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An effective method to compute closure ordering for nilpotent orbits of θ-representations

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ABSTRACT

We develop an algorithm for computing the closure of a given nilpotent G_0-orbit in $\mathfrak{g}_1$, where $\mathfrak{g}_1$ and G_0 are coming from a $\mathbb{Z}$ or a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ of a simple complex Lie algebra $\mathfrak{g}$.

1. Introduction

One of the main tasks of mathematics is to describe certain objects up to a certain equivalence relation. Often this relation is given by an algebraic group action. Then equivalence classes are orbits and orbit closures correspond to degenerations of our objects. Thus, describing orbits of algebraic actions, as well as deciding whether one orbit lies in the closure of another, is an important and interesting problem. However, this is possible only in a very few cases. One of these instances is provided by the θ-groups introduced by the second author in the seventies, see [35,36].

Let G be a connected reductive complex algebraic group and $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra. Let θ be a diagonalisable automorphism of $\mathfrak{g}$ that either defines a $\mathbb{Z}$ or a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where the grading components $\mathfrak{g}_i$ are the eigenspace of θ. Note that $\mathfrak{g}_0 = \mathfrak{g}^\theta$ is the subset of θ-stable points. Let $G_0 \subset G$ be a connected algebraic subgroup such that $\operatorname{Lie} G_0 = \mathfrak{g}_0$. If θ extends to an automorphism of G, then $G_0 = (G^\theta)^\circ$. The group G_0 is reductive and its natural action on $\mathfrak{g}_1$ is called a *θ-representation*; the group G_0, together with its action on $\mathfrak{g}_1$, is called a *θ-group*.

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An orbit $G_0 x \subset \mathfrak{g}_1$ is said to be *semisimple* if it is closed, and *nilpotent* if its closure $\overline{G_0 x}$ contains 0. This is the case if and only if x is semisimple (respectively nilpotent) as an element of $\mathfrak{g}$. The elements of $\mathfrak{g}_1$ inherit the Jordan decomposition $x = s + n$ from $\mathfrak{g}$. Besides, G_0-orbits $G_0(s+n)$ with the semisimple part s being fixed up to conjugation are classified by the nilpotent orbits of the θ-group coming from the pair $(\mathfrak{g}_s, \theta|_{\mathfrak{g}_s})$, where $\mathfrak{g}_s \subset \mathfrak{g}$ is the centraliser of s (and it is a reductive Lie algebra) and $\theta|_{\mathfrak{g}_s}$ is the restriction of θ to $\mathfrak{g}_s$. This indicates that nilpotent orbits are especially interesting. The θ-groups have several remarkable properties, one of them is that there are only finitely many nilpotent G_0-orbits in $\mathfrak{g}_1$ and there is a method to classify them [36].

From now on suppose that $\mathfrak{g}$ is simple. We will say that a θ-group is exceptional (respectively classical), if $\mathfrak{g}$ is exceptional (respectively classical). The classical case allows a more or less uniform treatment, since here everything is determined by the canonical embedding into an appropriate $\mathfrak{gl}_n$, see e.g. [35]. For inner automorphisms of $\mathfrak{gl}_n$, the nilpotent orbits as well as their closures are described by Kempken [25]. The complete answer, including the closure ordering for all classical types and all automorphisms, is not known, but there does not seem to be any profound difficulty in getting it, see e.g. [28].

More interesting representations arise in the context of exceptional θ-groups. Here several orbit classifications were carried out along the lines of [36]. To mention a few [39,1,2,16]. In these papers, all orbits, not only the nilpotent ones, were described. More recently, Pervushin treated one θ-group in type E_7 [29], he also got the closure diagram of the nilpotent orbits [30].

Despite the possibility to treat each particular exceptional θ-group by hand, the "classical" uniformity is lost and one faces a long list of different examples. Dealing with all of them by hand is at least difficult. Several computer algorithms for classifying nilpotent G_0-orbits in $\mathfrak{g}_1$ have been developed, see [26] and [20]. In this paper, we give a method how to check whether a nilpotent orbit $G_0 x$ lies in the closure $\overline{G_0 y}$ of another nilpotent orbit $G_0 y$.

Each nilpotent element $e \in \mathfrak{g}_1$ can be included into an $\mathfrak{sl}_2$-triple (e, h, f) with $h \in \mathfrak{g}_0$. Our method relies on the fact that this h is also a characteristic of e in the sense of Kempf and Hesselink, i.e., it gives rise to a one-dimensional torus in G_0 that takes e to zero fastest. Another important ingredient is that $G_0 e$ coincides with a Hesselink stratum, the set of all elements in $\mathfrak{g}_1$ having h as a Hesselink characteristic, see [38, Section 5]. Therefore $\overline{G_0 e} = G_0(V_{\geqslant 2}(h))$, where $V_{\geqslant 2}(h)$ is the linear span of all vectors $v \in \mathfrak{g}_1$ such that $[h, v] = kv$ with $k \geqslant 2$.

An orbit $G_0 e'$ lies in $\overline{G_0 e}$ if and only if its intersection with $V_{\geqslant 2}(h)$ is non-empty. When examining $G_0 e' \cap V_{\geqslant 2}(h)$, we replace G_0 by the union of its Bruhat cells. Further, let h' be a characteristic of e' and W_0 the Weyl group of G_0. Then Proposition 3.1 assures that $G_0 e'$ is contained in $\overline{G_0 e}$ if and only if there is $w \in W_0$ such that $U(w) = V_2(h') \cap V_{\geqslant 2}(wh)$ contains a point of $G_0 e'$ (here $V_2(h')$ is the set of all vectors $v \in \mathfrak{g}_1$ such that $[h', v] = 2v$). If this is indeed the case, then $U(w) \cap G_0 e'$ is an open dense subset of $U(w)$ and by taking a random $u \in U(w)$ we can find an element of $G_0 e'$ with probability almost one. In order to prove that the intersection in question is empty, we compute the dimension of a maximal $Z(h')$-orbit intersecting $U(w)$ for the centraliser $Z(h') \subset G_0$ of h'. Recall that $\dim Z(h')v < \dim Z(h')e'$ for all elements v in $V_2(h') \backslash G_0 e'$ (see Lemma 2.5). To loop over an orbits of the Weyl group, we use its parametrisation as a tree with edges given by simple reflections (Section 4). Other tools are described in Sections 2.1, 5, and 6. In particular, to prove a non-inclusion $G_0 e' \not\subset \overline{G_0 e}$ for some orbits, we use Theorem 2.9, which is a general statement on $\mathbb{Z}$-graded reductive Lie algebras and is interesting in itself. It already appeared in the literature and was proved by Kac in a particular case [24], see Remark 2.10 for a detailed discussion.

First examples of θ-groups are provided by the simple Lie algebras themselves, i.e., in the case where the automorphism is the identity. Then one asks for the Hasse (closure) diagram of the nilpotent orbits in $\mathfrak{g}$. The two most difficult, largest exceptional Lie algebras, of types E_7 and E_8, were treated by Mizuno [27]. Later his results were verified and corrected by Beynon and Spaltenstein [4]. The implementation of our method in GAP also works for $\mathfrak{g}$. We have computed the Hasse diagrams for the Lie algebras of exceptional type, and obtained the same diagrams as in Spaltenstein's book [33].

The same problem for real exceptional Lie algebras has been studied by Djoković in a series of papers [8–15]. If $\mathfrak{g}_{\mathbb{R}}$ is a non-compact real form of $\mathfrak{g}$ and $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{R}}$ is the Lie algebra of a maximal compact subgroup in $G_{\mathbb{R}}$, then the complexification $\mathfrak{k}(\mathbb{C})$ of $\mathfrak{k}$ is a symmetric subalgebra, i.e., $\mathfrak{k}(\mathbb{C}) = \mathfrak{g}^{\theta}$ for θ of order two. The Kostant–Sekiguchi correspondence (see e.g. [6, §9.5]) establishes a bijection between nilpotent $G_{\mathbb{R}}$ orbits in $\mathfrak{g}_{\mathbb{R}}$ and nilpotent G_0-orbits in $\mathfrak{g}_1$. Moreover, according to [3], this bijection preserves the closure ordering. For each automorphism of order 2 of each exceptional complex Lie algebra, Djoković gives the closure diagram for the nilpotent orbits. With the implementation of our method in GAP we have also computed these diagrams. The results of our computations were the same as those of Djoković, except in one case in type E_8. The difference is described in Section 8.1.

The finite order automorphisms of $\mathfrak{g}$ have been classified by Kac [23], up to conjugacy. A conjugacy class of automorphisms is identified by its Kac diagram. Here we briefly indicate how this works for inner automorphisms, for more information we refer to [37, Chapter 3, §3] and [22, Chapter X]. Let Φ be the root system of $\mathfrak{g}$ with a basis $\{\alpha_1, \dots, \alpha_l\}$. Let α_0 denote the lowest root of Φ. The Dynkin diagram of the roots $\alpha_0, \alpha_1, \dots, \alpha_l$ is the extended Dynkin diagram of Φ (or of $\mathfrak{g}$). Let $n_i \in \mathbb{N}$ be such that $\alpha_0 = -\sum_{i=0}^{l} n_i \alpha_i$ and set $n_0 = 1$. Take $l+1$ non-negative integers $s_0, \dots, s_l$ with $\gcd(s_0, \dots, s_l) = 1$ and set $m = \sum_{i=0}^{l} n_i s_i$. Let $\omega \in \mathbb{C}$ be a primitive m-th root of unity. Then a linear map $\theta : \mathfrak{g} \to \mathfrak{g}$ that multiplies vectors in the root space $\mathfrak{g}_{\alpha_i}$ $(0 \leqslant i \leqslant l)$ by ω^{s_i} uniquely defines an automorphism of $\mathfrak{g}$ of order m. The Kac diagram of this automorphism (or, more precisely, of its conjugacy class) is the extended Dynkin diagram with labels $s_0, \dots, s_l$. The automorphisms that will appear in the examples in this paper all have the labels s_i equal to 0 or 1. We will give the Kac diagram of such an automorphism by colouring the nodes of the extended Dynkin diagram: a black node means that the corresponding label is 1, otherwise it is 0.

There is also an easy way to read the θ-representation from the Kac diagram of an inner θ. The group G_0 contains a maximal torus of G and the semisimple part of $\mathfrak{g}_0$ is generated by all root spaces $\mathfrak{g}_{\alpha_i}$ $(0 \leqslant i \leqslant l)$ with $s_i = 0$. The lowest weights of $\mathfrak{g}_1$ (with respect to G_0) are in one-to-one correspondence with the roots labelled with 1.

There are two instances of θ-groups, one in E_7 and one in E_8, where G_0-orbits correspond to isomorphisms classes of two-step nilpotent (or metabelian) Lie algebras $\mathfrak{n}$ such that $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ is the centre of $\mathfrak{n}$ and either $\dim(\mathfrak{n}/\mathfrak{n}') \leqslant 6$, $\dim \mathfrak{n}' \leqslant 3$; or $\dim(\mathfrak{n}/\mathfrak{n}') \leqslant 5$, $\dim \mathfrak{n}' \leqslant 5$, see Section 8.3 and [16]. The nilpotent orbits correspond to those Lie algebras, whose structure tensor can be contracted to zero by a unimodular change of coordinates. Here taking closure of a nilpotent orbit can be interpreted as the degeneration of the encoded Lie algebra. The Lie algebra structures on a given vector space form an affine algebraic variety and some of its properties depend on the degenerations, see e.g. [34]. In Appendix A we present the Hasse diagrams for the nilpotent orbits of both these θ-representations.

We have also computed the closures of the nilpotent orbits of $\mathrm{SL}_9(\mathbb{C})$ in $\bigwedge^3(\mathbb{C}^9)$, see Figs. 1, 2. This is a θ-representation treated in [39].

Section 7 contains a few further observations on algebraic actions. We briefly discuss difficulties arising in developing a practical algorithm for describing the closure (Section 7.3); outline possible modifications in our algorithm; and present a parametrisation for a set of the double cosets of a Weyl group (Section 7.2), which appeared as a byproduct of our constructions.

2. Preliminaries

In this section we present some results, mainly taken from [38], on which our method is based.

