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Generator Ideals in Noetherian PI Rings

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INTRODUCTION

The present paper evolved from the examination of results in two seemingly unrelated areas. The first relates to the problem of whether, in a semi-prime Noetherian ring, a two-sided ideal principal on one side is also principal on the other. This question was posed by Montgomery [M] in 1988. A positive answer for prime affine Noetherian polynomial identity (PI) rings was provided by Guralnick *et al*. [GRS] in 1992. Earlier this question had already been settled in the classical order case and the homologically homogeneous case by Fröhlich *et al*. [FRU] and Vasconcelos [V] in their respective papers of 1974 and 1971.

The second area concerns localisation. In particular, Braun and Hajarnavis have shown in [BH, Proposition 4.3] that, in the presence of an Artinian quotient ring, a maximal two-sided ideal projective and generator as a right module is left localisable. This develops earlier results of Hajarnavis [H] for semi-prime Noetherian PI rings. Moreover, recently

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Chatters *et al.* [CHL] have managed to cast further light on this phenomenon. The relation between the two themes puzzled us and eventually led to the results of this paper.

Recall that M_R is a generator if R is a right direct summand of $M_R^{(n)}$ (the nth direct sum of M) for some positive n. Also, an ideal I satisfies the left Artin–Rees (AR) property if for every left ideal L in R, $I^k \cap L \subseteq IL$ for some k.

THEOREM A. *Let R be a Noetherian PI ring and let I be a two-sided ideal which is a right generator. Then I is right projective and satisfies the left AR property.*

Several corollaries are deduced from this result, and by combining two of them with results of [GM], we get the following result.

COROLLARY B. *Let R be a Noetherian PI ring with finite Krull dimension. Let I be a right generator two-sided ideal in R, such that $I_R^{(n)}$ is generated by n (or fewer) elements. Then $I_R^{(n)}$ is free and I^m is right stably free for some m.*

Theorem A depends on our next result, which seems to be the ultimate generalization (in the PI case) of Montgomery's question [M].

THEOREM C. *Let R be a semi-prime Noetherian PI ring and let I be a two-sided ideal which is a right generator. Then I is invertible and in particular ${}_RI$, I_R are both projective generators.*

We mention two corollaries of Theorem C.

COROLLARY D. *Let R be a semi-prime Noetherian PI ring and let ϕ be a ring endomorphism of R satisfying $\phi(z) = z$ for every z in $Z(R)$ the center of R. Then ϕ is an automorphism of R.*

Recall that this generalizes [V, Thm. 5.1].

COROLLARY E. *Let R be a semi-prime Noetherian PI ring and let I be a two-sided ideal in R. Then I is right stably free if and only if I is left stably free.*

Our final major result goes in the opposite direction. Here we assume that I_R is projective and we deduce some generator properties of a power of I as follows.

THEOREM F. *Let R be a Noetherian PI ring and let I be a two-sided ideal in R which is right projective. Then either*

(i) $I^k = I^{k+1}$, *for some k, or*

(ii) $I^m S$ *is a proper two-sided ideal in S and is a right S-(pro)generator, where* $S = \mathrm{End}_R(I_R^m)$, *for some m.*

One should remark that by a standard result I^m is a left S-generator; however, Theorem F(ii) deals with a generator property on the right side. The next result is a non-trivial consequence.

THEOREM G. *Let R be an affine Noetherian PI ring and let I be a two-sided ideal which is right projective and contains a regular element. Then either*

(i) $I^k = I^{k+1}$, *for some k, or*

(ii) *little height* $(I) \leq 1$.

The definition of little height is given before Theorem 3.8. For further background material we refer to [GW, McR].

1. GENERATOR IDEALS ARE INVERTIBLE

The main result of the present section is that generator ideals are invertible in a semi-prime Noetherian PI ring. We shall then elaborate on some applications. In particular results of [V, GRS, GM] are generalized.

DEFINITION 1.1. We say that a right R-module M is a generator if there exists an integer n and an onto R-module map ν: $M^{(n)} \to R$. Equivalently, M is a generator if $M^*M = R$, where $M^* = \operatorname{Hom}_R(M_R, R_R)$ and $M^*M = \{\Sigma f(m) \mid f \in M^*, m \in M\} \equiv$ the trace ideal of M.

Let $N(R)$ or N denote the nil radical of R. For an ideal J we denote $\mathscr{C}(J) = \{c \in R \mid c \text{ is regular mod } J\}$.

LEMMA 1.2. *Let R be a right Noetherian PI ring and let I be a two-sided ideal in R with $I^*I = R$. Then $I \cap \mathscr{C}(N(R)) \neq \phi$.*

Proof. Let ρ stand for the right Goldie rank. Consider the sequence $\{\rho(I^i) \mid i = 1, 2, \ldots\}$. This, being a decreasing sequence of positive integers, must stabilize. Hence $\rho(I^m) = \rho(I^{m+1})$, that is, I^m/I^{m+1} is a right $\mathscr{C}(N)$-torsion module. Now the H-condition on R [GW, Proposition 8.9] implies that $I^m c \subseteq I^{m+1}$ for some $c \in \mathscr{C}(N(R))$. By applying I^* on the last inclusion one gets $I^{m-1}c \subseteq I^m$. Iterating this process $m - 1$ more times yields the required result $c \in I$. Q.E.D.

A similar argument verifies the next lemma. This will be used in Sections 2 and 3.

LEMMA 1.3. *Let R be a right Noetherian PI ring and let I and K be two-sided ideals in R. Suppose that $I^*I = R$. Then $(I \cap K)d \subseteq IK$ for some $d \in \mathscr{C}(N(R))$.*

Proof. One considers the decreasing sequence $\{\rho(I^iK)\}$. So, as in the previous lemma, $I^mKd \subseteq I^{m+1}K$. This yields, after m successive applications of $I^*I = R$, that $Kd \subseteq IK$ for some $d \in \mathscr{C}(N(R))$. Consequently $(I \cap K)d \subseteq IK$. Q.E.D.

Remark 1.4. Let R be a semi-prime Noetherian ring and let I be a two-sided ideal in R satisfying $I \cap \mathscr{C}(0) \neq \phi$. Then we can and will identify $I^* \equiv \mathrm{Hom}_R(I_R, R_R)$ with $\{q \in Q(R) \mid qI \subseteq R\}$, where $Q(R)$ stands for the classical Artinian quotient ring of R. In fact each $f \in I^*$ extends to a unique element in $\mathrm{Hom}_{Q(R)}(Q(R)_{Q(R)}, Q(R)_{Q(R)})$ and it can clearly be identified with left multiplication by elements of $Q(R)$. Similarly $I^+ \equiv \mathrm{Hom}_R({}_RI, R_R) = \{q \in Q(R) \mid Iq \subseteq R\}$.

LEMMA 1.5. *Let R be a semi-prime Noetherian ring and let I be a two-sided ideal in R with $I \cap \mathscr{C}(0) \neq \phi$. Let $S = \sum_{i=0}^{\infty} I^i(I^*)^i$, where $I^0 \equiv R \equiv (I^*)^0$. Then S is a subring of $Q(R)$ which contains R.*

Proof. The proof is left to the reader.

Our next result is crucial; we prefer to do it, at this stage, for prime rings only. We denote by $T(R)$ the trace ring of R.

