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# Note

## A Simple Way to Tell a Simple Polytope from Its Graph

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Let $P$ be a simple $d$-dimensional polytope and let $G(P)$ be the graph of $P$. Thus, $G(P)$ is an abstract graph defined in the set of vertices $V(P)$ of $P$. Two vertices $v$ and $u$ in $V(P)$ are adjacent in $G(P)$ if $[v, u]$ is a 1-dimensional face of $P$. Perles [P] conjectured and Blind and Mani [BM] recently proved that $G(P)$ determines the entire combinatorial structure of $P$. Here is a simple proof of this result. Let $f$ denote the number of non-empty faces of $P$.

We consider the class of acylic orientations (i.e., edge orientations with no oriented cycles) of $G(P)$. We will not distinguish between an acyclic orientation $O$ of $G(P)$ and the partial order induced by $O$ on $V(P)$. ($x \leqslant_O y$ iff there is an $O$-oriented path from $x$ to $y$.) Note that if $O$ is an acyclic orientation of $G(P)$ then the restriction of $G(P)$ to any non-empty subset $A$ of $V(P)$ has a sink (=element with out-degree zero) with respect to $O$.

An acyclic orientation $O$ of $G(P)$ is *good* if for every non-empty face $F$ of $P$, $G(F)$ has exactly *one sink*. Otherwise, $O$ is *bad*. The existence of good acyclic orientations of $G(P)$ is well known. Good acyclic orientations are obtained, e.g., by orienting the edges according to the value of a linear functional on $\mathbb{R}^d$ that is 1–1 on $V(P)$; see [B, Sect. 15]. Our first goal is to distinguish intrinsically between good and bad orientations of $G(P)$.

Let $O$ be an acyclic orientation of $G(P)$. Let $h_k^O$ be the number of vertices of $G(P)$ with indegree $k$ in $O$. Define

$$f^O = h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^k h_k^O + \cdots + 2^d h_d^O.$$

If $x$ is a vertex of $G(P)$ of indegree $k$ w.r.t. $O$ then $x$ is a sink in $2^k$ faces of $P$. (Every $i$ edges incident to $x$ determine an $i$-face $F$ of $P$ which includes them.) Since each face has at least one sink we obtain that



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(I) $f^O \geqslant f$, and

(II) $O$ is good if and only if $f^O = f$.

To distinguish between good and bad orientations from the knowledge of $G(P)$ only, compute $f^O$ for every acyclic orientation $O$. The good acyclic orientations of $G(P)$ are those having the minimal value of $f^O$.

Now we will show how to identify the faces of $P$. The criterion is very simple: An induced connected $k$-regular subgraph $H$ of $G$ is the graph of some $k$-face of $P$ if and only if its vertices are initial w.r.t. some good acyclic orientation $O$ of $G(P)$. Indeed, if $F$ is a face of $P$, it is well known that $V(F)$ is an initial set with respect to some good acyclic orientation: just consider a linear functional with respect to which the vertices of $F$ lie below all other vertices. (See [B, Sect. 18].) On the other hand, let $H$ be a connected $k$-regular subgraph of $G(P)$ and let $O$ be a good acyclic orientation with respect to which $V(H)$ is an initial set. Let $x$ be a sink of $H$ with respect to $O$. There are $k$ edges containing $x$ in $H$, all oriented towards $x$. Therefore $x$ is a sink in a $k$-face $F$ that contains these $k$ edges. Since the orientation $O$ is good, $x$ is the unique sink of $F$, and therefore all vertices of $F$ are $\leqslant x$, with respect to $O$. But $V(H)$ includes the set of all vertices that are $\leqslant x$ with respect to $O$. (Remember: $V(H)$ is an initial set with respect to $O$.) Thus, $V(F) \subset V(H)$. Since both $H$ and $G(F)$ are $k$-regular and connected, $V(F) = V(H)$ and $G(F) = H$. This completes the proof.

*Remarks.* 1. We do not have a practical way to distinguish between good and bad orientations. The algorithm suggested by the proof above is exponantial in $|V(P)|$. We do not know of an efficient way even for computing the face numbers of $P$ from $G(P)$.

2. It was observed already by Perles that the 2-skeleton of $P$ determines $P$ up to combinatorial isomorphism. His observation is based on the following fact: Let $x$ and $y$ be adjacent vertices in $G(P)$ and let $F$ be the facet of $P$ containing $x$ but not $y$. Let $z$ be a vertex adjacent to $x$, $z \neq y$. It is easy to identify the unique vertex $w$ which is adjacent to $z$ and does not belong to $F$. Let $M$ be the (unique) 2-face of $P$ containing $x$, $y$, and $z$. Then $w$ is the vertex adjacent to $z$ in $M$, different from $x$. This gives a quick way to identify the facets of $P$, hence the entire combinatorial structure of $P$, from the 2-skeleton of $P$. Perles also observed that all induced 3-gons, 4-gons, and 5-gons in $G(P)$ correspond to 2-faces of $P$.

3. Perles [P] proved that simplical $d$-polytopes are determined by their $[d/2]$-skeleton. (Dancis [D] extended this result to a large class of simplicial manifolds.) Perles also proved that simple polytopes are determined by the incidence relations between their 1-faces and 2-faces. The proof described above can be extended to show that the combinatorial structure of a simple $d$-polytope is determined by the incidence relations

between its $i$-faces and $(i+1)$-faces, whenever $i < [d/2]$. It is also possible to show that $(d-k)$-simple polytopes are determined by their $k$-skeleton. ($P$ is $(d-k)$-simple if every $(k-1)$-face is included in exactly $d-k+1$ facets.) Details will appear elsewhere. (Note that general $d$-polytopes are determined by their $(d-2)$-skeleton, and this is best possible even for quasi-simplicial polytopes, [G, Chap. 12].)

4. Perles asked whether every connected $(d-1)$-regular subgraph of $G(P)$ which does not separate $G(P)$ is the graph of a facet of $P$. This is still unknown.

5. I am thankful to Micha A. Perles and Zeev Smilansky for helpful comments.

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