Throughout we let $\mathfrak{h}_0$ be a fixed Cartan subalgebra of $\mathfrak{g}_0$. The Weyl group of the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$ will be denoted W_0. We have $W_0 \cong N_{G_0}(\mathfrak{h}_0)/Z_{G_0}(\mathfrak{h}_0)$. Hence every $w \in W_0$ can be lifted to $g \in G_0$ such that $g|_{\mathfrak{h}_0} = w$. Usually we will denote these two elements by the same symbol. The group G is assumed to be simple unless explicitly stated to the contrary.

We say that an $\mathfrak{sl}_2$-triple (h, e, f) is *homogeneous* if $e \in \mathfrak{g}_1$, $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$.

Let us recall a few useful facts (see [35,36]):

(1) For a nilpotent element $e \in \mathfrak{g}_1$ there exist $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$ such that (h, e, f) is an $\mathfrak{sl}_2$-triple. The element h is called a (*Dynkin*) *characteristic* of e.
(2) Let (h', e', f'), (h, e, f) be two homogeneous $\mathfrak{sl}_2$-triples. Then e' and e are G_0-conjugate if and only if (h', e', f'), (h, e, f) are G_0-conjugate, if and only if h' and h are G_0-conjugate.

Thereby a nilpotent orbit $G_0 e$ corresponds to a unique G_0-conjugacy class of homogeneous $\mathfrak{sl}_2$-triples (h, e, f). Also, we may assume that h lies in $\mathfrak{h}_0$. Furthermore, after possibly replacing h by a

W_0-conjugate, we may assume that h lies in a fixed Weyl chamber C_0 of $\mathfrak{h}_0$. Then h is uniquely determined by the orbit G_0e.

Throughout we will write V for the space $\mathfrak{g}_1$. Then for $h \in \mathfrak{h}_0$ we set

$$V_k(h) = \{v \in V \mid [h, v] = kv\}, \qquad V_{\geqslant k}(h) = \bigoplus_{l \geqslant k} V_l(h).$$

Also we consider the parabolic subalgebra $\mathfrak{p}(h) \subset \mathfrak{g}_0$, which is the sum of the eigenspaces of h with non-negative eigenvalues. Let $P(h)$ denote the connected subgroup of G_0 with Lie algebra $\mathfrak{p}(h)$. We let $\mathfrak{z}(h)$ be the centraliser of h in $\mathfrak{g}_0$. Let $\tilde{\mathfrak{z}}(h)$ denote the orthogonal complement of h in $\mathfrak{z}(h)$, with respect to the Killing form of $\mathfrak{g}$. Let $Z(h)$ and $\tilde{Z}(h)$ be connected subgroups of G_0 with Lie algebras $\mathfrak{z}(h)$ and $\tilde{\mathfrak{z}}(h)$, respectively.

Now we will borrow two theorems from [38, Section 5].

Theorem 2.1. *(See [38, Theorem 5.4].) Let $e \in \mathfrak{g}_1$ be nilpotent and non-zero. Let $h \in \mathfrak{h}_0$ be such that $e \in V_{\geqslant 2}(h)$. Then h is a characteristic of e if and only if the projection of e on $V_2(h)$ is not a nilpotent element with respect to the action of the group $\tilde{Z}(h)$.*

Remark 2.2. In [38, Section 5], the term "characteristic" is used in a different sense, it is not necessarily a Dynkin characteristic. However, following the lines of Example 3 in [38, Section 5.5], one can show that the orbit $\tilde{Z}(h)e$ is closed in V, if h is a Dynkin characteristic of e. Therefore a Dynking characteristic of e is also a characteristic in the sense of Theorem 2.1.

The next theorem is the second part of [38, Theorem 5.6] and Corollary 2.4 is an immediate consequence of Theorem 2.3.

Theorem 2.3. *Let $\mathcal{O} = G_0e$ be a nilpotent orbit in V; and let $h \in \mathfrak{g}_0$ be a characteristic of e. Then $\overline{\mathcal{O}} = G_0(V_{\geqslant 2}(h))$.*

Corollary 2.4. *Let $\mathcal{O}' = G_0e'$, $\mathcal{O} = G_0e$ be two nilpotent orbits in V. Let h', h be Dynkin characteristics of e', e, respectively. Then $\mathcal{O}' \subset \overline{\mathcal{O}}$ if and only if $V_{\geqslant 2}(h)$ contains a point of $\mathcal{O}'$.*

We use the notation $\mathfrak{g}_{i,x}$ for the intersection of $\mathfrak{g}_i$ and the centraliser $\mathfrak{g}_x \subset \mathfrak{g}$ of $x \in \mathfrak{g}$. Next we have two lemmas that we will use in the sequel. The first one is an immediate consequence of Theorem 2.3.

Lemma 2.5. *Let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple. Then $Z(h)e$ is dense in $V_2(h)$.*

Lemma 2.6. *Let (h, e, f) be a homogeneous $\mathfrak{sl}_2$-triple, and $\mathcal{O} = G_0e$. Then h is a (Dynkin) characteristic of all elements of $\mathcal{O} \cap V_2(h)$.*

Proof. In fact, we are going to prove that $Y = \mathcal{O} \cap V_2(h)$ is a single $Z(h)$-orbit, i.e., this intersection is equal to $Z(h)e$. Let y be an element of Y. Let $\mathfrak{f}_i$ be the eigenspace of h in $\mathfrak{g}_0$ with eigenvalue i and let $\mathfrak{f}_{i,y} = \mathfrak{f}_i \cap \mathfrak{g}_{0,y}$ be the centraliser of y in $\mathfrak{f}_i$. Since $Z(h)e$ is dense in $V_2(h)$, the element y lies in its closure. In particular, taking the limits one sees that $\dim \mathfrak{f}_{i,y} \geqslant \dim \mathfrak{f}_{i,e}$ for all i. On the other hand $\dim \mathfrak{g}_{0,e} = \dim \mathfrak{g}_{0,y}$, since these are the elements of the same G_0-orbit. Taking into account that $\mathfrak{g}_{0,y} = \bigoplus \mathfrak{f}_{i,y}$ and that $\mathfrak{f}_{0,y} = \mathfrak{z}(h)_y$, we conclude that the $Z(h)$-orbits of e and y have the same dimension. Since $Z(h)e$ is the unique $Z(h)$-orbit of the maximal dimension, $y \in Z(h)e$.

Now the statement about characteristics is obvious. □

2.1. *Reduced θ-groups*

We conclude Section 2 with a few statements concerning θ-groups appearing from $\mathbb{Z}$-gradings. In this part of the paper, G is an arbitrary (not necessarily simple) reductive group. A $\mathbb{Z}$-grading of $\mathfrak{g} = \operatorname{Lie} G$ is defined by a diagonalisable one-parameter subgroup of $\operatorname{Aut}(\mathfrak{g})$ and therefore by the eigenvalues of some $h \in \mathfrak{g}$, i.e., $\mathfrak{g}_s = \{\xi \in \mathfrak{g} \mid [h, \xi] = s\xi\}$, see e.g. [37, Chapter 3, Section 3.3]. Without loss of generality we may assume that $h \in [\mathfrak{g}, \mathfrak{g}]$. Here all elements of $\mathfrak{g}_1$ are nilpotent and therefore there is a dense open G_0-orbit in $\mathfrak{g}_1$.

Let $\rho : G \to \mathrm{GL}(W)$ be a faithful linear representation of G on a finite-dimensional vector space W. We use the same letter ρ for its differential $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ and define a non-degenerate G-invariant symmetric scalar product $(\, ,)$ on $\mathfrak{g}$ by setting $(x, y) := \operatorname{tr}(\rho(x)\rho(y))$ for $x, y \in \mathfrak{g}$. Note that the restriction of $(\, ,)$ to each non-abelian simple factor of $\mathfrak{g}$ is the Killing form multiplied by a positive rational number. One of the benefits of this choice is that $(h, h) > 0$, whenever $h \neq 0$, and this is assumed to be the case. More generally, $(s, s) > 0$ for all non-zero $s \in [\mathfrak{g}, \mathfrak{g}]$ that have rational eigenvalues on $\mathfrak{g}$.

Let $\tilde{\mathfrak{g}}_0 \subset \mathfrak{g}_0$ be the orthogonal complement of h with respect to $(\, ,)$ and $\tilde{G}_0 \subset G_0$ a connected algebraic group with $\operatorname{Lie} \tilde{G}_0 = \tilde{\mathfrak{g}}_0$. Then the action of $\tilde{G}_0$ on $\mathfrak{g}_1$ is said to be a *reduced* θ-representation and $\tilde{G}_0$ a *reduced* θ-group. Note that $G_0 = \tilde{G}_0(\exp(\mathbb{C}h))$.

Lemma 2.7. *Let $x \in \mathfrak{g}_1$. Then $\tilde{G}_0 x = G_0 x$ if and only if $[\tilde{\mathfrak{g}}_0, x] = [\mathfrak{g}_0, x]$, and the equality takes place if and only if the orbit $\tilde{G}_0 x$ is conical.*

Proof. If $\tilde{G}_0 x = G_0 x$, then clearly $[\tilde{\mathfrak{g}}_0, x] = [\mathfrak{g}_0, x]$. Other way around, the equality of tangent spaces implies that $\dim G_0 x = \dim \tilde{G}_0 x$. Since $\tilde{G}_0$ is a normal subgroup of G_0, the same holds for all elements in $G_0 x$ and the two orbits coincide. Finally, being conical means that $\mathbb{C}^\times x \subset \tilde{G}_0 x$, or, equivalently, $\mathbb{C}x \subset [\tilde{\mathfrak{g}}_0, x]$. Therefore $\tilde{G}_0 x$ is a conical orbit if and only if there is the equality of orbits or their tangent spaces. □

Lemma 2.8. *Suppose that $2h$ is a Dynkin characteristic of $x \in \mathfrak{g}_1$. Then $[\mathfrak{g}_0, x] = \mathfrak{g}_1$, but $[\tilde{\mathfrak{g}}_0, x] \neq \mathfrak{g}_1$.*

Proof. If $[\mathfrak{g}_0, x] \neq \mathfrak{g}_1$, then there is $v \in \mathfrak{g}_{-1}$ such that $([\mathfrak{g}_0, x], v) = 0$ and also $(\mathfrak{g}_0, [x, v])$, where $[x, v] \in \mathfrak{g}_0$. Since the scalar product is non-degenerate on $\mathfrak{g}_0$, we obtain $[x, v] = 0$, which contradicts the $\mathfrak{sl}_2$-theory.

There is an element $y \in \mathfrak{g}_{-1}$ such that y, $2h$, and x form an $\mathfrak{sl}_2$-triple. For this y we have $(y, [\tilde{\mathfrak{g}}_0, x]) = 0$, because $(2h, \tilde{\mathfrak{g}}_0) = 0$. The inequality follows. □

Theorem 2.9. *Let G be an arbitrary reductive group and the objects $\mathfrak{g}_1$, G_0, $\tilde{G}_0$ as above. Suppose that $x \in \mathfrak{g}_1$. Then $\tilde{G}_0 x \neq G_0 x$ if and only if $2h$ is a Dynkin characteristic of x.*

Proof. Let $\hat{h} \in \mathfrak{g}_0$ be a Dynkin characteristic of x. We can write it as $\hat{h} = ah + h_0$ with $a \in \mathbb{C}$ and $h_0 \in \tilde{\mathfrak{g}}_0$. Since $[\hat{h}, x] = 2x = ax + [h_0, x]$, either $\mathbb{C}x \subset [\tilde{\mathfrak{g}}_0, x]$ or $a = 2$ and $[h_0, x] = 0$. In the latter case $(\hat{h}, h_0) = 0$. Taking into account the equality $(h, h_0) = 0$, we get that $(h_0, h_0) = 0$. Since $\hat{h}$ is a Dynkin characteristic, it lies in $[\mathfrak{g}, \mathfrak{g}]$. Hence $h_0 \in [\mathfrak{g}, \mathfrak{g}]$, because h also does. Moreover, eigenvalues of $\operatorname{ad}(h)$ are integers by the construction, and the same holds for $\operatorname{ad}(\hat{h})$, because it comes from an $\mathfrak{sl}_2$-triple. Since $[h, \hat{h}] = 0$, the eigenvalues of $\operatorname{ad}(h_0)$ are integers as well. According to our choice of the scalar product, the equality $(h_0, h_0) = 0$ is possible only if $h_0 = 0$. One concludes that $2h$ is a Dynkin characteristic of x.