PROPOSITION 1.6. *Let R be a prime Noetherian PI ring and let I be a two-sided ideal. Let $S = \sum_{i=0}^{\infty} I^i(I^*)^i$ be as in Lemma 1.5. Then S has a finite left and right R-module.*

Proof. Let $n = \mathrm{PI}\deg R$. By [A2] there exist a non-zero evaluation δ of a central polynomial on R and a free basis of $Q(R)$ over $Q(Z(R))$, consisting of elements $u_1, \ldots, u_{n^2}$, of R, satisfying $\delta R \subseteq Z(R)u_1 + \cdots + Z(R)u_{n^2}$. Moreover we have $\delta A \subseteq Z(A)u_1 + \cdots + Z(A)u_{n^2}$, for any ring A such that $R \subseteq A \subseteq Q(R)$. Now $T(R) = RZ(T(R))$ and $T(S) = SZ(T(S))$. So by the previous observations we have

(i) $\delta T(R) \subseteq Z(T(R))u_1 + \cdots + Z(T(R))u_{n^2}$, and

(ii) $\delta T(S) \subseteq Z(T(S))u_1 + \cdots + Z(T(S))u_{n^2}$.

Let $c_j(x)$ denote the jth coefficient of the Cayley-Hamilton equation of x. Given $y \in S$ then $y = \sum a_i q_i$, where a_i is a product of i elements of I and q_i is a product of i elements of I^*. In particular $a_i \in I^i$ and $q_i \in (I^*)^i$. Observe that we do not exclude the possibility that different products $a_i q_i$ and $a_i' q_i'$, with the same i, will appear in the sum $y = \sum a_i q_i$. Now by [A1], $c_j(y) = c_j(\sum a_i q_i) =$ a polynomial in several c_α (product of $a_i q_i$'s), where α varies over several integers. Let $a_{k_1} q_{k_1} \cdots a_{k_t} q_{k_t}$ be such a typical product and let m denote $\max\{k_i, 1 \leq i \leq t\}$. Then it is easily seen, using $I^*I \subseteq R$, that $a_{k_1} q_{k_1} \cdots a_{k_t} q_{k_t} = aq$, where $a \in I^m$, $q \in (I^*)^m$. Consequently, $c_\alpha(a_{k_1}q_{k_1} \cdots a_{k_t}q_{k_t}) = c_\alpha(aq) = c_\alpha(qa) \in Z(T(R))$, where the last inclusion holds since $qa \in R$. Consequently

(iii) $c_j(y) \in Z(T(R))$, for each $y \in S$ and $1 \leq j \leq n$.

Let $a \in R$ be a fixed element of R with $\mathrm{tr}(a) \neq 0$, and let $x \in Z(T(S))$ be arbitrary. So $xa \in T(S) = S[c_i(s) \mid s \in S, 1 \leq i \leq n]$. Consequently $xa = \Sigma s_j p_j$, where $s_j \in S$ and p_j are monomials in $\{c_i(s) \mid s \in S, 1 \leq i \leq n\}$. By (iii), we have that for each j, $p_j, \mathrm{tr}(s_j) \in Z(T(R))$. Therefore $x\,\mathrm{tr}(a) = \mathrm{tr}(xa) = \mathrm{tr}(\Sigma s_j p_j) = \Sigma_j \mathrm{tr}(s_j)p_j \in Z(T(R))$, for each $x \in Z(T(S))$. Hence

(iv) $\mathrm{tr}(a)Z(T(S)) \subseteq Z(T(R))$.

Consequently, using (ii) and (iv), we have

$$\begin{aligned}\mathrm{tr}(a)\delta T(S) &\subseteq \mathrm{tr}(a)Z(T(S))u_1 + \cdots + \mathrm{tr}(a)Z(T(S))u_{n^2}\\ &\subseteq Z(T(R))u_1 + \cdots + Z(T(R))u_{n^2}\\ &\subseteq T(R)u_1 + \cdots + T(R)u_{n^2}.\end{aligned}$$

Therefore, $T(S)$ is contained in a finite $T(R)$-module. Now R being Noetherian implies that $T(R)$ is a finite R-module, and we conclude that $T(S)$ is contained in a finite R-module. The similar statement about S is now obvious. Q.E.D.

The next result is a major consequence of the above.

THEOREM 1.7. *Let R be a prime Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$. Then I is invertible and in particular $II^+ = R$, $I^* = I^+$ and I is projective on both sides.*

Proof. Let $S = \Sigma_{i=0}^{\infty} I^i(I^*)^i$. Then by Proposition 1.6, we have $S = \Sigma_{i=0}^{k-1} I^i(I^*)^i$ for some k; that is, $I^k(I^*)^k \subseteq \Sigma_{i=0}^{k-1} I^i(I^*)^i$. We now multiply this by $(I^*)^{k-1}$ on the left and by I^{k-1} on the right. Using Remark 1.4 and the hypothesis $I^*I = R$, we obtain

$$II^* = (I^*)^{k-1} I^k (I^*)^k I^{k-1} \subseteq \sum_{i=0}^{k-1} (I^*)^{k-1} I^i (I^*)^i I^{k-1} \subseteq R.$$

Similarly, we have $I^{k+m-1}(I^*)^{k+m-1} \subseteq \Sigma_{i=0}^{k-1} I^i(I^*)^i$, for each $m \geq 1$ which yields $I^m(I^*)^m \subseteq R$ for each m. It is easy to verify now, using $I^*I = R$, that $I^m(I^*)^m$ is an idempotent ideal in R for each m. Consequently by [RS, Theorem 3] we get $I^m(I^*)^m = I^{m+r}(I^*)^{m+r}$ for some positive integers m, r. Applying $(I^*)^{m+r-1}$ on the left and I^{m+r-1} on the right of the last equality gives $R = II^*$. This shows that $I^* \subseteq I^+$ and that $R = II^+$. The symmetric argument now gives the inclusion $I^+ \subseteq I^*$. Therefore

$I^* = I^+$, and I is invertible. Finally $1 \in R = II^*$ shows that I_R is projective and $1 \in R = I^+I$ shows that ${}_RI$ is projective. Q.E.D.

The next proposition generalizes [GRS].

PROPOSITION 1.8. *Let R be a prime Noetherian PI ring with $aR \supseteq Ra$ for some $a \in R$. Then $aR = Ra$.*

Proof. $I \equiv aR$ is a two-sided ideal in R and hence $1\text{-ann}_R(a) = \{0\}$. Consequently, since R is a prime PI ring, we have $\text{r-ann}(a) = 0$ and a is regular. Clearly $I^* = \{q \in Q(R) \mid qI \subseteq R\} = Ra^{-1}$, where $a^{-1} \in Q(R)$. So, by Theorem 1.7, $II^* = R$. Equivalently $(aR)(Ra^{-1}) = R$. Hence $aRa^{-1} \subseteq R$ and therefore $aR \subseteq Ra$. Q.E.D.

Our next result is an equivalent formulation using endomorphisms.

PROPOSITION 1.9. *Let R be a prime Noetherian PI ring and let ϕ be a ring endomorphism of R satisfying $\phi(z) = z$, for all $z \in Z(R)$. Then ϕ is an isomorphism.*

Proof. Let $q \in Q(R)$. Then $q = xc^{-1}$, for some $x \in R$, $c \in Z(R)$. Therefore, using the Ore condition, $\tilde{\phi}(q) = \phi(x)c^{-1}$ is a well-defined extension of ϕ to $Q(R)$. Moreover $\tilde{\phi}$ is a ring homomorphism which fixes $Z(Q(R)) = Q(Z(R))$ elementwise. Also, $\tilde{\phi}$ is injective since otherwise we would have $\ker \phi \cap Z(R) \neq \{0\}$. Now since $Q(R)$ is finite dimensional over $Z(Q(R))$ we get that $\tilde{\phi}$ is onto. By the Skolem–Noether theorem $\tilde{\phi}(x) = yxy^{-1}$, for each $x \in Q(R)$ and some fixed regular y in $Q(R)$. Now $y = ae^{-1}$, $a \in R$, $e \in Z(R)$, and this implies that $\tilde{\phi}(x) = axa^{-1}$, for each $x \in Q(R)$. Consequently $\phi(r) = ara^{-1}$ for each $r \in R$. Now since, for all r in R, $\phi(r) \in R$, we have $aRa^{-1} \subseteq R$; that is, $aR \subseteq Ra$. Proposition 1.8 grants that $aR = Ra$, which shows that ϕ is onto. Q.E.D.

