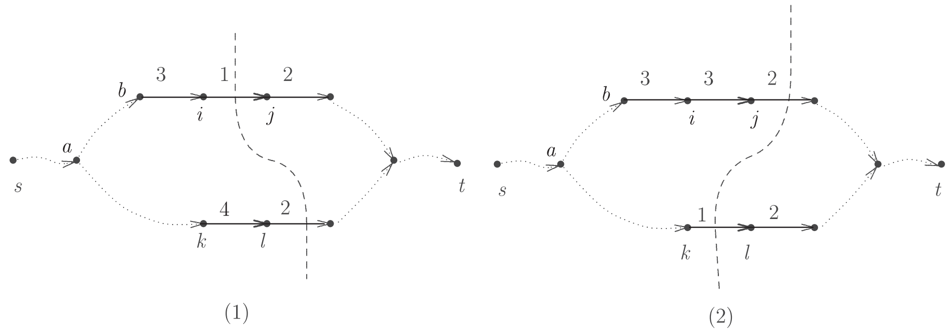

Fig. 4. G' for Case 1.

Case 3: Either e_1 or e_2 is a front arc and the other is a back arc, and their capacities are allowed to change in the same direction.

We will now construct a parametric capacity function $c(\cdot;\cdot)$ for each of the above three cases. Since $e_1 = (i, j)$ and $e_2 = (k, l)$ are parallel, there are simple paths p_1 from s to i, p_2 from s to k, p_3 from j to t and p_4 from l to t such that $p_1 \cap p_4 = \emptyset$, $p_1 \cap p_3 = \emptyset$, $p_2 \cap p_3 = \emptyset$ and $p_2 \cap p_4 = \emptyset$. Let G' be the subgraph of G spanned by arcs (i, j), (k, l) and paths p_1, p_2, p_3 and p_4. Assign very small constant capacities to arcs not in G' so that the total capacity of these arcs is less than 1. Consider a minimum cut, $(X, \overline{X})$, in G for a capacity function yet to be constructed. It disconnects t from s in G' as well as in G. Thus $(X, \overline{X})$ contains a subset of arcs, A', which forms a cut set of arcs for G'. In the capacity function being constructed, each arc in G' will be assigned an integer capacity greater than or equal to 1. Therefore, A' must be a minimum cut set of arcs for G'. Otherwise, a minimum cut in G' can be augmented with all arcs in $G \backslash G'$ to form a new cut in G whose capacity is smaller than the capacity of $(X, \overline{X})$.

If for two values of the parameter, say $\lambda_1 < \lambda_2$, the corresponding minimum cuts are unique and cannot be ordered, then the minimum cuts in G for λ_1 and λ_2 cannot be ordered. Thus it suffices to show that unique minimum cuts cannot be ordered in G' for two values, λ_1 and λ_2 in Λ.

In the following counterexamples, assign constant values to capacities of arcs other than e_1 and e_2 in G'. G' corresponding to Case 1 is illustrated in Figs. 3 and 4. G' corresponding to Cases 2 and 3 is illustrated in Figs. 5 and 6, respectively. The dotted lines represent paths in G'. Each such path may be empty. Let $c(\lambda; e) = \infty$ for each arc e in the paths represented by the dotted lines and for each $\lambda \in \mathbb{R}$. Values of $c(\lambda_1;\cdot)$ and $c(\lambda_2;\cdot)$ for other arcs in G' are as follows.

In Case 1, if the capacities of e_1 and e_2 are allowed to change in the same direction, then the values of $c(\lambda_1;\cdot)$ and $c(\lambda_2;\cdot)$ are shown in Fig. 3. If the capacities of e_1 and e_2 are allowed to change in opposite directions, then the values of $c(\lambda_1;\cdot)$ and $c(\lambda_2;\cdot)$ are shown in Fig. 4. In Cases 2 and 3, the values of $c(\lambda_1;\cdot)$ and $c(\lambda_2;\cdot)$ are illustrated, respectively, in Figs. 5 and 6. In each figure, Diagram (1) corresponds to λ_1 and Diagram (2) corresponds to λ_2. It can be seen that the direction of change in capacities of e_1 and e_2 is consistent with the one set forth in the corresponding case.

The dashed lines represent the minimum cuts. In each case, the unique minimum cuts for λ_1 and λ_2 are not ordered. This completes the proof. □

The following corollary presents a necessary and sufficient condition for the existence of a totally ordered selection of minimum cuts for every capacity function in a G^{3}T class, $\mathscr{F}(A_v, \emptyset, \emptyset)$, with no directional restrictions.

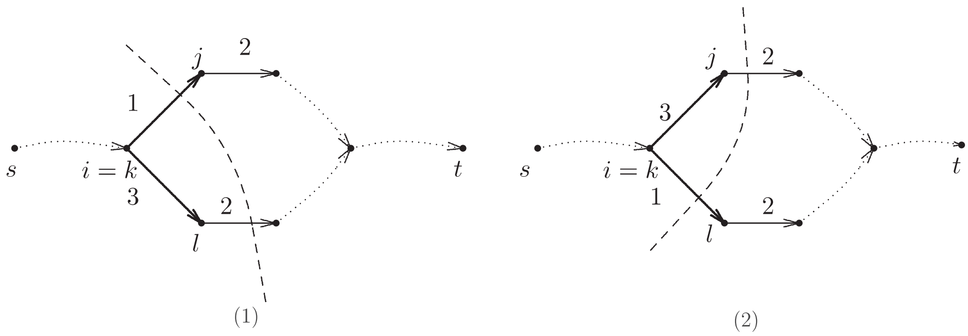

Fig. 5. G' for Case 2.