We have shown that if $2h$ is not a Dynkin characteristic of x, then $[h_0, x] = bx$ with $b \in \mathbb{C}^\times$, in particular, $\tilde{G}_0 x$ is a conical orbit. By Lemma 2.7, $G_0 x = \tilde{G}_0 x$.

If $2h$ is a Dynkin characteristic of x, then $[\tilde{\mathfrak{g}}_0, x] \neq [\mathfrak{g}_0, x]$ by Lemma 2.8 and therefore $G_0 x \neq \tilde{G}_0 x$. □

Remark 2.10. In case $\mathfrak{g}$ is simple and the representation of G_0 on $\mathfrak{g}_1$ is irreducible, Theorem 2.9 was proved by V.G. Kac, see [24, Proposition 3.2]. It is also mentioned without a proof in [38, Section 8.5] that the statement holds for an arbitrary reduced θ-group. Since we could not find a general case proof in the literature, we decided to include it here.

3. Criteria for inclusion

In this section we state and prove the main criterion (Proposition 3.1) that we use for deciding whether a given nilpotent orbit is contained in the closure of another given nilpotent orbit. This reduces the problem of checking inclusion to a finite number of checks, each corresponding to an element of a certain orbit of the Weyl group W_0. Subsequently we give some observations that help when using the criterion.

Proposition 3.1. *Let the notation be as in Corollary 2.4. Then $\mathcal{O}' \subset \overline{\mathcal{O}}$ if and only if there is a $w \in W_0$ such that $U = V_2(h') \cap V_{\geqslant 2}(wh)$ contains a point of $\mathcal{O}'$. Moreover, in that case the intersection of U and $\mathcal{O}'$ is dense in U.*

Proof. The "if" part follows directly from Theorem 2.3. Therefore suppose that $\mathcal{O}' \subset \overline{\mathcal{O}}$.

By the Bruhat decomposition we have that

$$G_0 = \bigcup_{w \in W_0} P\big(h'\big) w P(h).$$

By Theorem 2.3,

$$\begin{aligned}\overline{\mathcal{O}} &= G_0\big(V_{\geqslant 2}(h)\big) \\ &= \bigcup_{w \in W_0} P\big(h'\big) w P(h)\big(V_{\geqslant 2}(h)\big) \\ &= \bigcup_{w \in W_0} P\big(h'\big) w\big(V_{\geqslant 2}(h)\big).\end{aligned}$$

Let (h', e', f') be a homogeneous $\mathfrak{sl}_2$-triple. Then it follows from the above that there exist $w \in W_0$, $p \in P(h')$, and $x \in V_{\geqslant 2}(h)$ with $e' = pwx$, or, equivalently, $p^{-1}e' = wx$.

Next $P(h') = Z(h') \ltimes N$, where N is a connected subgroup of G_0 whose Lie algebra is the sum of the eigenspaces of h' in $\mathfrak{g}_0$ with positive eigenvalues. In particular, $p^{-1} = ln$ with $l \in Z(h')$ and $n \in N$. Since $e' \in V_2(h')$, we have $ne' = e' + y$, where $y \in V_{\geqslant 3}(h')$. Now $p^{-1}e' = le' + ly$, with $le' \in V_2(h')$ and $ly \in V_{\geqslant 3}(h')$. In particular, $p^{-1}e'$ lies in $V_{\geqslant 2}(h')$. Since $p^{-1}e' = wx$ and $wx \in V_{\geqslant 2}(wh)$, it also lies in $\tilde{U} = V_{\geqslant 2}(h') \cap V_{\geqslant 2}(wh)$.

The elements h' and wh commute and thereby $\tilde{U}$ is stable under the action of h'. That is, $\tilde{U}$ is the direct sum of h'-eigenspaces. It follows that $\tilde{U}$ contains le', which is obviously an element of $\mathcal{O}'$. Moreover, $le' \in V_2(h')$ and hence $le' \in U$, where $U = V_2(h') \cap V_{\geqslant 2}(wh)$.

By Theorem 2.1, an element $v \in U$ lies in $\mathcal{O}'$ if and only if it is not nilpotent with respect to the action of $\tilde{Z}(h')$. Therefore if the intersection of U and $\mathcal{O}'$ is not empty, then it has to be open and dense. $\square$

Proposition 3.2. *Let (h', e', f'), (h, e, f) be homogeneous $\mathfrak{sl}_2$-triples, with $e' \in \mathcal{O}'$, $e \in \mathcal{O}$. Let κ denote the Killing form of $\mathfrak{g}$. If $\kappa(h', h) < \kappa(h', h')$ then $V_2(h') \cap V_{\geqslant 2}(h)$ contains no points of $\mathcal{O}'$.*

Proof. Note that $h \in \mathfrak{z}(h')$, hence $h = ah' + t$, where $a \in \mathbb{C}$ and $t \in \tilde{\mathfrak{z}}(h')$. Moreover,

$$a = \frac{\kappa(h', h)}{\kappa(h', h')},$$

which is in $\mathbb{Q}$ and < 1. Hence t has only positive eigenvalues on $V_2(h') \cap V_{\geqslant 2}(h)$. Let T be the connected subgroup of G_0 whose Lie algebra is spanned by t. Then all elements of $V_2(h') \cap V_{\geqslant 2}(h)$

are nilpotent with respect to T, and in particular with respect to $\tilde{Z}(h')$. Hence by Theorem 2.1 and Lemma 2.6, the former space contains no points of $\mathcal{O}'$. □

Let $\mathfrak{l}$ be a Lie algebra acting on a vector space M. Then for $v \in M$ we denote its stabiliser by $\mathfrak{l}_v$, i.e.,

$$\mathfrak{l}_v = \{x \in \mathfrak{l} \mid x \cdot v = 0\}.$$

The set of $v \in M$ with $\dim \mathfrak{l}_v$ minimal is open and dense in M.

Proposition 3.3. *Let (h', e', f'), (h, e, f) be homogeneous $\mathfrak{sl}_2$-triples, with $e' \in \mathcal{O}'$, $e \in \mathcal{O}$. Let $d = \dim \mathfrak{z}(h')_{e'}$. Let d' be the minimal dimension of $\mathfrak{z}(h')_u$, for $u \in V_2(h') \cap V_{\geqslant 2}(h)$. Then $d \leqslant d'$. Moreover, $V_2(h') \cap V_{\geqslant 2}(h)$ contains a point of $\mathcal{O}'$ if and only if $d = d'$.*

Proof. By Lemma 2.5, the stabiliser of e' in $\mathfrak{z}(h')$ has minimal possible dimension.

Furthermore, if $d = d'$ then there is $u \in V_2(h') \cap V_{\geqslant 2}(h)$ such that $\dim \mathfrak{z}(h')_u = \dim \mathfrak{z}(h')_{e'}$. Hence the dimension of the $Z(h')$-orbit of u is the same as the dimension of $Z(h')e'$. So $Z(h')u$ is dense in $V_2(h')$ as well. The conclusion is that $Z(h')e' = Z(h')u$, and u lies in $\mathcal{O}'$. □

Proposition 3.4. *Let (h', e', f'), (h, e, f) be homogeneous $\mathfrak{sl}_2$-triples, with $e \in \mathcal{O}$, $e' \in \mathcal{O}'$. Set $U = V_2(h') \cap V_{\geqslant 2}(h)$. Let $\mathfrak{n} = N_{\mathfrak{g}_0}(U) = \{x \in \mathfrak{g}_0 \mid [x, U] \subset U\}$. Let $u \in U$; if $[\mathfrak{n}, u] = U$, and $u \notin \mathcal{O}'$, then U has no points of $\mathcal{O}'$.*

Proof. Indeed, if U has a point of $\mathcal{O}'$, then the intersection of $\mathcal{O}'$ and U is dense in U. But also the $N_{G_0}(U)$-orbit of u is dense in U. So the two sets must intersect, which is not possible. □

4. Orbits of the Weyl group

In our algorithm we need to loop over an orbit $W_0 h$, where $h \in \mathfrak{h}_0$. In this section we briefly describe how this is done. For simplicity we assume that the centre of $\mathfrak{g}_0$ is zero. If this is not the case then $\mathfrak{g}_0$ has to be replaced by its derived subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$, and $\mathfrak{h}_0$ by its intersection with $[\mathfrak{g}_0, \mathfrak{g}_0]$.

We let κ denote the Killing form of $\mathfrak{g}$. Since it is non-degenerate on $\mathfrak{h}_0$ it gives an isomorphism $\mathfrak{h}_0^* \to \mathfrak{h}_0$, $\alpha \mapsto \hat{\alpha}$. This yields an inner product on $\mathfrak{h}_0^*$ by $(\alpha, \beta) = \kappa(\hat{\alpha}, \hat{\beta})$.

Let Φ_0 be the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$. Let $\Delta_0 = \{\alpha_1, \dots, \alpha_l\}$ be a fixed basis of Φ_0. The corresponding set of positive roots will be denoted Φ_0^+.

For $\alpha \in \Phi_0$ we set

$$\alpha^\vee = \frac{2\hat{\alpha}}{(\alpha, \alpha)} \in \mathfrak{h}_0.$$

The Weyl group W_0 is generated by the simple reflections $s_i = s_{\alpha_i}$. For $h \in \mathfrak{h}_0$ we have $s_i(h) = h - \alpha_i(h)\alpha_i^\vee$.

We use a basis $h_1, \dots, h_l$ of $\mathfrak{h}_0$, defined by $\alpha_i(h_j) = \delta_{ij}$. Then, if $h = \sum_i a_i h_i$, we get $s_j(h) = h - a_j \alpha_j^\vee$. The elements h of which we compute the W_0-orbit, lie in an $\mathfrak{sl}_2$-triple. This implies that the coefficients of h with respect to this basis are integers. The dominant Weyl chamber C_0 consists of the elements of $\mathfrak{h}_0$ having non-negative coefficients with respect to the basis $h_1, \dots, h_l$.

Now let $h \in \mathfrak{h}_0$ be the element of which we want to compute the orbit $W_0 h$. Since every orbit of W_0 has a unique point in C_0, we may assume that $h \in C_0$. Let $\hat{h} \in W_0 h$, then we define the length of $\hat{h}$, denoted $\ell(\hat{h})$, as the length of a shortest $w \in W_0$ with $\hat{h} = wh$. Then

$$\ell(\hat{h}) = \left|\left\{\alpha \in \Phi_0^+ \;\middle|\; \alpha(\hat{h}) < 0\right\}\right|.$$

This implies that $\ell(s_i\hat{h}) = \ell(\hat{h}) + 1$ if and only if $a_i > 0$, where $\hat{h} = \sum_i a_i h_i$. We use a criterion due to Snow [32]:

Lemma 4.1. *Let $\tilde{h} = \sum_i a_i h_i$ be an element of $W_0 h$ of length $k+1$. Then there is a unique $\hat{h}$ of length k in $W_0 h$ such that*

- *there is a simple reflection s_i with $s_i(\hat{h}) = \tilde{h}$,*
- *$a_j \geqslant 0$ for $i < j \leqslant l$.*

Let $\tilde{h}$, $\hat{h}$ be as in the previous lemma. Then we say that $\hat{h}$ is the *predecessor* of $\tilde{h}$, and conversely, that $\tilde{h}$ is a *successor* of $\hat{h}$. Let $\hat{h} = \sum_i b_i h_i$ be a given element of $W_0 h$ of length k. Then it is straightforward to determine its successors. Indeed, let i be such that $b_i > 0$, and write $s_i(\hat{h}) = \sum_j a_j h_j$. Then this element is of length $k+1$, and it is a successor of $\hat{h}$ if and only if $a_j \geqslant 0$ for $i < j \leqslant l$.

This means that we can define a tree: the nodes are the elements of $W_0 h$, and there is an edge from $\hat{h}$ to $\tilde{h}$ if and only if $\tilde{h}$ is a successor of $\hat{h}$. By traversing this tree, we can efficiently loop over $W_0 h$. Every element of $W_0 h$ comes at the cost of applying one reflection. Moreover, we do not obtain the same element of $W_0 h$ twice.

Remark 4.2. We finish this section with an observation that will be used later. Let h' be an element of C_0. Let $\hat{h} \in W_0 h$ be of length k and suppose that $s_j\hat{h}$ is of length $k+1$. Write $\hat{h} = \sum_i a_i h_i$; then, as seen above, $a_j > 0$. Hence

$$\kappa\left(s_j\hat{h}, h'\right) = \kappa\left(\hat{h}, h'\right) - a_j \kappa\left(\alpha_j^\vee, h'\right) \leqslant \kappa\left(\hat{h}, h'\right).$$

Furthermore, equality happens if and only if $\alpha_j(h') = 0$, which is equivalent to s_j lying in the stabiliser of h'.