Our next natural task will be the generalization of Theorem 1.7 to the semi-prime case. This in turn will provide a generalization of Proposition 1.9 to the semi-prime case as well.

LEMMA 1.10. *Let R be a right Noetherian PI ring and let I be a two-sided ideal in R satisfying $I^*I = R$. Let K be either $N(R)$ or a minimal prime in R. Then each $f \in I^*$ induces a well-defined map $\bar{f} \in (\bar{I})^*$ where $\bar{I} = (I + K)/K$, and $\bar{R} \equiv R/K$. Moreover $(\bar{I})^*(\bar{I}) = \bar{R}$.*

Proof. Let $a \in I$ and $\bar{a} = a + K \in \bar{R}$. Define $\bar{f}(\bar{a}) \equiv f(a) + K$. If $a \in I \cap K$, then by Lemma 1.3, $ad \in IK$ for some $d \in \mathscr{C}(N(R))$. So $f(a)d = f(ad) \in f(IK) \subseteq f(I)K \subseteq K$. Hence, since $d \in \mathscr{C}(N(R))$, we get $f(a) \in K$. Consequently $\bar{f}$ is a well-defined $\bar{R}$-module map on $\bar{I}$; that is $\bar{f} \in (\bar{I})^*$. Now $I^*I = R$ translates to $\Sigma_{i=1}^{k} f_i(a_i) = 1$, for some $f_i \in I^*$, $a_i \in I$. Consequently $\Sigma \bar{f}_i(\bar{a}_i) = \Sigma f_i(a_i) + K = 1 + K = \bar{1}$. That is, $(\bar{I})^*\bar{I} = \bar{R}$. Q.E.D.

COROLLARY 1.11. *Let R be a Noetherian semi-prime PI ring, P a minimal prime ideal, and I a two-sided ideal in R with $I^*I = R$. Then $\overline{I^*} \subseteq (\bar{I})^*$, where $\bar{I} \equiv (I+P)/P$ and $\overline{I^*}$ is the image of I^* in $Q(R)/PQ(R) \equiv \overline{Q(R)} \cong Q(R/P)$.*

Proof. By Remark 1.4 every $f \in I^*$ is identified with a left multiplication by an element $q \in Q(R)$. It is now clear that $\bar{f}$, appearing in Lemma 1.10, is identified with $\bar{q} \equiv q + PQ(R)$ in $\overline{Q(R)}$. Q.E.D.

Our next result is the generalization of Theorem 1.7 to the semi-prime case.

THEOREM 1.12. *Let R be a semi-prime Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$. Then I is invertible.*

Proof. Let $\{P_1, \dots, P_t\}$ be the set of minimal prime ideals in R. It is standard that $R \subseteq R/P_1 \oplus \cdots \oplus R/P_t$ is a central extension of R. Let $\overline{I}_i \equiv I + P_i/P_i$ for each $1 \le i \le t$. By Lemma 1.10 we have that $(\overline{I}_i)^*\overline{I}_i \equiv R/P_i$, for each $1 \le i \le t$. Consequently, by Theorem 1.7, $\overline{I}_i(\overline{I}_i)^* = R/P_i$, for each $1 \le i \le t$. So, by Corollary 1.11, $II^* \subseteq R/P_1 \oplus \cdots \oplus R/P_t$. Similarly $I^i(I^*)^i \subseteq R/P_1 \oplus \cdots \oplus R/P_t$, for each i. Let $S \equiv \sum_{i=0}^{\infty} I^i(I^*)^i$ be as in Theorem 1.7. The finiteness of $R/P_1 \oplus \cdots \oplus R/P_t$ as a right (and left) R-module shows that $S = \sum_{i=0}^{k-1} I^i(I^*)^i$, for some k. The rest of the argument, identical to the one used in Theorem 1.7, is therefore omitted. Q.E.D.

The following is an attractive reformulation of Theorem 1.12.

THEOREM 1.12′. *Let R be a semi-prime Noetherian PI ring. Suppose that $JI = zR$ for some $z \in Z(R) \cap \mathscr{C}(0)$, where I, J are two-sided ideals in R. Then $IJ = JI$.*

Proof. Clearly $z^{-1}J \subseteq I^*$ implies $I^*I = R$. Hence, by Theorem 1.12, $II^* = R$. Therefore $z^{-1}J = (z^{-1}J)II^* = I^*$. So, by using $II^* = R$ again, we get $IJ = zR$. Q.E.D.

We shall now draw several corollaries out of Theorem 1.12.

PROPOSITION 1.13. *Let R be a semi-prime Noetherian PI ring and let I be a two-sided ideal satisfying $I_R^{(n)} \cong R_R^{(t)}$, for some integers n and t. Then $n = t$ and ${}_RI^{(n)} \cong {}_RR^{(n)}$.*

Proof. By Lemma 1.2 we have $I \cap \mathscr{C}(0) \neq \phi$. So $\rho(R/I) = 0$, where ρ is the right Goldie rank for R-modules. Equivalently, we have $\rho(R) = \rho(I)$, which implies that $n\rho(R) = n\rho(I) = \rho(I^{(n)}) = \rho(R^{(t)}) = t\rho(R)$. So $n = t$ as claimed. Now $I_R^{(n)} \cong R_R^{(n)}$ clearly shows that $I^*I = R$ and therefore, by

Theorem 1.12, I is invertible. The conclusion ${}_R I^{(n)} \cong {}_R R^{(n)}$ is standard, using for example [CR2, Sect. 55] or [GM, Lemma 2.4]. Q.E.D.

Recall that a right R-module M is said to be stably free if $M \oplus R^{(s)} \cong R^{(t)}$, for some integers s and t.

PROPOSITION 1.14. *Let R be a semi-prime Noetherian PI ring and let I be a two-sided ideal which is a stably free right R-module. Then I is a stably free left R-module.*

Proof. By [L, Thm. 1], there exists an integer m such that $I_R^{(n)}$ is free for each $n \geq m$. Therefore, by Proposition 1.13, ${}_R I^{(m)} \cong {}_R R^{(m)}$ and ${}_R I^{(m+1)} \cong {}_R R^{(m+1)}$. So, by substitution, we get ${}_R I \oplus {}_R R^{(m)} \cong {}_R R^{(m+1)}$ and ${}_R I$ is left stably free. Q.E.D.

Remark. Note that there are stably free two-sided ideals which are not free (e.g., [GM, Prop. 5.7], [C, Sect. 6], or [CP, Sect. 3]).

The next result generalizes [GM, Theorems 3.6, 3.7] by removing the affine and semi-prime assumptions on R and the invertibility assumption on I. We do use, however, the results of [GM].