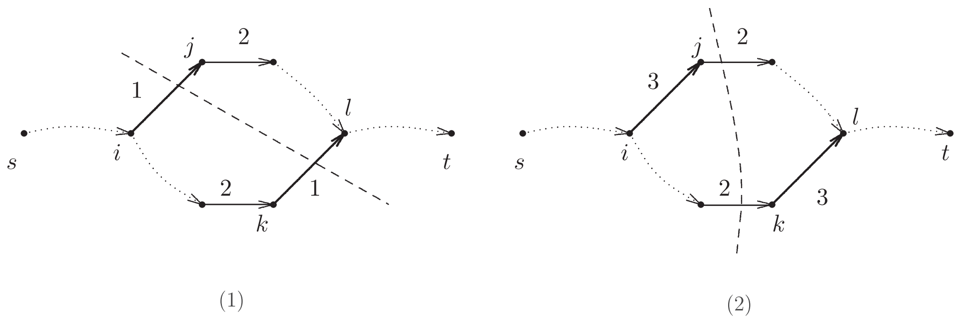

Fig. 6. G' for Case 3.

Corollary 13. *Let A_v be a set of arcs in a directed network $G(N, A)$. Then a totally ordered selection of minimum cuts exists for each parametric capacity function $c(\cdot;\cdot)$ in $\mathscr{F}(A_v, \emptyset, \emptyset)$, if and only if for each pair of parallel arcs in A_v, one arc is both a front arc and a back arc.*

Proof. For each $c(\cdot;\cdot) \in \mathscr{F}(A_v, \emptyset, \emptyset)$, only arcs in A_v can change in λ, but the direction of change is unrestricted. By Theorem 12, a totally ordered selection of minimum cuts exists for every $c(\cdot;\cdot)$ in $\mathscr{F}(A_v, \emptyset, \emptyset)$, if and only if each pair of parallel arcs in A_v satisfies at least one of items (a), (b) and (c) in Condition II. Conditions (a) and (b) require each pair of parallel arcs to be in A_v^+ or A_v^-, which are empty sets in this setting. Thus (a) and (b) cannot be satisfied and (c) must hold. This completes the proof. □

Next, we present some special cases of the class of parametric flow problems presented in Theorem 12.

Corollary 14. *Let $G(N, A)$ be a capacitated network and let v_1 and v_2 be two arbitrary nodes therein. Suppose capacities of all arcs terminating at v_1 are either all nondecreasing functions of λ or are all nonincreasing functions of λ, and similarly, capacities of all arcs originating from v_2 are either all nondecreasing functions of λ or are all nonincreasing functions of λ. Further, the capacities of the arcs on a directed path p from v_1 to v_2 are arbitrary positive functions of λ, and capacities of all other arcs are constant. Let A_v be the set of all variable arcs. If each pair of parallel arcs in A_v either both originate from v_2 or both terminate at v_1, then there exists a totally ordered selection of minimum cuts when λ changes.*

Proof. From the assumptions, each pair of parallel arcs must share the same head or the same tail, and their capacities change in the same direction. By Theorem 12, a totally ordered selection of minimum cuts exists. □

For the parametric flow problem studied by AUK (1993) [1], if the capacities of arcs incident to a single node v are all nondecreasing functions of λ, then a totally ordered selection of minimum cuts exists. This result is a special case of Corollary 14 as can be seen by identifying nodes v_1 and v_2 and letting the path p from v_1 to v_2 to be empty. For the same problem, if node v is allowed to have a capacity which changes in λ, a totally ordered selection of minimum cuts still exists. Indeed, one can transform this node-and-arc-capacitated network into an arc-capacitated network by splitting node v into nodes v_1 and v_2. All arcs terminating at v will now terminate at v_1 and all arcs originating from v will now originate from v_2. A directed arc (v_1, v_2) is added which has the capacity of node v. This arc-capacitated network is a special case of Corollary 14, since in this case the path p is the single arc (v_1, v_2).

It can be shown that in the AUK setting, a totally ordered selection of minimum cuts does not exist if we further allow capacities on nodes adjacent to v to change. For a counterexample, see [12].

In Corollary 14, if one lets v_1 be s and v_2 be t, then the set of arcs terminating at v_1 or originating from v_2 will be empty, and the following result can be obtained.

Corollary 15. *Let p be a path from the source s to the sink t in a network $G(N, A)$. Suppose no two arcs in p are parallel to each other. Then, when the capacities of arcs on p are parametrically changed, a totally ordered selection of minimum cuts exists.*

In an *s*–*t* series–parallel network, no pairs of arcs on a path from s to t are parallel. Thus by Corollary 15, when the capacities on an *s*–*t* path are arbitrarily changed in such a network, a totally ordered selection of minimum cuts exists. Liu [12] applied Corollary 15 to conduct a qualitative analysis of a parametric extended selection problem.[4] He has shown therein that when the costs of a nested sequence of facilities are parameterized, a totally ordered selection of optimal solutions exists.

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[4] See also [3] for a discussion of extended selection problems.