5. Complement of the dense orbit

According to Proposition 3.1, we will have to check whether a subspace $U \subset V_2(h)$ contains a point of the dense orbit $Z(h)e$. If U contains a point of $Z(h)e$ then the intersection of U and $Z(h)e$ is dense in U (Proposition 3.1). So in that case, by trying random elements of U, we quickly find a vector $u \in U$ lying in $Z(h)e$; thus proving that the intersection is non-empty. The most difficult part of the problem is to prove that U contains no points of $Z(h)e$. Here we present two possible solutions.

Let $v_1, \ldots, v_s$ and $x_1, \ldots, x_n$ be bases of $V_2(h)$ and $\mathfrak{z}(h)$ respectively. Let also $w_1, \ldots, w_s$ (with $w_i \in V_2(h)^*$) be the dual basis. Let B denote the action matrix for the representation of $\mathfrak{z}(h)$ on $V_2(h)^*$. To be more explicit, the entries of B are elements of $V_2(h)^*$, $b_{ij} = x_i \cdot w_j$. For $v \in V$, let B_v denote the restriction of B to v. The entries of this new matrix are $[x_i \cdot w_j](v) = w_j([v, x_i])$. In the same spirit, we can define the restriction of B to U, B_U, to be a matrix with entries in U^*. The rank of B_U is calculated over the field $\mathbb{C}(U)$ (note that $U^* \subset \mathbb{C}(U)$).

Using the fact that $[\xi, v] = 0$ (with $\xi \in \mathfrak{z}(h)$) if and only if $w_i([\xi, v]) = 0$ for all i, one can easily deduce that

$$\begin{aligned}
&\text{(i)} \quad \dim \mathfrak{z}(h)_v = n - \operatorname{rank} B_v \quad \text{for all } v \in V_2(h);\\
&\text{(ii)} \quad \dim Z(h)v = \operatorname{rank} B_v;\\
&\text{(iii)} \quad \max_{u \in U} \dim Z(h)u = \operatorname{rank} B_U;\\
&\text{(iv)} \quad U \cap Z(h)e \neq \varnothing \quad \text{if and only if} \quad \operatorname{rank} B_U = s.
\end{aligned} \tag{5.1}$$

Depending on s and n, computing the rank of B_U over a function field may turn out to be rather time consuming. For this reason we also consider an alternative method, based on another characterisation of the elements in $V_2(h)\backslash Z(h)e$, which comes from Theorem 2.9.

Proposition 5.1. *Take $v \in V_2(h)$. Then the three conditions: $Z(h)v = \tilde{Z}(h)v$, $[\mathfrak{z}(h), v] = [\tilde{\mathfrak{z}}(h), v]$, and $v \in V_2(h)\backslash Z(h)e$, are equivalent.*

Proof. We are going to identify $\tilde{Z}(h)$ with a reduced θ-group. To this end, for each $i \in \mathbb{Z}$, set $\hat{i} := i \bmod m$, if θ has a finite order m; and $\hat{i} := i$ otherwise. Then $\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$, where $\mathfrak{l}_i = (\mathfrak{g}_{\hat{i}})_{2i}(h)$, is a $\mathbb{Z}$-graded Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{l}_1 = V_2(h)$ and $\mathfrak{l}_0 = \mathfrak{z}(h)$. Let $L \subset G$ be a connected subgroup with $\operatorname{Lie} L = \mathfrak{l}$. Since κ defines a non-degenerate pairing between $(\mathfrak{g}_{\hat{i}})_{2i}(h)$ and $(\mathfrak{g}_{\hat{j}})_{-2i}(h)$ with $j = -i$, we get a non-degenerate L-invariant scalar product $(\,,\,) := \kappa|_{\mathfrak{l}}$ on $\mathfrak{l}$. In particular, $\mathfrak{l}$ is a reductive subalgebra. Here $e \in (\mathfrak{g}_1)_2(h) = \mathfrak{l}_1$, $f \in (\mathfrak{g}_{-1})_{-2}(h) = \mathfrak{l}_{-1}$ and therefore $h \in [\mathfrak{l}, \mathfrak{l}]$. Note that the $\mathbb{Z}$-grading on $\mathfrak{l}$ is defined by the eigenvalues of $h/2$.

Recall that $\mathfrak{g}$ is assumed to be simple. Restricting the adjoint action of G to L we get a faithful representation ρ of L on $\mathfrak{g}$ such that $(x, y) = \operatorname{tr}(\rho(x)\rho(y))$ for $x, y \in \mathfrak{l}$ and $(\,,\,) = \kappa|_{\mathfrak{l}}$. Thus, we are in the setting of Section 2.1 and can apply Theorem 2.9 to the $\mathbb{Z}$-graded reductive Lie algebra $\mathfrak{l}$. Here $\tilde{L}_0 = \tilde{Z}(h)$ and $\mathfrak{l}_1 = (\mathfrak{g}_1)_2(h) = V_2(h)$.

We have $[\mathfrak{z}(h), v] = [\tilde{\mathfrak{z}}(h), v]$ if and only if the $\tilde{Z}(h)$-orbit $\tilde{Z}(h)v$ is conical. Besides, h is a Dynkin characteristic of all elements in $Z(h)e$. Therefore both equivalences follow from Theorem 2.9. □

Assume that the basis $x_1, \ldots, x_n$ is chosen in such a way that $x_1 = h$ and $x_2, \ldots, x_n$ form a basis of $\tilde{\mathfrak{z}}(h)$. Let $\tilde{B}$ be a submatrix of B consisting of the last $n-1$ rows (corresponding to the Lie subalgebra $\tilde{\mathfrak{z}}(h)$). Let also $\tilde{B}_U$ be the restriction of $\tilde{B}$ to U. Since $\dim Z(h)v = \operatorname{rank} B_v$ and $\dim \tilde{Z}(h)v = \operatorname{rank} \tilde{B}_v$, Proposition 5.1 gives us the following:

$$U \cap Z(h)e = \varnothing \quad \text{if and only if} \quad \operatorname{rank} B_U = \operatorname{rank} \tilde{B}_U. \tag{5.2}$$

In other words, either $\operatorname{rank} B_u = \operatorname{rank} \tilde{B}_u$ or u is an element of $Z(h)e$ and $\operatorname{rank} B_u = s$. The equality in Eq. (5.2) is satisfied if and only if the first row of B_U lies in the linear span of the rows of $\tilde{B}_U$. In order to check this we use the following steps.

(1) Take a random $u \in U$, compute the rank of $\tilde{B}_u$, say $\operatorname{rank} \tilde{B}_u = r$.
(2) Find an $r \times r$ non-zero minor of $\tilde{B}_u$, without loss of generality suppose that it is given by the first r rows and the first r columns.
(3) Check whether the first row of B_U is contained in the span of the first r rows of $\tilde{B}_U$.

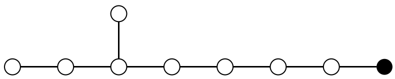
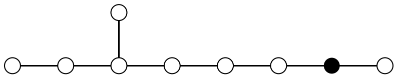
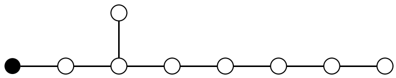
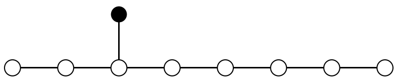
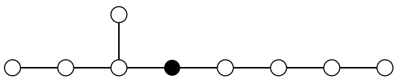
If $u \in U$ is generic, i.e., $\operatorname{rank} \tilde{B}_u = \operatorname{rank} \tilde{B}_U$, then the first r rows of $\tilde{B}_U$ span the row space of $\tilde{B}_U$. Hence step (3) verifies whether the first row of B_U is contained in the row space of $\tilde{B}_U$. Moreover, this will be the case if and only if $U \cap Z(h)e$ is empty. (Also note that the check in the third step can be done by computing $s - r$ minors of size $r + 1$.) Even if u is not a generic element, it may still be true that the first row of B_U is contained in the span of the first r rows of $\tilde{B}_U$, and the above procedure will prove that $U \cap Z(h)e = \varnothing$.

In many cases it is easier to carry out this procedure than to check the inequality $\operatorname{rank} B_U < s$. For example, some 32×38-matrices B_U appeared while checking non-inclusions for a half-spin representation of D_8 (line 3 in Table 1) and 2 760 681 minors would have to be computed for them. In other cases it may be easier to deal with the whole matrix, if, for example, B_U contains a zero column.

It is not obvious beforehand which choice is the best. In the implementation of our algorithm we do the following: if $s - n < s - r$, then it is checked whether $\operatorname{rank} B_U < s$. Otherwise we check whether the first row of B_U is contained in the first r rows of $\tilde{B}_U$, using the procedure outlined above. We do not claim that this always gives the best choice, but *some* choice is better than none.

If it turns out that the first row of B_U is not contained in the span of the first r rows of $\tilde{B}_U$, then it may still be the case that the intersection is empty (if this happens, then necessarily $\operatorname{rank} \tilde{B}_U >$

Table 1
Running times (in seconds) for the algorithm applied to several automorphisms of the Lie algebra of type E_8 and A_n. The first column lists the order of θ, and the second column its Kac diagram. The third column has the number of nilpotent orbits. The fourth and fifth columns display, respectively, the time needed for executing the GAP part of the program, and the time spent in the MAGMA part.

$\lvert\theta\rvert$	Kac diagram of θ	# orbits	GAP	MAGMA
1		69	1003	397
2		36	124	0
2		115	900	0.18
3		101	444	0.09
5		105	335	0.04
2	$A_9(0,5)$	160	581	0
2	$A_{10}(0,5)$	212	2703	0
2	$A_{11}(0,6)$	360	29 292	0

rank $\tilde{B}_u$). Then we will have to compute the rank of B_U. However, the probability of this event can be made arbitrarily small.

6. The main algorithm

Here we describe our algorithm for deciding whether one of the two given nilpotent G_0-orbits in $\mathfrak{g}_1$ lies in the closure of the other.

First we consider the following problem: given a homogeneous $\mathfrak{sl}_2$-triple (h,e,f) and $e' \in V_2(h)$, decide whether $e' \in G_0 e$. We have a straightforward solution for that, based on Lemma 2.6. The existence of $f' \in \mathfrak{g}_{-1}$ with $[h,f'] = -2f'$ and $[e',f'] = h$ is equivalent to a system of linear equations. We solve this system; if it has a solution then e' lies in $G_0 e$, otherwise it does not.

Throughout we fix a basis of the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$. Then the Weyl group W_0 is generated by the reflections corresponding to the elements of this basis. Furthermore, this choice also fixes a dominant Weyl chamber $C_0 \in \mathfrak{h}_0$. As before we let κ denote the Killing form of $\mathfrak{g}$.

Algorithm 1. Input: two homogeneous $\mathfrak{sl}_2$-triples, (h',e',f'), (h,e,f), such that $h', h \in C_0$.
Output: TRUE if $\mathcal{O}' = G_0 e'$ is contained in the closure of $\mathcal{O} = G_0 e$, FALSE otherwise.

(1) If $\kappa(h',h) < \kappa(h',h')$ then return FALSE. Else go to the next step.
(2) For all elements $wh \in W_0 h$ do the following:
  (a) If $\kappa(h',wh) \geqslant \kappa(h',h')$ then:
    (i) Select a random $u \in U = V_2(h') \cap V_{\geqslant 2}(wh)$.
    (ii) If $u \in \mathcal{O}'$ then return TRUE. Otherwise go to the next step.
    (iii) Set $\mathfrak{n} = N_{\mathfrak{g}_0}(U)$. If $[\mathfrak{n},u] \neq U$ then decide whether $U \cap Z(h)e$ is empty using the methods of Section 5. If the intersection is not empty then return TRUE.
(3) If in the previous loop TRUE was never returned, then return FALSE.

Proposition 6.1. *The previous algorithm terminates correctly.*

Proof. It is obvious that the algorithm terminates, we must show that the output is correct. We claim that the algorithm checks whether there is an element $wh \in W_0 h$ such that $U(wh) = V_2(h') \cap V_{\geqslant 2}(wh)$ contains a point of $\mathcal{O}'$. Then by Proposition 3.1 the output is correct.