THEOREM 1.15. *Let R be a right Noetherian PI ring with finite Krull dimension and let I be a two-sided ideal satisfying $I_R^{(n)} \cong R_R^{(n)}$, for some n. Then I^m is a stably free right R-module, for some m. Moreover, if R is in addition affine, $m = n^u((PI \deg R)!)$ and $u > \log_2(K.\dim R)$.*

Proof. $I_R^{(n)} \cong R_R^{(n)}$ implies that $I^*I = R$ and also that I_R is projective. By Lemma 1.3, $(I \cap N(R))d \subseteq IN(R)$, for some $d \in \mathscr{C}(N(R))$. This shows that $(I \cap N(R))/IN(R)$ is a torsion $R/N(R)$-submodule of $I/IN(R)$. However, the latter, being a finitely generated projective $R/N(R)$-module, is torsion free. Consequently, $I \cap N(R) = IN(R)$. Similarly $I^i \cap N(R) = I^i N(R)$ for each i. Let $\bar{I} \equiv (I + N(R))/N(R) \subseteq R/N(R) \equiv \bar{R}$. Then the previous observations show that $\bar{I}_{\bar{R}}^{(n)} \cong \bar{R}_{\bar{R}}^{(n)}$. Hence $(\bar{I})^*\bar{I} = \bar{R}$ and, by Theorem 1.12, $\bar{I}$ is invertible. Now the proof of [GM, Theorem 3.6] shows that $\bar{I}^m$ is a stably free right R-module for some m. That is, $\bar{I}_{\bar{R}}^m \oplus \bar{R}_{\bar{R}}^{(s)} \cong \bar{R}_{\bar{R}}^{(t)}$. Now $I^m \cap N(R) = I^m N(R)$ implies that $I^m/I^m N(R) = I^m/(I^m \cap N(R)) \cong (I^m + N(R))/N(R) = \overline{I^m} = \bar{I}^m$, as right R-module. Let $F_1 \equiv I^m \oplus R^{(s)}$, $F_2 = R^{(t)}$. Then F_1, F_2 are right projective R-modules which, by the previous observations, satisfy $F_1/F_1 N(R) \cong F_2/F_2 N(R)$ as right $\bar{R}$-modules. A standard result now implies that $F_1 \cong F_2$, as right R-modules, and so I^m is right stably free. Q.E.D.

Remark 1.16. In all of the main symmetry results of the present section (e.g., Theorem 1.12, Proposition 1.13, and Proposition 1.14) I is assumed to be a two-sided ideal. It is rather easy to give examples of two-sided ideals which are right projective but not left projective. However, it is still possible for the property of being free to be transferable to higher rank bimodules. In particular, let X be a central finitely generated (on both sides) bimodule over a prime Noetherian PI ring. Suppose that $X_R \cong R_R^{(n)}$ for some n. Is ${}_RX \cong {}_RR^{(n)}$? Recall that Theorem 1.7 provides a positive answer in the case $n = 1$. Actually it is proven in [AS, Corollary 1.13] that the left projectivity of a special central bimodule (of the form $R^{(n)}/y$, where $y \subset R^{(n)}$ is a central R-subbimodule of $R^{(n)}$) implies its right projectivity, when R is an affine PI algebra. However, this central bimodule is rather special and this result does not seem to bear on any result of the present paper.

2. GENERATOR IDEALS, PROJECTIVITY, AND THE AR PROPERTY

The main result of the present section asserts that a right generator two-sided ideal in a Noetherian PI ring is right projective and satisfies the left AR property. While the part referring to the AR property generalizes several earlier results in the literature, the conclusion of projectivity seems to be new.

Remark 2.1. Given R a Noetherian PI ring and I a two-sided ideal satisfying $I^*I = R$, we shall consider $\mathrm{End}_R(I_R) \subseteq \mathrm{Hom}_R(I_R, R_R) \equiv I^*$. Clearly both are R-bimodules, and the former is a ring as well. We shall use the convention of applying $f \in I^*$ on elements of I on the left. This will result in $(f \cdot g)(i) = f(g(i))$, for each $f, g \in \mathrm{End}_R(I_R)$, $i \in I$. We consider the map $x \to t_x$, where $t_x(i) = xi$, for $x \in R$, $i \in T$, is the left multiplication by x. Then $x \to t_x$ furnishes an anti-homomorphism from R to $\mathrm{End}_R(I_R)$. By abuse of notation we write $x \cdot f$ for $t_x \cdot f$, where $x \in R$, $f \in \mathrm{End}_R(I_R)$. Moreover, if $f, g \in I^*$ and $y \in I$, then $fyg \in I^*$ and $fyg = f(y)g$. Indeed, for each $i \in I$, $(fyg)(i) = (ft_y g)(i) = f(yg(i)) = f(y)g(i)$; that is, $fyg = t_{f(y)}g = f(y)g$.

Let $S \equiv R + II^*$. Here by R we mean $\{t_x \mid x \in R\}$. Clearly S is a subring of $\mathrm{End}_R(I)$. Moreover I is a left ideal in S, as well as in $\mathrm{End}_R(I_R)$. Indeed let $f \in \mathrm{End}_R(I_R)$, $a \in I$ and $i \in I$; then $(f \cdot t_a)(i) = f(ai) = f(a)i = t_{f(a)}(i)$, for each i. Hence $f \cdot t_a = t_{f(a)}$ and by abuse of notation $f \cdot a = f(a) \in I$.

LEMMA 2.2. *Let R be a right Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$. Then $f(I \cap N(R)) \subseteq N(R)$, for each $f \in I^*$.*

Proof. By Lemma 1.3, $(I \cap N(R))d \subseteq IN(R)$ for some $d \in \mathscr{C}N(R))$. Consequently $f(I \cap N(R))d = f((I \cap N(R))d) \subseteq f(IN(R)) \subseteq f(I)N(R) \subseteq$

$N(R)$. Now the fact that $d \in \mathscr{C}(N(R))$ forces the conclusion $f(I \cap N(R)) \subseteq N(R)$. Q.E.D.

Remark 2.3. Let R and I be as in Lemma 2.2 and let $f \in I^*$. Then f induces a well-defined map $\bar{f} \in (\bar{I})^*$, where $\bar{I} \equiv (I + N(R))/N(R) \subseteq \overline{R} = R/N(R)$, via $\bar{f}(\bar{x}) = f(x) + N(R)$, for each $x \in I$. Moreover $(\bar{I})^* \bar{I} = \overline{R}$.

Proof. The above is in fact the content of Lemma 1.10.

LEMMA 2.4. *Let R be a right Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$. Then $(\bar{I})^* = \{\bar{f} \mid f \in I^*\}$, where $\bar{f}$ is defined in Remark* 2.3.

Proof. Let $f_i \in I^*$ and $a_i \in I$ be such that $\Sigma f_i(a_i) = 1$. Hence $\Sigma \overline{f_i}(\overline{a_i}) = \bar{1}$, in $\overline{R}$. That is, $(\bar{I})^* \bar{I} = \overline{R}$. Consequently, by Theorem 1.12, $\bar{I}(\bar{I})^* = \overline{R}$. Now $(\bar{I})^* \subseteq Q(\overline{R})$, so each $g \in (\bar{I})^*$ is identified with left multiplication by an element of $Q(\overline{R})$. Say $\bar{f}_i$ is identified with $q_i \in Q(\overline{R})$, for each i. Then $\Sigma \bar{f}_i(\bar{a}_i) = \bar{1}$ yields $\Sigma q_i(\overline{a_i}) = \bar{1}$. Let $g \in (\bar{I})^*$ be an arbitrary element. Then $g = \bar{1}g = \Sigma q_i \bar{a}_i g = \Sigma q_i(\bar{a}_i g)$. Now $\overline{a_i}g \in \bar{I}(\bar{I})^* \subseteq \overline{R}$, for each i, implies that $g \in \Sigma q_i \overline{R} = \Sigma \overline{f_i}\overline{R} \subseteq \{\bar{f} \mid f \in I^*\}$. So, $(\bar{I})^* \subseteq \{\bar{f} \mid f \in I^*\}$. The reverse inclusion is given by Remark 2.3. Q.E.D.