First of all we note that since $h', h \in C_0$ we have that the maximal value of $\kappa(h', wh)$, for $wh \in Wh$, is $\kappa(h', h)$ (see Remark 4.2). Therefore, if $\kappa(h', h) < \kappa(h', h')$ then no space $U(wh)$ contains a point of $\mathcal{O}'$ (Proposition 3.2). So in this case we are immediately done.

Otherwise we inspect every $wh \in W_0 h$. If $\kappa(h', wh) < \kappa(h', h')$ then $U(wh)$ contains no points of $\mathcal{O}'$ by Proposition 3.2. So then we can discard it. Otherwise we select a random $u \in U(wh)$. If $u \in \mathcal{O}'$, then we are done. If not, and $[\mathfrak{n}, u] = U(wh)$, then $U(wh)$ has no points of $\mathcal{O}'$ by Proposition 3.4. Finally $U(wh)$ contains an element with a minimal dimensional stabiliser if and only if $U(wh)$ has a point of $\mathcal{O}'$ by Proposition 3.3. □

Remark 6.2. Note that if $U(wh)$ contains a point of $\mathcal{O}'$, then the set of such points is open and dense in $U(wh)$, by Proposition 3.1. Hence in that case the random choice has a high probability of finding an element $u \in \mathcal{O}'$.

Remark 6.3. Now we make some observations that help to execute the algorithm more efficiently.

- Of course, we apply the algorithm only if $\dim \mathcal{O}' < \dim \mathcal{O}$, as otherwise there is no inclusion.
- Inclusion also implies that $\dim \mathfrak{g}_{k,e'} \geqslant \dim \mathfrak{g}_{k,e}$ for all k; so if that condition is not fulfilled, we also do not apply the algorithm.
- When looping over the orbit $W_0 h$ we use the tree structure described in the previous section. When doing this, several shortcuts can be made. First of all if $\kappa(h', wh) < \kappa(h', h')$ then the entire subtree below wh can be discarded, by Remark 4.2. Second, if $V_2(h') \cap V_{\geqslant 2}(wh)$ contains no points of $\mathcal{O}'$, and the successor of wh is $s_i wh$, where s_i lies in the stabiliser of h', then also $V_2(h') \cap V_{\geqslant 2}(s_i wh)$ contains no points of $\mathcal{O}'$. Hence in that case we can immediately jump to the next element of the orbit.
- We collect the subspaces $U(wh) = V_2(h') \cap V_{\geqslant 2}(wh)$ that appear during the execution of the algorithm. If a certain such subspace is contained in one that was treated before, then we already know that it contains no points of $\mathcal{O}'$. So in that case we can immediately go to the next round. All calculations are done in a basis $v_1, \dots, v_s$ of $V_2(h')$ consisting of $\mathfrak{h}_0$-eigenvectors. Each subspace $U(wh)$ is a linear span of v_i such that $(wh) \cdot v_i = a_i v_i$ with $a_i \geqslant 2$. Therefore storing and verifying inclusions among the $U(wh)$ is a binary problem.

We have implemented this algorithm in the language of the computer algebra system GAP4 [17], on top of the SLA package [19], which has implementations of algorithms to list the nilpotent orbits of a θ-group. One of the main problems for the practical computation lies in the methods of Section 5, where minors of a matrix with entries in a function field have to be computed. For these computations we use the computer algebra system Magma [5]. We have chosen this system, because it has very efficient implementations of algorithms to compute the determinant of a matrix with entries in a function field.

In Table 1 we collect some experimental data with respect to the implementation of our algorithm. All computations have been performed on a 3.16 GHz machine.

In Table 1, we let $A_n(0, \lceil \frac{n}{2} \rceil)$ denote the Kac diagram of an inner involution (automorphism of order 2) of $\mathfrak{sl}_{n+1}$, having all labels equal to 0, except for two labels which are 1, and which are located as far apart in the diagram as possible.

From the table we see that in the examples concerning E_8 the GAP part essentially has no problems. On some occasions it is not necessary to execute the Magma part, as with the second automorphism in the table. On other occasions this part has a trivial running time, as with the last three examples involving E_8. However, it also happens that a fair amount of time is spent in the Magma part, as with the first example. In the examples involving A_n ($n = 9, 10, 11$) the number of orbits increases rapidly, as well as the dimension of the homogeneous components of the grading. For these examples we see a sharp increase in the running time.

Relying on this evidence, one can conclude that the program will work for all automorphisms of the exceptional Lie algebras. It is difficult to say until what rank of $\mathfrak{g}$ it can go in the classical case, but here everything can be computed by hand.

7. Further remarks on groups and orbits

Here we collect some theoretical observations that could have been used in the algorithm, but turned out not to be of much practical value.

Let (f, h, e) be a homogeneous $\mathfrak{sl}_2$-triple as before, we also keep all the previous notation, including $V_2(h)$. First we consider the actions of $Z(h)$ and $\tilde{Z}(h)$ on $V_2(h)$ more closely. As was already mentioned, $Z(h)$ acts on $V_2(h)$ with a dense open orbit, $Z(h)e$.

7.1. The semi-invariant P

The stabiliser $\mathfrak{z}(h)_e$, being the centraliser of an $\mathfrak{sl}_2$-subalgebra generated by e, h, and f, is reductive and therefore the orbit $Z(h)e$ is an affine space. This implies that the complement $V_2(h)\backslash Z(h)e$ is a divisor and is a zero-set of a single semi-invariant polynomial, say P. To check whether a subspace $U = U(wh') = V_2(h) \cap V_{\geqslant 2}(wh')$ intersects the dense orbit, one just has to look on the restriction of P to U. In this terms, $\mathcal{O}$ lies in the closure of $\mathcal{O}'$ if and only if there is $U(wh')$ such that P is non-zero on it. This could be a replacement for both: choosing a random element in U and generic rank considerations.

One can try to compute a polynomial P by hand. This may involve typing errors and time consuming calculations. It is also possible to get P from the matrix B with entries $b_{ij} = x_i \cdot w_j$, where $\{x_i\}$ is a basis of $\mathfrak{z}(h)$ and $\{w_j\}$ is a basis of $V_2(h)^*$ (this matrix was already considered in Section 5). The polynomial P is the greatest common divisor of the largest, $\dim V_2(h) \times \dim V_2(h)$, minors of B. In some cases the resulting formula is rather bulky and not easy to deal with, in some other Magma was unable to finish the calculation. It turns out that Magma checks much more easily that the restriction B_U of B to U does not have the maximal rank, $\dim V_2(h)$, than it computes the greatest common divisor of minors. Thus we gave up the idea of using P.

7.2. Double cosets of Weyl groups

Suppose that we have two characteristics h and h' lying in the dominant chamber of W_0. Parametrisation of $W_0 h$ involves a certain numbering of simple roots $\alpha_1, \ldots, \alpha_l$, see Section 4. This numbering can be arbitrary. The stabiliser $W_{0,h'}$ is a Weyl subgroup generated by s_i with $\alpha_i(h') = 0$. Assume that $\operatorname{rank} W_{0,h'} = r$ and the simple roots orthogonal to h' have numbers from $l - r + 1$ to l.

Lemma 7.1. *Keep the above notation and enumeration of simple roots. Let* $\mathbf{T}$ *be a tree parametrising the orbit* $W_0 h$*, constructed according to the principles of Lemma 4.1. Then the nodes* h *and* $s_i \hat{h}$ *of* $\mathbf{T}$ *with* $i \leqslant l - r$ *are in one-to-one correspondence with the double cosets* $W_{0,h'} \backslash W_0 / W_{0,h}$.

Proof. First note that h or $s_i \hat{h}$ with $i \leqslant l - r$ lies in the dominant chamber of $W_{0,h'}$. Secondly, an element $s_i \hat{h}$ with $l - r < i$ does not bring a new double coset, because here s_i lies in $W_{0,h'}$. $\square$

This, of course, is not a very effective way for listing the double cosets as the whole orbit $W_0 h$ has to be constructed. However, if some time consuming calculation has to be performed for representatives of double cosets, such a treatment may be useful.

In our situation, collecting subspaces $U(wh)$ turned out to be much more effective than refining the Weyl-group tree. The explanation is that one and the same $U = U(wh)$ arises for many different elements $wh \in W_0 h$.

7.3. Other algebro-geometric methods

Our algorithm is designed for θ-groups and works quite well. There are known some other, more general, approaches, which unfortunately have a rather small range of application.

To begin with, consider a linear action $a: Q \times V \to V$ of an affine algebraic group Q on a vector space V. For $x \in V$, the map $g \mapsto gx$ from Q to V is regular, i.e., given by polynomials. Let $I(Qx) \lhd \mathbb{C}[V]$ be the ideal of Qx, i.e., a set of all polynomials vanishing on Qx. In [7], algorithmic methods are described for computing generators of the vanishing ideal of the image of a regular map. In particular, this can be applied to $I(Qx)$. Here $I(Qx)$ equals $(a^*)^{-1}(I(Q \times \{x\}))$, where $I(Q \times \{x\}) \lhd \mathbb{C}[\mathrm{GL}(V) \times \{x\}]$ is the defining ideal of the product of the image of Q in $\mathrm{GL}(V)$ and the point x. Once the generators of $I(Qx)$ are known, it is straightforward to decide whether a point (and hence the orbit of that point) lies in $\overline{Qx}$.

In order to use the algorithms for getting generators of $I(Qx)$, we need as input the polynomials defining Q as a subgroup of $\mathrm{GL}(V)$. In our setting, $V = \mathfrak{g}_1$ and Q is the image in $\mathrm{GL}(V)$ of G_0, acting on V. In order to get equations for Q, methods from [18] can be used. However, both the algorithm for obtaining the polynomials defining Q and the one for computing $I(Qx)$ heavily rely on Gröbner basis computations. These are extremely time consuming. For this reason this method is only applicable to very small examples (e.g., when the semisimple part of $\mathfrak{g}_0$ is of type A_1, and $\mathfrak{g}_1$ is of dimension 5).

Now let $G \subset \mathrm{GL}[V]$ be reductive. Then in the above considerations G can be replaced by its big open cell, BwB, where $w \in W$ is the longest element in the Weyl group W of G and $B \subset G$ is a Borel subgroup. In [31], V.L. Popov suggested an algorithm, based on this observation, for deciding whether Gy lies in $\overline{Gx}$. That algorithm uses a system of linear equations in $n\binom{n+2d-3}{n-1}$ variables, where $n = \dim V$ and d is the degree of G as a subvariety of $\mathrm{L}(V)$. For an irreducible five-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$, the number of variables equals 56 794 400. This makes it difficult, if not impossible, to use the algorithm for practical computations.

8. Examples

In this section we show the output of our programs on several examples. The θ-groups of our examples have all previously been studied in the literature, for various reasons.

Here we describe the examples; the next section contains the Hasse diagrams that we computed with the algorithm, as well as tables giving the characteristics of the nilpotent orbits.

8.1. Symmetric pairs

As was mentioned in the Introduction, the order two case, or, in other words, the symmetric case, was studied by Djoković, because of its relationship with simple real Lie algebras. We have checked all symmetric pairs arising from the exceptional Lie algebras. The result is that Djoković diagrams are basically correct, if one takes into account the necessary alteration that he found himself [13,15]. Our calculations confirm these corrections. Apart from this, there are two inclusions missing for one automorphism in type E_8. For the involution in question, $\mathfrak{g}_0$ is of type D_8 and $\mathfrak{g}_1$ is a half-spin representation, the corresponding Kac diagram is the third one in Table 1.

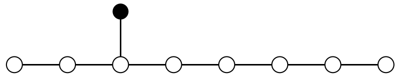
According to our calculations, in Table 2 of [14], $59 \to 53$, and $95 \to 92$ should be added (notation as in the mentioned paper: $59 \to 53$ means that orbit number 53 is contained in the closure of orbit number 59). This then results in several other changes. For example, [14, Table 2] states "$99 \to 92, 94, 95$" and 92 has to be removed from here, because the orbit number 92 does not give rise to an irreducible component of the boundary $\partial\mathcal{O}_{99}$.

Neither $(59, 53)$ nor $(95, 92)$ appears in the list [14, Table 5] of the critical pairs, i.e., pairs (a, b) such that the non-inclusion $a \nrightarrow b$ is (or has to be) proved. In both cases, our program immediately found that the space $U = V_2(h') \cap V_{\geqslant 2}(h)$ contains a point of $\mathcal{O}'$.[1]

[1] This can be even verified by hand, would someone wish to do so.

8.2. Trivectors of a nine-dimensional space

In [39], the orbits of $\mathrm{SL}_9(\mathbb{C})$ acting on $\bigwedge^3(\mathbb{C}^9)$ were obtained. This is known as the classification of the trivectors of a nine-dimensional space. The orbits were obtained by realising this representation as a θ-representation. Here θ is an automorphism of order 3 of the Lie algebra of type E_8, with Kac diagram



We have that $\mathfrak{g}_0 \cong \mathfrak{sl}_9(\mathbb{C})$ and $\mathfrak{g}_1 \cong \bigwedge^3(\mathbb{C}^9)$.

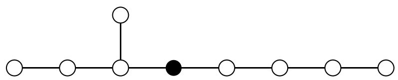
In Table 4 we list the characteristics of the (non-zero) nilpotent orbits. A characteristic h is given by the values of $\alpha_i(h)$, where $\{\alpha_1, \dots, \alpha_8\}$ is a basis of the root system of $\mathfrak{g}_0$. Moreover, all the characteristics h lie in the dominant Weyl chamber with respect to this basis.

Figs. 1, 2 contain, respectively, the top half and the bottom half of the Hasse diagram.

8.3. The classification of metabelian Lie algebras

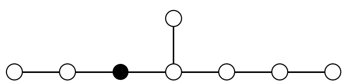
A finite-dimensional Lie algebra L is said to be *metabelian* (or two-step nilpotent), if $[L,[L,L]] = 0$. In [16], Galitski and Timashev described the G_0-orbits for the two particular θ-representations in order to obtain the classification of the metabelian Lie algebras of dimensions up to 9 (over algebraically closed fields of characteristic 0). With our algorithm we have computed the closure diagram of the nilpotent orbits in both these cases.

For the first θ-group, θ is an automorphism of order 5 of the Lie algebra of type E_8, with Kac diagram



Characteristics of the nilpotent orbits are given in Table 5. The closure diagram is displayed in Figs. 3, 4.

For the second θ-group, θ is the automorphism of order 3 of the Lie algebra of type E_7, with Kac diagram



Characteristics of the nilpotent orbits are given in Table 6. The closure diagram is displayed in Figs. 5, 6.

As was already mentioned, in [16] the orbits of these particular θ-groups are used for a classification of metabelian Lie algebras. Every orbit corresponds to one such Lie algebra (up to isomorphism). Every metabelian Lie algebra L has a signature, that is a pair (m,n) where $m = \dim L/[L,L]$ and $n = \dim[L,L]$. Let $Z \lhd L$ be a maximal abelian ideal such that $Z \cap [L,L] = 0$. Then $L \cong L/Z \oplus Z$. In the closure diagrams we indicate the signature of L/Z as it was computed in [16]. Mostly this is done by writing the label of the node in a particular font, according to Tables 2 (for Figs. 3, 4) and 3 (for Figs. 5, 6). For the orbits corresponding to signatures not present in these tables we have put the signature in the diagram, next to the node.

Table 2
Fonts for the Hasse diagram in Figs. 3, 4.

Font	Example	Signature
roman	10	$(5,5)$
bold face	**14**	$(5,4)$
italics	*70*	$(5,3)$
underline	$\underline{82}$	$(4,4)$
typewriter	`92`	$(4,3)$

Table 3
Fonts for the Hasse diagram in Figs. 5, 6.

Font	Example	Signature
roman	10	$(6,3)$
bold face	**35**	$(6,2)$
italics	*41*	$(5,3)$
typewriter	`68`	$(4,3)$
overline	$\overline{64}$	$(4,2)$

Taking the closure of a given nilpotent orbit corresponds to the degeneration of the encoded two-step nilpotent Lie algebra. Let us explain this. Let W be a vector space (over $\mathbb{C}$). Then a Lie bracket on W can be seen as an element of $\mathrm{Hom}(\bigwedge^2 W, W)$. The group $\mathrm{GL}(W)$ acts on $\mathrm{Hom}(\bigwedge^2 W, W)$, and the orbits of this action are in one-to-one correspondence with the isomorphism classes of Lie algebra structures on W. Let $\lambda, \mu \in \mathrm{Hom}(\bigwedge^2 W, W)$; if μ is contained in the closure of the $\mathrm{GL}(W)$-orbit of λ, then μ is said to be a *degeneration* of λ. We refer to [21] for an introduction into this concept.

In relation to the variety of metabelian Lie algebras one considers two vector spaces U and V. A metabelian Lie bracket on $U \oplus V$ is viewed as an element of $\mathrm{Hom}(\bigwedge^2 U, V)$. Let L be the Lie algebra defined by such an element; then $[L, L] \subset V$. The group $\mathrm{GL}(U) \times \mathrm{GL}(V)$ acts on $\mathrm{Hom}(\bigwedge^2 U, V)$. Two metabelian Lie algebra structures on $U \oplus V$ are isomorphic if and only if the corresponding elements of $\mathrm{Hom}(\bigwedge^2 U, V)$ lie in the same $\mathrm{GL}(U) \times \mathrm{GL}(V)$-orbit.

Write $m = \dim U$, $n = \dim V$, and let W be a vector space of dimension $m + n$. If $\lambda \in \mathrm{Hom}(\bigwedge^2 W, W)$ is a metabelian Lie bracket, defining a Lie algebra L on W of a signature (m, n), then by setting $V = [L, L]$ and taking U to be a complement of V in W, we get an element of $\mathrm{Hom}(\bigwedge^2 U, V)$. This construction preserves isomorphism, i.e., a $\mathrm{GL}(W)$-orbit of metabelian Lie brackets in $\mathrm{Hom}(\bigwedge^2 W, W)$ is mapped to a $\mathrm{GL}(U) \times \mathrm{GL}(V)$-orbit in $\mathrm{Hom}(\bigwedge^2 U, V)$. Thus metabelian Lie algebras of signature (m, n) are classified by the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-orbits in $\mathrm{Hom}(\bigwedge^2 U, V)$. Moreover, degenerations of these Lie algebras are given by the orbits closures.

Two instances of θ-groups considered in [16] correspond to the signatures $(5, 5)$ and $(6, 3)$. Strictly speaking, the group G_0 is semisimple in both cases. Therefore, if $0 \notin \overline{G_0 x}$ with $x \in \mathfrak{g}_1$, then the one-parameter family of G_0-orbits $G_0(ax)$ with $a \in \mathbb{C}^\times$ gives only one isomorphism class of metabelian Lie algebras. For the nilpotent G_0-orbits, there is no difference between $G_0 x$ and $(\mathrm{GL}(U) \times \mathrm{GL}(V))x$.

Note that the quotients L/Z with signatures being smaller than or equal to $(5, 3)$ appear in both θ-groups. Therefore, the lower parts of both Hasse diagrams (Figs. 3, 4; 5, 6) are the same.

The affine variety of metabelian Lie algebras structures with signature $(5, 5)$ on a ten-dimensional vector space W is irreducible, because of the equivalence with θ-group orbits. According to [16], it has a one-parameter family of the maximal $\mathrm{GL}(W)$-orbits. The same holds for the second case, where signatures are $(6, 3)$ and $\dim W = 9$, only there is a two-parameter family of the maximal orbits [16]. Since isomorphism classes of metabelian Lie algebras with smaller signatures are parametrised by nilpotent orbits, there are only finitely many of them. In addition, our diagrams show that for each signature (m, n), where either $m \leqslant 5$, $n \leqslant 5$ and $(m, n) \neq (5, 5)$, or $m \leqslant 6$, $n \leqslant 3$ and $(m, n) \neq (6, 3)$, the affine variety of metabelian Lie algebra structures with signature (m, n) on W with $\dim W = m + n$ is irreducible. For example, all metabelian Lie algebras with signature $(5, 4)$ are degenerations of L/Z

(including L/Z itself), where L corresponds to the orbit number 14 of the first θ-group (in E_8), up to isomorphism, see Fig. 3. The metabelian Lie algebras with signature $(5,3)$ are degenerations of L/Z, where L is encoded by the orbit number 62 of the first θ-group as well as by the orbit number 27 of the second θ-group, see Figs. 3, 5.

Appendix A. Diagrams and tables

Table 4
Characteristics of the nilpotent orbits in the case of 3-vectors.

No.	Characteristic	No.	Characteristic
1	6 6 6 6 6 6 6 12	2	6 6 6 0 6 6 6 6
3	6 6 6 0 6 0 6 6	4	6 0 6 0 6 6 0 6
5	0 6 0 6 0 6 0 6	6	6 1 5 1 5 6 1 5
7	6 0 6 0 6 0 0 6	8	0 6 0 0 6 0 6 0
9	0 0 6 0 0 6 0 6	10	6 1 5 1 5 0 1 5
11	1 5 1 1 4 1 5 1	12	2 2 2 2 2 4 2 2
13	0 1 5 0 1 5 1 5	14	2 2 2 2 2 2 2 2
15	1 1 4 1 1 5 1 4	16	6 0 0 0 6 0 0 6
17	3 0 3 3 0 6 0 3	18	0 6 0 0 0 6 0 0
19	2 2 0 2 2 2 2 2	20	6 0 1 0 5 0 1 5
21	0 3 0 3 0 3 3 0	22	1 4 0 1 1 4 1 1
23	1 4 1 1 1 3 1 1	24	2 0 4 2 0 6 0 4
25	0 0 6 0 0 0 0 6	26	6 1 0 1 4 1 0 5
27	3 0 3 0 3 0 3 0	28	0 4 0 2 0 4 2 0
29	1 1 2 1 1 1 1 4	30	0 1 5 0 0 1 0 5
31	2 2 2 2 0 2 0 2	32	1 0 5 0 1 0 1 4
33	2 0 2 2 0 2 0 4	34	0 0 0 0 6 0 0 0
35	1 1 4 1 0 1 0 4	36	2 0 2 0 2 0 2 2
37	0 0 1 0 5 0 0 1	38	2 1 1 1 1 1 1 2
39	0 1 0 1 4 0 1 0	40	1 0 1 0 4 0 1 1
41	3 0 3 3 0 0 0 3	42	3 0 0 0 3 0 3 0
43	1 1 0 1 3 1 0 1	44	0 3 0 0 0 3 0 3
45	1 0 1 1 2 1 1 1	46	3 0 1 0 2 1 2 0
47	2 0 4 2 0 0 0 4	48	1 1 1 1 1 1 1 1
49	2 0 0 0 4 0 2 0	50	0 2 0 0 2 2 0 2
51	1 1 0 1 1 2 1 1	52	2 0 1 0 3 1 1 0
53	1 1 1 1 0 1 1 2	54	0 3 0 0 3 0 0 0
55	0 0 0 3 0 3 0 0	56	2 0 2 0 0 2 0 2
57	0 0 0 0 0 0 0 6	58	0 4 0 0 2 0 0 0
59	0 0 3 0 0 0 3 0	60	1 1 1 1 1 0 1 1
61	0 1 0 2 0 3 0 1	62	0 0 0 0 1 0 0 5
63	1 1 1 0 1 0 2 1	64	0 1 2 0 1 0 2 0
65	1 2 1 1 0 0 1 1	66	0 2 0 0 2 0 0 2
67	1 0 1 1 0 3 1 0	68	0 0 1 0 0 0 1 4
69	1 1 0 1 1 0 1 1	70	0 1 0 0 1 0 0 4
71	0 2 2 0 0 0 2 0	72	2 0 1 0 1 1 0 1
73	1 0 1 1 0 1 1 0	74	1 0 0 1 0 0 1 3
75	0 3 0 0 0 0 0 3	76	1 2 0 0 0 1 0 2
77	0 0 2 0 0 4 0 0	78	0 1 0 0 1 0 1 2
79	0 2 0 0 0 0 0 4	80	1 1 0 1 0 1 0 1
81	0 0 2 0 0 2 0 0	82	1 1 0 0 0 1 0 3
83	0 0 0 3 0 0 0 0	84	1 0 0 1 0 1 0 2
85	0 1 0 2 0 0 0 1	86	2 1 0 1 0 0 0 2
87	0 0 0 2 0 0 0 2	88	0 1 0 1 0 0 1 1
89	0 0 0 0 0 0 3 0	90	2 0 0 1 0 1 0 0
91	2 0 0 1 0 0 0 3	92	1 0 1 0 0 1 0 1
93	0 0 0 1 0 0 2 0	94	0 1 0 0 0 1 1 0
95	1 0 0 1 0 0 1 0	96	0 0 0 0 0 2 0 0
97	0 0 1 0 0 1 0 0	98	3 0 0 0 0 0 0 0
99	2 0 0 0 0 0 1 0	100	1 0 0 0 1 0 0 0
101	0 0 1 0 0 0 0 0		

Table 5
Characteristics of the nilpotent orbits.