LEMMA 2.5. *Let R be a Noetherian PI ring and let $S \equiv R + II^*$, where R is considered as a subring of* $\operatorname{End}_R(I_R)$ *as explained in Remark* 2.1. *Then S is a Noetherian PI ring.*

Proof. S is an R-subbimodule of $\operatorname{End}_R(I_R)$. Now by [SZ, Theorem 3.5] $\operatorname{End}_R(I_R)$ is a finitely generated left as well as right R-module. Q.E.D.

LEMMA 2.6. *Let R be a Noetherian PI ring with $I^*I = R$ and $S = R + II^*$. Then IS is a two-sided ideal in S satisfying $(IS)^*(IS) = S$.*

Proof. By Remark 2.1, I is a left ideal in S (recall that we identify here I with $\{t_x \mid x \in I\}$). So IS is a two-sided ideal in S. Let $g \in I^*$, $f \in S$, and $a \in I$. We define $g(af) \equiv gaf$, where gaf is understood as the composition of the three maps g, t_a, f (which is in S). It is clearly well defined and the considerations in Remark 2.1 show that $gaf = g(a)f$. It is also evident that g induces a well-defined element in $(IS)^*$. Finally, choose $g_i \in I^*$, $a_i \in I$, satisfying $\Sigma g_i(a_i) = 1$. Then $\Sigma g_i(a_i f) = \Sigma g_i(a_i) f = f$, for each $f \in S$. Consequently $\Sigma g_i(a_i \cdot 1_s) = 1_s$ and $(IS)^*(IS) = S$. Q.E.D.

COROLLARY 2.7. *Let R be a Noetherian PI ring and $S = R + II^*$, where I is a two-sided ideal in R satisfying $I^*I = R$. Then $IS \cap \mathscr{C}(N(S)) \neq \phi$.*

Proof. By Lemma 2.5 S is a Noetherian PI ring. By Lemma 2.6, $(IS)^*(IS) = S$. So by Lemma 1.2 (applied to IS and S), we get

$$IS \cap \mathscr{C}(N(S)) \neq \phi. \qquad \text{Q.E.D.}$$

LEMMA 2.8. *Let R be a Noetherian PI ring and let I be a two-sided ideal in R satisfying $I^*I = R$. Then $IN(R)I^*$ is a two-sided nilpotent ideal in $S \equiv R + II^*$.*

Proof. It is clear, using $SI = I$ and $I^*S = I^*$, that $IN(R)I^*$ is a two-sided ideal in S. Let t satisfy $N(R)^t = \{0\}$. Then,

$$(IN(R)I^*)^t = IN(R)(I^*I)N(R)(I^*I)\cdots(I^*I)N(R)I^*$$
$$\subseteq I\big(N(R)^t\big)I^* = \{0\}. \qquad \text{Q.E.D.}$$

The next result is crucial.

PROPOSITION 2.9. *Let R be a Noetherian PI ring and let I be a two-sided ideal in R satisfying $I^*I = R$. Let $S = R + II^*$. Then $N(R)S \subseteq N(S)$.*

Proof. We shall first show that $II^*N(R)^{i+1}II^* \subseteq N(S)$, for each $i \geq 0$. Indeed $N(R)^{i+1}I \subseteq I \cap N(R)$ and hence, by Lemma 2.2, $I^*N(R)^{i+1}I \subseteq I^*(I \cap N(R)) \subseteq N(R)$. So, by Lemma 2.8, $I(I^*N(R)^{i+1}I)I^* \subseteq IN(R)I^* \subseteq N(S)$.

Next let t satisfy $N(R)^t = \{0\}$. We shall show that $(N(R)S)^{2t+1} \subseteq N(S)$. This will settle the claim, since $N(R)S$ is a right ideal in S. Now $N(R)S = N(R) + N(R)II^*$. In the expansion of $(N(R) + N(R)II^*)^{2t+1}$ we either have terms where $N(R)II^*$ appears at least twice or terms where $N(R)II^*$ appears at most once. In the former possibility we have a typical term of the form $N(R)^{\alpha}(N(R)II^*)^{\beta}N(R)^{\gamma}(N(R)II^*)^{\delta}N(R)^{\varepsilon}$, where γ, β, δ are positive integers. Consequently it has a subexpression of the form $II^*N(R)^{i+1}II^*$, where $i \geq 0$. By the first paragraph $II^*N(R)^{i+1}II^*$ is in $N(S)$ and therefore the relevant term is in $N(S)$. In considering the latter possibility, one observes that all the terms where $N(R)II^*$ appears at most once are contained in $N(R)^{2t+1} + \sum_{i=0}^{2t+1}N(R)^i(N(R)II^*)N(R)^{2t+1-i-1}$. It is obvious that either $N(R)^i = 0$ or $N(R)^{2t+1-i-1} = 0$, for each $0 \leq i \leq 2t+1$, and the result is established. Q.E.D.

We shall now prove a major result of the present paper. It says that a right generator two-sided ideal is always right projective.

THEOREM 2.10. *Let R be a Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$ (that is, I_R is a generator). Then I_R is projective.*

Proof. By Remark 2.3 (or Lemma 1.10), $(\bar{I})^*\bar{I} = \bar{R}$, where $\bar{I} \equiv (I + N(R))/N(R)$ and $\bar{R} \equiv R/N(R)$. Consequently, by Theorem 1.12, $\bar{I}(\bar{I})^* = \bar{R}$. That is, $\bar{1} = \sum\overline{a_i}g_i$, where $a_i \in I$ and $g_i \in (\bar{I})^*$. By Lemma 2.4, we can find, for each i, $f_i \in I^*$, such that $\bar{f_i} = g_i$. So $\bar{1} = \sum\overline{a_i}\overline{f_i}$. Let u be an arbitrary element in I. Then $\bar{u} = \sum\overline{a_i}\overline{f_i}(\bar{u})$. Now, by definition, $\bar{f_i}(\bar{u})$

$= \overline{f_i(u)}$. Hence $\bar{u} = \Sigma \bar{a}_i \overline{f_i(u)}$, for each u in I. Equivalently, $u - \Sigma a_i f_i(u) \in N(R)$ for each $u \in I$. Let $h \equiv 1_I - \Sigma a_i f_i \in S$, where 1_I is the identity map on I (which is also the unit element of S). Then $h(u) \in N(R)$ for each $u \in I$. Therefore, $h \cdot u \cdot 1_s = h(u) 1_s \in N(R) \cdot 1_s$, for each $u \in I$. Hence $hIS \subseteq N(R)S \subseteq N(S)$, where the last inclusion is given by Proposition 2.9. Now Corollary 2.7 implies that $h \in N(S)$. Hence $1_I - h$ is invertible in S with inverse w; that is, $1_I = (1_I - h)w = \Sigma a_i f_i w$. It is clear that, for each i, $f_i w \in I^*$, and therefore $\{a_i, f_i w\}$ is a dual basis for I_R. Q.E.D.

Remark. Here we have used the two-sided Noetherian assumption on R, primarily to establish that $\phi \neq IS \cap \mathscr{C}(N(S))$ in Lemma 2.7. We wonder whether one can prove Theorem 2.10 with only a right Noetherian hypothesis on R. Our next result confirms this belief in the case $I = aR$. This result is, of course, a corollary of Theorem 2.10 if R is assumed to be Noetherian on both sides.