No.	Characteristic	No.	Characteristic
1	10 10 10 10 10 10 10 20	2	10 10 0 10 10 10 10 10
3	10 0 10 10 10 0 10 10	4	0 10 0 10 10 10 0 10
5	10 0 10 0 0 10 0 10	6	1 9 1 9 10 10 1 9
7	10 0 3 7 7 3 7 3	8	3 7 3 7 3 4 3 3
9	0 10 0 0 10 0 0 10	10	0 0 10 0 0 0 10 0
11	1 9 0 1 10 0 1 9	12	2 0 8 2 2 2 6 2
13	0 3 7 0 3 0 7 3	14	0 0 0 0 0 0 0 10
15	1 8 1 1 10 1 1 8	16	3 3 4 3 3 3 1 3
17	5 0 5 0 0 5 5 0	18	1 2 7 1 3 1 6 3
19	0 5 0 5 0 5 0 5	20	0 0 1 0 0 0 1 9
21	2 2 2 4 2 4 2 2	22	6 0 4 0 0 6 4 0
23	3 3 1 3 3 1 2 4	24	0 3 0 7 4 3 0 3
25	1 0 1 0 0 1 1 8	26	0 1 0 1 0 1 0 9
27	0 8 2 0 10 2 0 8	28	4 2 2 2 4 0 2 4
29	0 8 2 0 0 0 2 0	30	2 2 2 2 2 2 2 2
31	1 2 1 6 4 3 1 2	32	1 1 0 1 1 0 1 8
33	1 0 1 1 1 1 1 7	34	3 2 0 5 3 5 0 2
35	3 2 0 5 2 0 3 0	36	1 1 1 1 1 1 1 6
37	2 0 0 0 0 2 0 8	38	0 2 0 0 2 0 2 6
39	0 0 5 0 0 5 0 0	40	3 1 2 1 3 1 2 1
41	2 2 0 6 4 4 0 2	42	1 2 1 3 2 1 1 2
43	2 0 1 0 1 1 0 8	44	3 3 0 4 1 0 3 0
45	0 5 0 0 0 0 0 5	46	0 0 0 3 3 0 0 7
47	1 1 1 1 1 1 2 4	48	1 1 0 1 2 1 1 6
49	0 2 2 2 2 2 0 2	50	0 4 0 0 0 0 0 6
51	2 0 2 0 2 0 2 4	52	0 0 3 0 0 3 0 4
53	1 1 1 2 1 1 1 4	54	1 3 1 1 0 1 1 3
55	0 1 0 2 3 0 1 6	56	0 1 2 1 1 2 0 4
57	1 2 1 1 0 1 1 4	58	0 3 2 0 0 2 0 3
59	1 1 1 1 1 2 1 3	60	1 0 1 1 3 1 0 6
61	0 2 0 2 2 0 2 2	62	0 0 0 0 0 0 5 0
63	2 1 1 2 1 1 1 1	64	0 2 2 0 0 2 0 4
65	1 1 1 1 2 1 1 2	66	2 0 1 2 1 1 1 2
67	0 0 1 0 0 1 4 0	68	0 3 0 0 0 0 3 1
69	5 0 0 0 5 0 0 0	70	0 1 0 1 1 0 4 0
71	1 2 0 1 0 1 2 1	72	0 0 0 5 0 0 0 0
73	2 1 0 1 3 0 1 2	74	1 0 1 0 1 1 3 0
75	3 0 2 0 1 0 2 0	76	1 1 1 1 1 1 1 1
77	0 2 0 0 0 0 4 0	78	0 1 0 4 0 0 0 1
79	0 0 2 0 4 0 0 6	80	2 1 1 0 1 0 2 1
81	1 1 0 1 0 1 3 0	82	0 0 0 4 0 0 0 2
83	1 0 1 1 1 1 2 0	84	0 1 2 0 2 0 1 1
85	0 1 0 3 0 0 1 1	86	2 0 0 0 2 0 3 0
87	2 0 1 1 1 1 0 1	88	0 0 2 0 2 0 2 0
89	0 2 0 0 0 2 1 0	90	2 0 1 0 1 0 3 0
91	1 0 1 2 0 1 0 1	92	0 0 0 3 0 0 2 0
93	1 1 0 1 1 1 1 0	94	0 1 0 2 0 1 1 0
95	0 1 0 0 0 3 0 0	96	1 0 1 1 1 0 1 0
97	1 0 0 1 1 2 0 0	98	3 0 0 0 0 0 0 1
99	0 0 0 2 0 2 0 0	100	2 0 0 1 0 0 1 0
101	0 0 2 0 0 0 1 0	102	0 1 0 1 1 1 0 0
103	1 0 1 0 0 1 0 0	104	0 0 0 1 2 0 0 0
105	0 1 0 0 1 0 0 0		

Table 6
Characteristics of the nilpotent orbits.

No.	Characteristic	No.	Characteristic
1	6 6 6 6 6 6 12	2	6 6 6 0 6 6 6
3	0 6 0 6 6 6 6	4	0 6 0 6 0 6 12
5	2 4 2 6 4 6 6	6	6 0 6 0 6 0 6
7	6 1 5 1 5 1 5	8	0 6 0 6 0 0 6
9	0 6 0 0 6 6 0	10	4 0 2 4 2 6 6
11	0 0 6 0 0 0 6	12	1 5 0 1 5 6 1
13	1 5 1 1 4 5 1	14	2 2 2 2 4 4 2
15	2 0 4 0 2 0 6	16	2 2 2 2 2 2 2
17	0 1 5 0 1 1 5	18	2 4 2 0 4 6 0
19	0 6 0 0 0 6 6	20	2 0 2 2 2 2 4
21	1 1 4 1 1 1 4	22	2 1 3 1 1 1 5
23	0 0 0 6 0 6 0	24	0 3 0 3 0 0 6
25	2 2 0 2 2 2 2	26	1 0 2 3 1 3 3
27	0 0 0 0 6 0 0	28	3 0 3 3 0 0 3
29	0 6 0 0 0 0 0	30	0 1 0 5 0 6 1
31	1 2 1 0 2 3 3	32	1 2 0 3 1 1 5
33	2 0 4 2 0 0 4	34	1 4 0 1 1 1 1
35	0 0 1 0 5 0 1	36	0 3 0 3 0 3 0
37	2 0 2 0 2 2 2	38	0 4 0 2 0 2 0
39	0 1 0 1 4 1 0	40	2 2 0 0 2 4 2
41	0 0 0 0 0 0 6	42	0 2 0 0 4 0 0
43	2 1 1 1 1 1 2	44	0 2 0 4 0 2 4
45	1 0 1 0 4 1 1	46	0 2 0 2 0 2 2
47	1 1 0 1 3 0 1	48	0 0 0 1 0 1 5
49	1 0 1 1 2 1 1	50	3 0 0 0 3 3 0
51	0 1 0 1 0 1 4	52	2 0 0 0 4 2 0
53	1 0 2 0 1 3 0	54	0 3 0 0 0 0 3
55	3 0 1 0 2 2 0	56	0 2 0 0 2 0 2
57	2 0 1 0 3 1 0	58	0 2 0 0 0 0 4
59	0 0 2 0 2 2 0	60	0 1 0 0 2 0 3
61	0 1 0 0 0 2 4	62	1 1 0 1 1 1 1
63	0 1 0 1 0 2 2	64	0 0 0 3 0 0 0
65	1 0 0 1 1 1 2	66	0 1 0 2 0 0 1
67	1 0 1 1 0 1 0	68	0 0 0 2 0 0 2
69	0 1 0 1 0 1 1	70	2 0 0 1 0 0 0
71	0 0 2 0 0 0 0	72	1 0 1 0 0 0 1
73	0 0 0 0 0 3 0	74	0 0 0 1 0 2 0
75	0 1 0 0 0 1 0		

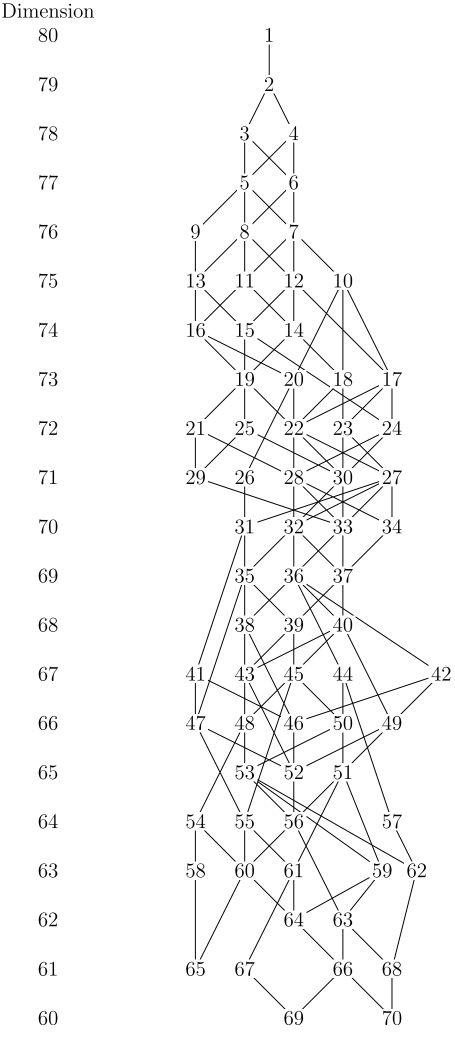


Fig. 1. Hasse diagram of nilpotent orbits in the case of trivectors; top half.

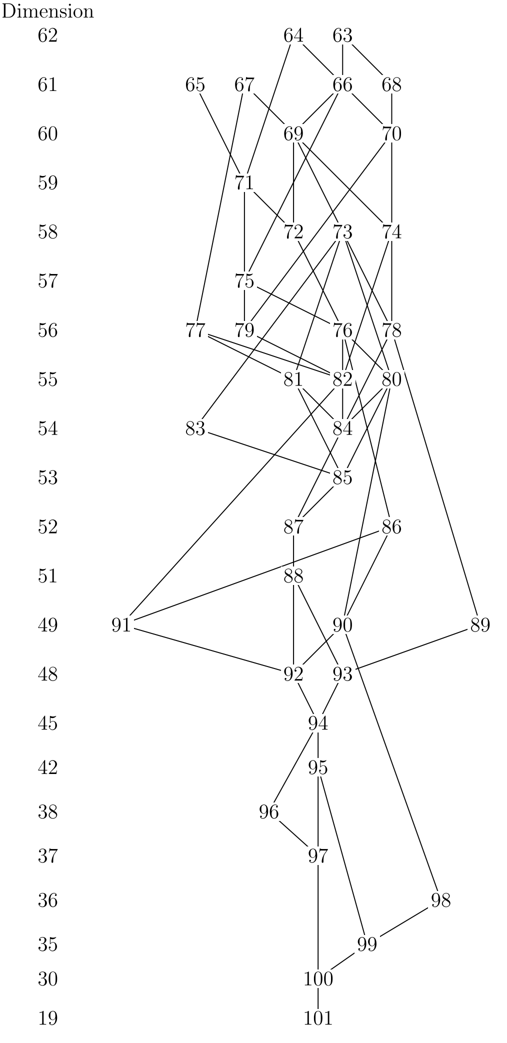


Fig. 2. Hasse diagram of nilpotent orbits in the case of trivectors; bottom half.

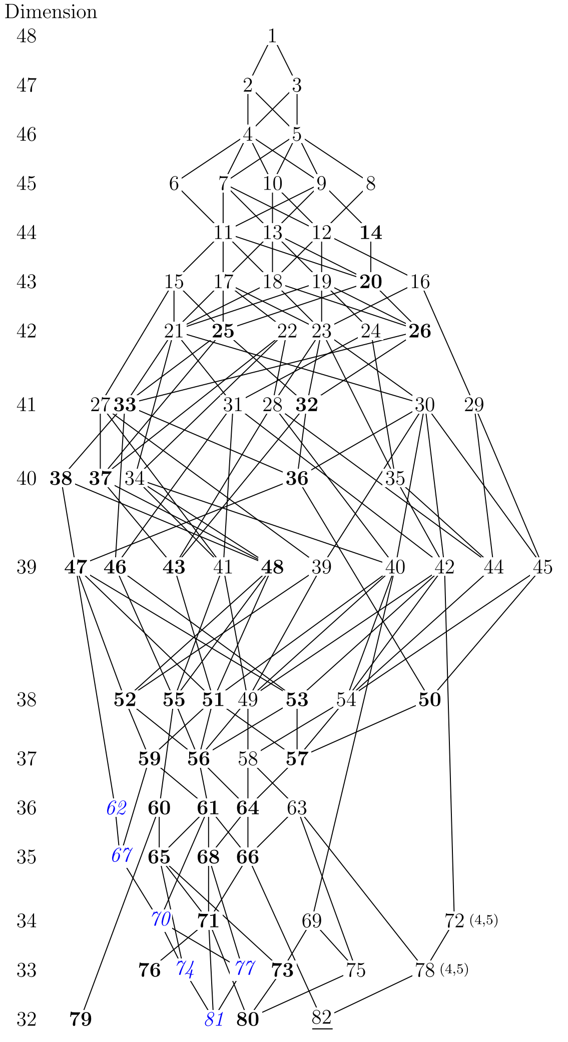


Fig. 3. Hasse diagram of nilpotent orbits; top.

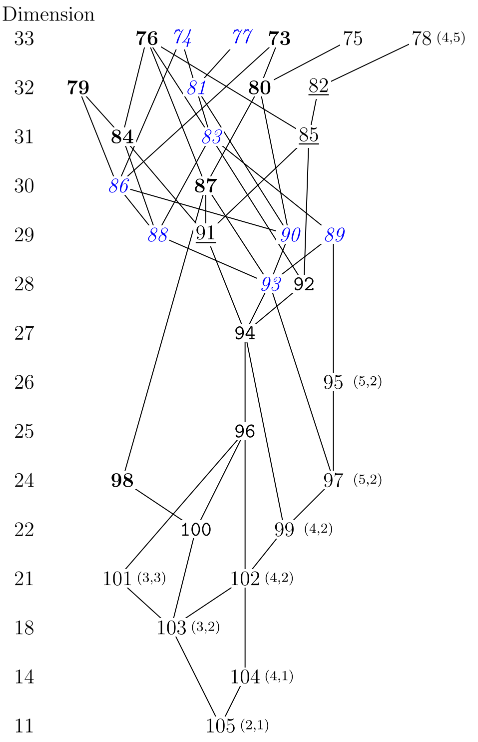


Fig. 4. Hasse diagram of nilpotent orbits; bottom.

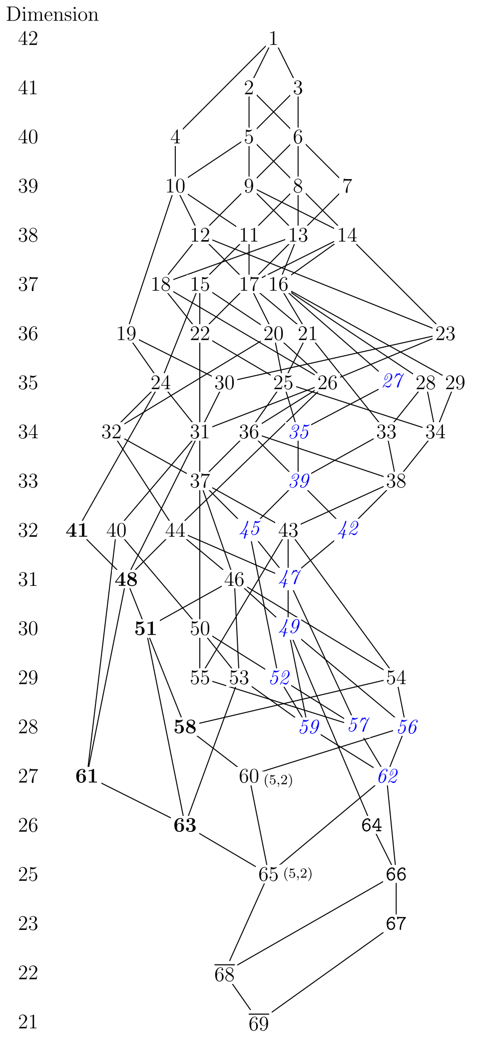


Fig. 5. Hasse diagram of nilpotent orbits, E_7, top.

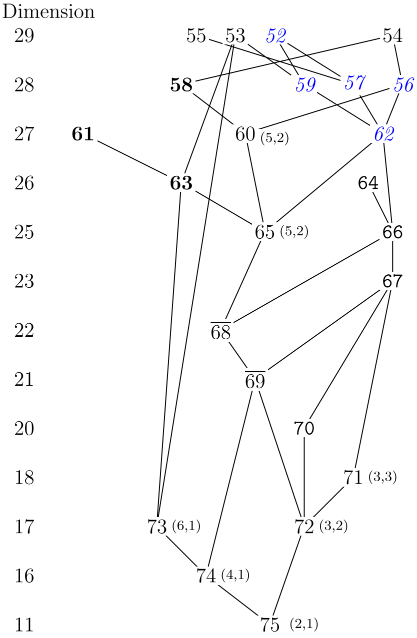


Fig. 6. Hasse diagram of nilpotent orbits, E_7, bottom.

References

[1] L.V. Antonyan, Classification of four-vectors of an eight-dimensional space, Trudy Sem. Vektor. Tenzor. Anal. 20 (1981) 144–161.

[2] L.V. Antonyan, A.G. Èlashvili, Classification of spinors of dimension sixteen, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 70 (1982) 5–23.

[3] D. Barbasch, M.R. Sepanski, Closure ordering and the Kostant–Sekiguchi correspondence, Proc. Amer. Math. Soc. 126 (1) (1998) 311–317.

[4] W.M. Beynon, N. Spaltenstein, Green functions of finite Chevalley groups of type E_n ($n = 6, 7, 8$), J. Algebra 88 (1984) 584–614.

[5] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, in: Computational Algebra and Number Theory, London, 1993, J. Symbolic Comput. 24 (3–4) (1997) 235–265.

[6] D.H. Collingwood, W.M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[7] D. Cox, J. Little, D. O'Shea, Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer-Verlag, New York, Heidelberg, Berlin, 1992.

[8] Dragomir Ž. Đoković, The closure diagrams for nilpotent orbits of real forms of F_4 and G_2, J. Lie Theory 10 (2) (2000) 491–510.

[9] Dragomir Ž. Đoković, The closure diagram for nilpotent orbits of the real form EIX of E_8, Asian J. Math. 5 (3) (2001) 561–584.

[10] Dragomir Ž. Đoković, The closure diagram for nilpotent orbits of the split real form of E_7, Represent. Theory 5 (2001) 284–316 (electronic).

[11] Dragomir Ž. Đoković, The closure diagrams for nilpotent orbits of real forms of E_6, J. Lie Theory 11 (2) (2001) 381–413.

[12] Dragomir Ž. Đoković, The closure diagrams for nilpotent orbits of the real forms E VI and E VII of E_7, Represent. Theory 5 (2001) 17–42 (electronic).

[13] Dragomir Ž. Đoković, Correction to: "The closure diagrams for nilpotent orbits of the real forms E VI and E VII of E_7" [Represent. Theory 5 (2001) 17–42], Represent. Theory 5 (2001) 503 (electronic).

[14] Dragomir Ž. Đoković, The closure diagram for nilpotent orbits of the split real form of E_8, Cent. Eur. J. Math. 1 (4) (2003) 573–643 (electronic).

[15] Dragomir Ž. Đoković, Corrections for "The closure diagram for nilpotent orbits of the split real form of E_8" [Cent. Eur. J. Math. 1 (4) (2003) 573–643] (electronic), Cent. Eur. J. Math. 3 (3) (2005) 578–579 (electronic).

[16] L.Yu. Galitski, D.A. Timashev, On classification of metabelian Lie algebras, J. Lie Theory 9 (1999) 125–156.

[17] The GAP Group, GAP – Groups, algorithms, and programming, Version 4.4, http://www.gap-system.org, 2004.

[18] Willem A. de Graaf, Constructing algebraic groups from their Lie algebras, J. Symbolic Comput. 44 (2009) 1223–1233.

[19] Willem A. de Graaf, SLA – Computing with simple Lie algebras. A GAP package, http://science.unitn.it/~degraaf/sla.html, 2009.

[20] Willem A. de Graaf, Computing representatives of nilpotent orbits of θ-groups, J. Symbolic Comput. 46 (2011) 438–458.
[21] Fritz Grunewald, Joyce O'Halloran, Varieties of nilpotent Lie algebras of dimension less than six, J. Algebra 112 (2) (1988) 315–325.
[22] Sigurdur Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Pure Appl. Math., vol. 80, Academic Press Inc., Harcourt Brace Jovanovich Publishers, New York, 1978.
[23] V.G. Kac, Automorphisms of finite order of semisimple Lie algebras, Funktsional. Anal. i Prilozhen. 3 (3) (1969) 94–96.
[24] V.G. Kac, Some remarks on nilpotent orbits, J. Algebra 64 (1) (1980) 190–213.
[25] G. Kempken, Eine Darstellung des Köchers $\tilde{A}_K$, Dissertation, Rheinischen Friedrich-Wilhelms-Universität, Bonn, 1981, Bonner Math. Schriften 137 (1982).
[26] P. Littelmann, An effective method to classify nilpotent orbits, in: Algorithms in Algebraic Geometry and Applications, Santander, 1994, in: Progr. Math., vol. 143, Birkhäuser, Basel, 1996, pp. 255–269.
[27] Kenzo Mizuno, The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8, Tokyo J. Math. 3 (2) (1980) 391–461.
[28] T. Ohta, Orbits, rings of invariants and Weyl groups for classical θ-groups, Tohoku Math. J. 62 (2010) 527–558.
[29] D.D. Pervushin, Invariants and orbits of the standard $(\mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}))$-module, Izv. Ross. Akad. Nauk Ser. Mat. 64 (5) (2000) 133–146; English translation: Izv. Math. 64 (5) (2000) 1003–1015.
[30] D.D. Pervushin, On the closures of the nilpotent orbits of fourth-order matrix pencils, Izv. Ross. Akad. Nauk Ser. Mat. 66 (5) (2002) 183–192; English translation: Izv. Math. 66 (5) (2002) 1047–1055.
[31] V.L. Popov, Two orbits: When is one in the closure of the other?, in: Mnogomernaya Algebraicheskaya Geometriya, Tr. Mat. Inst. Steklova 264 (2009) 152–164; English translation: Proc. Steklov Inst. Math. 264 (1) (2009) 146–158.
[32] D.M. Snow, Weyl group orbits, ACM Trans. Math. Software 16 (1) (1990) 94–108.
[33] Nicolas Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Math., vol. 946, Springer-Verlag, Berlin, 1982.
[34] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, Bull. Soc. Math. France 98 (1970) 81–116.
[35] È.B. Vinberg, The Weyl group of a graded Lie algebra, Izv. Akad. Nauk SSSR Ser. Mat. 40 (3) (1976) 488–526; English translation: Math. USSR-Izv. 10 (1976) 463–495.
[36] È.B. Vinberg, Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra, Trudy Sem. Vektor. Tenzor. Anal. 19 (1979) 155–177; English translation: Selecta Math. Sov. 6 (1987) 15–35.
[37] È.B. Vinberg, V.V. Gorbatsevich, A.L. Onishchik, Lie Groups and Lie Algebras, III, Sovrem. Probl. Mat. Fund. Naprav., vol. 41, VINITI, Moskva, 1990; English translation: Encyclopaedia Math. Sci., vol. 41, Springer, Berlin, 1994.
[38] È.B. Vinberg, V.L. Popov, Invariant theory, in: Algebraic Geometry, IV, in: Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 137–314; English translation: Encyclopaedia Math. Sci., vol. 55, Springer, Berlin, 1994.
[39] È.B. Vinberg, A.G. Èlašvili, A classification of the three-vectors of nine-dimensional space, Trudy Sem. Vektor. Tenzor. Anal. 18 (1978) 197–233; English translation: Selecta Math. Sov. 7 (1988) 63–98.