PROPOSITION 2.11. *Let R be a right Noetherian PI ring and let $I = aR$ be a two-sided ideal in R satisfying $I^* I = R$. Then I_R is free* (*that is* $\text{r-ann}_R(a) = \{0\}$).

Proof. By assumption $Ra \subseteq aR$. Hence $I^*(Ra) \subseteq I^*(aR) = I^* I = R$. By Lemma 1.2, $I \cap \mathscr{C}(N(R)) \neq \phi$ and consequently $a \in \mathscr{C}(N(R))$. Moreover, by Lemma 1.10, $(\bar{I})^* \bar{I} = \bar{R}$, where $\bar{R} = R/N(R)$ and $\bar{I} = (I + N(R))/N(R)$. (This can be shown directly, since $\bar{a}$ is regular in $\bar{R}$ and $\bar{R}(\bar{a})^{-1} = (\bar{I})^*$.) Consequently, by Theorem 1.12, $\bar{I}$ is invertible, which implies that $\bar{I}(\bar{I})^* \subseteq \bar{R}$. Equivalently $(\bar{a}\bar{R})(\bar{R}(\bar{a})^{-1}) \subseteq \bar{R}$, which shows that $\bar{a}\bar{R} = \bar{R}\bar{a}$. Therefore $aR \subseteq Ra + N(R)$ and hence, using $a \in \mathscr{C}(N(R))$, we have $aR \subseteq Ra + aN(R)$. So $R = I^*(aR) \subseteq I^*(Ra) + I^*(aN(R)) \subseteq I^*(Ra) + N(R)$. Hence $1 = x + n$, where $x \in I^*(Ra)$ and $n \in N(R)$. So $x = 1 - n$ is an invertible element in the left ideal $I^*(Ra)$, implying that $R = I^*(Ra)$. It is now evident that $\text{r-ann}_R(a) = \{0\}$. Q.E.D.

The next result is a curious consequence of Proposition 2.11.

PROPOSITION 2.12. *Let R be a right Noetherian PI ring and let I be a two-sided ideal satisfying $I^* I = R$. Then $I_R^{(n)}$ is free if and only if $I_R^{(n)}$ is generated by n* (*or fewer*) *elements*.

Proof. By adding elements we may assume that $I_R^{(n)} = e_1 R + \cdots + e_n R$. Let $\check{e}_i$ denote e_i written in column form, for each i. So in matrix form, we have

$$\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} = \left(\check{e}_1 \,|\, \check{e}_2 \,|\, \cdots \,|\, \check{e}_n \right) \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix}.$$

Consequently $M_n(I) = \alpha M_n(R)$, where $\alpha = (\check{e}_1 \mid \check{e}_2 \mid \cdots \mid \check{e}_n)$. Clearly $M_n(I)$ is a two-sided ideal in $M_n(R)$. Also $M_n(I^*) \subseteq M_n(I)^*$ and therefore $M_n(I)^* M_n(I) = M_n(R)$. So all the conditions of Proposition 2.11 are now satisfied and we conclude that r-ann$(\alpha) = 0$. Equivalently $\{e_1, \dots, e_n\}$ are free right generators for $I_R^{(n)}$. Conversely assume that $I_R^{(n)} \cong R_R^{(t)}$. Then, as in the proof of Proposition 1.13, $n = t$ and the result follows. Q.E.D.

We shall now proceed to prove the other major result of the present section, namely that I satisfies the left AR property [McR, 4.2.3] provided that $I^*I = R$. This will be a consequence of Theorem 2.10. We start with the following corollary of Theorem 2.10.

LEMMA 2.13. *Let R be a Noetherian PI ring and let I be a two-sided ideal in R satisfying $I^*I = R$. Then $I \cap N(R) = IN(R)$.*

Proof. By Lemma 1.3, $(I \cap N(R))/IN(R)$ is an $R/N(R)$-torsion submodule of $I/IN(R)$. However, by Theorem 2.10, I_R is projective and hence $I/IN(R)$ is a finitely generated projective $R/N(R)$-module. This is impossible unless $(I \cap N(R))/IN(R) = 0$. Q.E.D.

PROPOSITION 2.14. *Let R be a Noetherian PI ring and I a two-sided ideal in R. Suppose that $\bar{I}$ satisfies the left AR property in $\bar{R} \equiv R/N(R)$ and $I^k \cap N(R) \subseteq IN(R)$, for some k. Then I satisfies the left AR property in R.*

Proof. Clearly $\bar{I}^k$ satisfies the left AR property in $\bar{R}$. Let L be a left ideal in R. Then $(\bar{I}^k)^t \cap \bar{L} \subseteq \bar{I}^k \bar{L}$ for some t depending on L. That is, $(I^k)^t \cap L \subseteq I^k L + N(R)$. Consequently $(I^k)^t \cap L \subseteq I^k L + (I^k \cap N(R))$. Hence $I^{kt} \cap L \subseteq I^k L + IN(R)$. Let V, W be two prime ideals in R satisfying $V \supseteq I$ and $V \rightsquigarrow W$, where $\rightsquigarrow$ means the existence of an ideal A in R satisfying $VW \subseteq A \subsetneqq V \cap W$, r-ann$_R((V \cap W)/A) = W$, and l-ann$_R((V \cap W)/A) = V$. By standard results in Jategaonkar's localization theory [e.g., GW], we need to show that $I \subseteq W$. Take $L \equiv V \cap W$ in the above. Then $I^{kt} \cap (V \cap W) \subseteq I^k(V \cap W) + IN(R) \subseteq VW$. Consequently $(V \cap W)I^{kt} \subseteq I^{kt} \cap (V \cap W) \subseteq VW \subseteq A$. Then $I^{kt} \subseteq W$, which shows that $I \subseteq W$.

Q.E.D.

Remark. It would be interesting to find out the extent of validity of the previous result.

THEOREM 2.15. *Let R be a Noetherian PI ring and let I be a two-sided ideal satisfying $I^*I = R$. Then I satisfies the left AR property.*

Proof. By Lemma 1.10, $(\bar{I})^*\bar{I} = \bar{R}$ in $\bar{R} = R/N(R)$. Consequently, by Theorem 1.12, $\bar{I}$ is invertible and therefore satisfies the left AR property

[McR, 4.2.6]. Now by Lemma 2.13, $I \cap N(R) = IN(R)$. So, all the ingredients of Proposition 2.14 are in place, and I therefore satisfies the left AR property. Q.E.D.

Remark. The previous theorem generalizes [BH, Proposition 4.3; L, Theorem 5.11; CHL, Theorems A and B], where it is assumed that I is maximal as well as right projective.

3. FROM PROJECTIVE TO GENERATOR

Let I be a right projective, two-sided ideal in a Noetherian PI ring R. We shall show here that if $\{I^i\}$ does not stabilize, then, for some m, $I^m S$ is a proper two-sided ideal in $S \equiv \mathrm{End}_R(I_R^m)$ and is a right generator as well as a right projective module. This will enable us to deduce properties of I from those of $I^m S$. A noteworthy application is the following dichotomy. Let R be in addition an affine ring, and let I satisfy $I \cap \mathscr{C}(0) \neq \phi$. Then either $I^k = I^{k+1}$ for some k, or little height $(I) \leq 1$. Moreover, the above-mentioned procedure provides a method of producing non-trivial generator two-sided ideals.

LEMMA 3.1. *Let I be a two-sided ideal of a ring R. Suppose that I_R is projective and finitely generated. Then $(I^*)^k = (I^k)^*$ for each k.*

Proof. Recall that $I^* \equiv \mathrm{Hom}_R(I_R, R_R)$ and $f \cdot g \in (I^*)^2$ is considered as an element of $(I^2)^*$ via $f \cdot g(ab) = f(g(a)b)$. Consequently $(I^*)^k \subseteq (I^k)^*$ holds trivially, for each k. Recall also that if $g, h \in (I^k)^*$ and $d \in I^k$, then $gdh = g(d)h$ (simply evaluate on any $x \in I^k$). Assume by induction that $(I^*)^{k-1} = (I^{k-1})^*$. Now, I_R^{k-1} being projective [McR, 7.3.9(i)] implies that $1_{I^{k-1}} = \Sigma c_j g_j$, where $c_j \in I^{k-1}$, $g_j \in (I^{k-1})^* = (I^*)^{k-1}$. Also $1_I = \Sigma a_i f_i$, where $a_i \in I$, $f_i \in I^*$, since I_R is projective. We claim that $1_{I^k} = \Sigma_i a_i(\Sigma c_j g_j) f_i$. Indeed if $x \in I$, $y \in I^{k-1}$, then $\{\Sigma_i a_i(\Sigma c_j g_j) f_i\}(xy) = \{\Sigma_i a_i(\Sigma c_j g_j)(f_i(x)y)\} = \Sigma a_i f_i(x) y$, where the last equality is due to $\Sigma c_j g_j = 1_{I^{k-1}}$. Hence, since $\Sigma a_i f_i = 1_I$, we establish the claim. Finally, let $g \in (I^k)^*$; then

$$g = g \cdot 1_{I^k} = g \cdot \Sigma_{i,j} a_i c_i g_j f_i = \sum_{i,j} g(a_i c_i) g_j f_i \in R(I^*)^{k-1} I^* = (I^*)^k.$$

Q.E.D.

LEMMA 3.2. *Let I_R be as in Lemma 3.1. Then $(I^*)^k I^k$ is an idempotent ideal for each k.*

Proof. $1_{I^k} \in I^k(I^*)^k$ by the previous lemma. Hence $[(I^*)^k I^k]^2 = (I^*)^k(I^k(I^*)^k)I^k \supseteq (I^*)^k 1_{I^k} I^k \supseteq (I^*)^k I^k$. The reverse inclusion is trivial. Q.E.D.

LEMMA 3.3. *Let I_R be as in Lemma* 3.1. *Then $II^* = \mathrm{End}_R(I_R) \equiv S$ and $SI = I$.*

Proof. $SI = I$ follows trivially since, for each $x \in S$ and $i \in I$, $x(i) \in I$, so since I is considered inside S via left multiplication, $(xi)(j) = x(ij) = x(i)j = x(i)(j)$ for each $j \in J$. Now, by the projectivity of I_R, we have $1_I = \Sigma a_i f_i$, when $a_i \in I$ and $f_i \in I^*$. So $x = x \cdot 1_I = \Sigma x a_i f_i = \Sigma x(a_i) f_i \in II^*$. So $S \subseteq II^*$. The reverse inclusion is trivial. Q.E.D.

THEOREM 3.4. *Let R be a ring with a finite number of idempotent ideals and let I be a two-sided ideal such that I_R is projective and finitely generated. Then for some m, $I^m S$ is a right progenerator two-sided ideal in $S \equiv \mathrm{End}_R(I_R^m)$.*

Proof. By assumption and Lemma 3.2, $(I^*)^{m+i} I^{m+i} = (I^*)^m I^m$ for some m and all $i \geq 0$. Hence $(I^{2m})^* I^{2m} = (I^m)^* I^m$. Since I_R^m is projective we change notation and replace I^m by I. So $(I^2)^* I^2 = I^* I$. Thus, by Lemma 3.1, $S = SS = (II^*)(II^*) = I(I^*I)I^* = I((I^2)^*I^2)I^* = I((I^*)^2 I^2)I^* = (II^*)I^*I(II^*) = SI^*IS$. Now by Lemma 3.3, IS is a two-sided ideal in S. Finally, clearly $SI^* \subseteq (IS)^*$. Hence $S = (IS)^* IS$, which implies that IS is a right generator. Now the right projectivity of IS follows since $1 \in II^* \subseteq (IS)(SI^*) \subseteq (IS)(IS)^*$. Q.E.D.

Remark. It is evident from the proof of Theorem 3.4 that one only needs the stabilization of the descending sequence $\{(I^*)^n I^n\}$.

LEMMA 3.5. *Let I be as in Theorem* 3.4. *Then $I^m = I^{2m}$ if and only if $I^m S = S$.*

Proof. Suppose first that $I^m S = S$. Then $1_{I^m} \in I^m S = I^m I^m (I^*)^m = I^{2m}(I^*)^m$. Applying it to I^m yields $I^m \subseteq I^{2m}(I^*)^m I^m \subseteq I^{2m}$. That is, $I^m = I^{2m}$. Conversely, if $I^m = I^{2m}$, then $(I^m)^* \equiv \mathrm{Hom}_R(I_R^m, R_R) = \mathrm{End}_R(I_R^m) = S$. Since, by the projectivity of I_R^m [McR, 7.3.9], $1_{I^m} \in I^m(I^m)^* = I^{2m}(I^m)^*$ we get $S = 1_{I^m} \cdot S \subseteq I^m S \cdot S = I^m S$. Q.E.D.

The next result is a corollary of the previous results.

THEOREM 3.6. *Let R be a Noetherian PI ring and let I be a two-sided ideal which is right projective. Then either $I^k = I^{k+1}$ for some k or, for some m, $I^m S$ is a proper two-sided ideal in $S \equiv \mathrm{End}(I_R^m)$ and $I^m S$ is a right progenerator S-module.*

Proof. By [RS, Thm. 3], R has only a finite number of idempotent ideals. The rest follows from Theorem 3.4 and Lemma 3.5 Q.E.D.

Here is another consequence.

PROPOSITION 3.7. *Let R be a prime Noetherian PI ring and let I_R be a non-zero right projective two-sided ideal. Then exactly one of the following holds*:

(i) $I^k = I^{k+1}$ *for some* k,

(ii) $\bigcap_i I^i = \{0\}$.

Proof. We saw, by Theorem 3.6, that if (i) does not hold, then $I^m S$ is a proper two-sided ideal in $S = \mathrm{End}_R(I_R^m)$ and is a right S (pro)generator. Now by Lemma 3.3, $S = I^m(I^*)^m$ and since R is prime, $I^* \subseteq Q(R)$ (by identifying each $g \in I^*$ with a left multiplication by an element of $Q(R)$). In particular $S \subseteq Q(R)$ and therefore S is prime. Also, by standard results, ${}_R I^*$ is a finitely generated left R-module and consequently S is a Noetherian prime PI ring. Therefore, by Theorem 1.12, $I^m S$ is an invertible (proper) ideal. So, by [HL, Cor. 2.2], $\bigcap_i (I^m S)^i = \{0\}$. Now $\bigcap_i I^i = \{0\}$ since $R \subset S$. Q.E.D.

Remark. A direct proof of Proposition 3.7 can be given as well (using duality).

Our next task is to show that if I_R is projective and is not a virtual idempotent, then its height is small in some sense. To this end we need the following definition, which appears as "little rank" in [K, p. 98].

DEFINITION. The *little height* of an ideal I, ℓ-height(I), is the length of the shortest saturated chain of prime ideals descending from a minimal prime above I to a minimal prime in R.

Remark. If R is an affine prime PI ring then, by Schelter's catenarity result [MR, Theorem 13.10.12], ℓ-height(I) = height(I) for each prime ideal I of R.

THEOREM 3.8. *Let R be an affine Noetherian PI ring and let I be a two-sided ideal which is right projective. Suppose, in addition, that $I \cap \mathscr{C}(0) \neq \phi$. Then one of the following holds*:

(i) $I^k = I^{k+1}$ *for some* k, *or*

(ii) ℓ-*height*$(I) \leq 1$.

Proof. Let $a \in \mathscr{C}(0) \cap I$ and let P be an arbitrary minimal prime in R. Hence $aP \subseteq IP$. Let ρ denote the right Goldie rank function. So $\rho(P) \geq \rho(IP) \geq \rho(aP) = \rho(P)$. This implies that $\rho(P/IP) = 0$ and consequently $\rho((I \cap P)/IP) = 0$. Hence $(I \cap P)/IP$ is a torsion R/P-submodule of the finitely generated projective R/P-module I/IP. This is a contradiction, unless $(I \cap P)/IP = 0$, since I/IP is a torsion free R/P-module. Therefore, $I \cap P = IP$.

Suppose now that (ii) is violated. In particular, we have $I \cap \mathscr{C}(N(R)) \neq \phi$ and therefore $I \not\subset P$, for each minimal prime P in R. Consequently $\bar{I} \equiv (I + P)/P \cong I/(I \cap P) = I/IP$ is a right projective two-sided ideal in $\overline{R} \equiv R/P$. We shall show next that $\bar{I}^k = \bar{I}^{k+1}$ for some k. Suppose by negation that no such k exists. Then by Theorem 3.4 (and the proof of Proposition 3.7), $\bar{I}^m\overline{S}$ is a proper invertible two-sided ideal in the ring $\overline{S} \equiv \mathrm{End}_{\overline{R}}(\bar{I}^m_{\overline{R}})$. Moreover, as in the proof of Proposition 3.7, since $\overline{S}$ is a finitely generated left $\overline{R}$-module, $\overline{S}$ is a prime affine Noetherian PI ring which contains $\overline{R}$. In particular $\overline{S}$ is catenary and $K.\dim \overline{R} = K.\dim \overline{S}$. Also by [CH] height $(\bar{I}^m\overline{S}) = 1$. Consequently $K.\dim \overline{R} - 1 = K.\dim \overline{S}/\bar{I}^m\overline{S} = K.\dim(\overline{R}/(\bar{I}^m\overline{S} \cap \overline{R})) \leq K.\dim \overline{R}/\bar{I}^m \leq K.\dim \overline{R} - 1$. Hence $K.\dim \overline{R} - 1 = K.\dim \overline{R}/I^m$ and $\mathrm{height}(\bar{I})^m = \mathrm{height}(\bar{I}) = 1$. Let $P_1 \supseteq I + P$ be a minimal prime above $I + P$. Consequently $\mathrm{height}(P_1/P) = \mathrm{height}(\bar{I}) = 1$. Therefore the proper sequence $P \subset P_1$ of prime ideals with P_1 minimal above I shows that ℓ-height$(I) \leq 1$, a contradiction. Therefore $\bar{I}^k = \bar{I}^{k+1}$ for some k. In particular $\bar{I}^k = (\bar{I})^{2k}$. Since R has only a finite number of minimal primes, there exists an integer k such that $I^k = I^{2k} + P \cap I^k$ for each minimal prime P. Let $I^k \equiv I_1$ and let $\{P_1, \ldots, P_t\}$ be the set of minimal primes in R. Suppose by induction on n that $I_1^{n-1} \subseteq I_1^n + (P_1 \cap I_1) \cdots (P_{n-1} \cap I_1)$. Multiplying the last inclusion by I_1, on the right, yields $I_1^n \subseteq I_1^{n+1} + (P_1 \cap I_1) \cdots (P_{n-1} \cap I_1)I_1 \subseteq I_1^{n+1} + (P_1 \cap I_1) \cdots (P_{n-1} \cap I_1)(I_1^2 + P_n \cap I_1)$, where the last inclusion uses the fact that $I_1 = I_1^2 + (P_n \cap I_1)$. By expanding the last term we get $I_1^n \subseteq I_1^{n+1} + (P_1 \cap I_1) \cdots (P_n \cap I_1)$, proving the induction. Consequently $I_1^t \subseteq I_1^{t+1} + N(R)$, which implies that $I_1^t = I_1^{2t} + N(R) \cap I_1^t$. Let $I_2 \equiv I_1^t$. Now, using the fact that $N(R)^s = 0$ for some s and the previous argument, we conclude that $I_2^s = I_2^{s+1} + (N(R) \cap I_2)^s = I_2^{s+1}$. That is, $I^{kts} = I^{kt(s+1)}$. This shows that (i) holds. Q.E.D.

Remarks. 1. We believe that the assumption $I \cap \mathscr{C}(0) \neq \phi$, in Theorem 3.8, is superfluous.

2. Our next example shows that ℓ-height cannot be replaced by height, even in the semi-prime case.

EXAMPLE 3.9. There exists a semi-prime, affine, Noetherian PI ring with a two-sided ideal I which is right projective and $I^k \neq I^{k+1}$ for each k. Moreover, there exists a prime ideal X minimal over I with $\mathrm{height}(X) = 2$ and ℓ-height$(X) = 1$.

Consider a prime affine Noetherian PI ring A with P a maximal ideal in A so that $P^2 = P$, $\mathrm{height}(P) = 1$, and P_A is projective. Let B be a prime affine Noetherian PI ring with a maximal ideal M, so that $M^2 = M$, $\mathrm{height}(M) = 2$, and M_B is projective. Suppose also that $A/P \stackrel{\mathrm{g}}{\cong} B/M$.

One can find such examples, e.g., among affine Noetherian PI rings with global dimension 2. Let R be the pullback of A and B along $A/P = B/M$. That is, $R = \{(x, y) \mid x + P \stackrel{\sigma}{=} y + M\} \subseteq A \oplus B$. Let $X \equiv (P, M)$. It is clear that X is a maximal ideal in R and $R/X \cong A/P \cong B/M$. Let $0 \subset Q_1 \subset M$ be a maximal chain of prime ideals in B; then $(P, 0) \subset (P, Q_1) \subset (P, M) = X$ is a maximal chain of prime ideals in R. Consequently height$(X) = 2$. Also $(0, M) \subset (P, M) = X$ is also a maximal chain of prime ideals inside X, implying that ℓ-height$(X) = 1$. Let $z \in P$ be a non-zero central element in A. Let $I_1 = zP$. Clearly I_{1_A} is projective. Moreover $\bigcap_i I_1^i \subseteq \bigcap_i z^i A = \{0\}$ and hence $I_1^k \neq I_1^{k+1}$ for each k. Let $I \equiv (I_1, M)$. Clearly $X = (P, M)$ is a minimal prime ideal over I. Also if $I^k = I^{k+1}$ for some k then $I_1^k = I_1^{k+1}$ for some k, which was excluded. Last, we need to show that I_R is projective. Now it is easily seen that $I_1^* = z^{-1}P^* \subseteq Q(A)$ and therefore $I_1^* I_1 = z^{-1}P^* zP = P^*P = P$. Consider $(I_1^*, M^*) \subseteq Q(A) \oplus Q(B) = Q(A \oplus B) = Q(R)$. Then $(I_1^*, M^*)(I) = (I_1^*, M^*)(I_1, M) \subseteq (I_1^* I_1, M^*M) = (P, M) = X \subseteq R$. Therefore $(I_1^*, M^*) \subseteq I^*$. Finally $1_R = (1_A, 1_B) \in (I_1 I_1^*, MM^*) = (I_1, M_1)(I_1^*, M^*) \subseteq II^*$ shows that I_R is projective.

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