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Codes and designs in Grassmannian spaces

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Abstract

The notion of t-design in a Grassmannian space $\mathcal{G}_{m,n}$ was introduced by the first and last authors and G. Nebe in a previous paper. In the present work, we give a general lower bound for the size of such designs. The method is inspired by Delsarte, Goethals and Seidel work in the case of spherical designs. This leads us to introduce a notion of f-code in Grassmannian spaces, for which we obtain upper bounds, as well as a kind of duality tight-designs/tight-codes. The bounds are in terms of the dimensions of the irreducible representations of the orthogonal group $O(n)$ occurring in the decomposition of the space $L^2(\mathcal{G}^\circ_{m,n})$ of square integrable functions on $\mathcal{G}^\circ_{m,n}$, the set of oriented Grassmanianns.

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1. Introduction

There are various combinatorial problems related to finite sets of Euclidean spheres. Among those, two, in a sense dual to each other, have received much attention, namely the notions of spherical t-design (t an integer), and spherical A-codes (A a finite set in $[-1,1]$). The notion of spherical design was motivated by numerical integration: a spherical t-design is a finite subset X of a sphere S^{d-1}, such that the integral over S^{d-1} of a polynomial function up to degree t coincides with its average value at the points of X. It is thus important, for instance for applications, to find designs with

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smallest possible cardinality. So the question of finding a *lower* bound for the size of a spherical design is central. As for A-codes, it is natural conversely to ask for an *upper* bound of their size: an A-code is a finite set in a sphere S^{d-1} such that the scalar products of pairwise distinct elements belong to a fixed set $A \subset [-1, 1]$. When $A = [-1, 1/2]$, finding an upper bound is equivalent to the kissing number problem, known as the problem of the thirteen spheres when $n = 3$. In their landmark paper [6], Delsarte, Goethals and Seidel proposed a general method, based on harmonic analysis on the orthogonal group, to study both questions.

The problem of packings, and related combinatorial questions, in the Grassmanian spaces $\mathscr{G}_{m,n}$ of m-dimensional subspaces of $\mathbb{R}^n$ have been investigated in a series of recent papers (see [4,3]). In [1], a theory of designs was developed in that framework. One task of the present paper is to define a notion of f-code in Grassmannian spaces, which reduces to A-code when $m = 1$ (the codes in the first Grassmannian $\mathscr{G}_{1,n}$ are in one-to-one correspondence with the antipodal codes of the unit sphere). Then, inspired by Delsarte, Goethals and Seidel's works, we establish lower/upper bounds for the size of such designs/codes, which involve the dimensions of some irreducible representations of $O(n)$.

2. Zonal functions on Grassmannian spaces

Let $\mathscr{G}_{m,n} \simeq O(n)/O(m) \times O(n-m)$ be the Grassmannian space of m-dimensional subspaces of $\mathbb{R}^n$. Recall (see [1,9]), that the orbits under $O(n)$ of pairs $(p, q) \in \mathscr{G}_{m,n} \times \mathscr{G}_{m,n}$ are parametrized by the m-tuples

$$1 \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m \geqslant 0.$$

Namely, to a couple (p, q) of m-dimensional subspaces, one associates the m-tuple $t_1 = \cos\theta_1, \ldots, t_m = \cos\theta_m$, where $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_m \leqslant \pi/2$ are the *principal angles* between p and q. One way to compute the t_i, is as follows: denoting by p_0 the subspace generated by the first m vectors of the canonical basis of $\mathbb{R}^n$, and writing $p = g \cdot p_0$, $q = h \cdot p_0$, with suitable g, h in $O(n)$, then the $y_i := t_i^2$ are the eigenvalues of the $m \times m$ symmetric matrix AA^t, where A is the m-size block appearing in the block-decomposition

$$hg^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

Moreover, g and h are defined up to multiplication by an element in $\mathrm{Stab}(p_0) \simeq O(m) \times O(n-m)$, and may be chosen so that

$$A = \begin{pmatrix} \cos\theta_1 & 0 & \ldots & 0 \\ 0 & \cos\theta_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \cos\theta_m \end{pmatrix},$$

$$C = \begin{pmatrix} \sin\theta_1 & 0 & \dots & 0 \\ 0 & \sin\theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sin\theta_m \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \qquad (2)$$

Besides $\mathscr{G}_{m,n}$, we have to consider the set $\mathscr{G}_{m,n}^{\circ}$ of *oriented* m-dimensional subspaces of $\mathbb{R}^n$. We may view the elements of $\mathscr{G}_{m,n}^{\circ}$ as couples $\tilde{p}=(p,s)$, with p an m-dimensional subspace, and s an element in $\bigwedge^m p$. The action of $O(n)$ on these couples is given by

$$g.(p,s) := (gp, gs),$$

so that if we fix an orientation s_0 on p_0, the stabilizer of $\tilde{p_0} = (p_0, s_0)$ identifies with $SO(m) \times O(n-m)$. Consequently

$$\mathscr{G}_{m,n}^{\circ} \simeq SO(n)/SO(m) \times SO(n-m)) \simeq O(n)/SO(m) \times O(n-m),$$

which is a 2 to 1 covering of $\mathscr{G}_{m,n}$. The orbits under $O(n)$ of pairs $(\tilde{p},\tilde{q}) \in \mathscr{G}_{m,n}^{\circ} \times \mathscr{G}_{m,n}^{\circ}$ can be likewise parametrized by $(m+1)$-tuples $(\varepsilon, t_1, \dots, t_m)$, where $t_1, \dots, t_m$ are defined as above, in terms of the principal angles between p and q, regardless to the orientation, and $\varepsilon \in \{\pm 1\}$ is defined as follows: if the block A in (1) is non-singular, we set $\varepsilon = \det A/|\det A|$, otherwise we set $\varepsilon = +1$. We still have a canonical block-decomposition like (2), but with top-left block

$$A = \begin{pmatrix} \varepsilon\cos\theta_1 & 0 & \dots & 0 \\ 0 & \cos\theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \cos\theta_m \end{pmatrix}. \qquad (2')$$

Both $\mathscr{G}_{m,n}$ and $\mathscr{G}_{m,n}^{\circ}$ inherit from the Haar measure of $O(n)$, a measure denoted $\mathrm{d}p$ and $\mathrm{d}\tilde{p}$ respectively. Since we will be mainly interested in non-oriented Grassmanians, we normalize it so as $\int_{\mathscr{G}_{m,n}} \mathrm{d}p = 1$ (whence $\int_{\mathscr{G}_{m,n}^{\circ}} \mathrm{d}\tilde{p} = 2$). Accordingly, the space $L^2(\mathscr{G}_{m,n}^{\circ})$ of square integrable functions on $\mathscr{G}_{m,n}^{\circ}$ is endowed with the scalar product

$$\langle f, g\rangle = \frac{1}{2}\int_{\mathscr{G}_{m,n}^{\circ}} f(\tilde{p})\overline{g(\tilde{p})}\,\mathrm{d}\tilde{p},$$

so that its restriction to $L^2(\mathscr{G}_{m,n})$ satisfies

$$\langle f, g\rangle = \int_{\mathscr{G}_{m,n}} f(p)\overline{g(p)}\,\mathrm{d}p, \quad f \in L^2(\mathscr{G}_{m,n}),\ g \in L^2(\mathscr{G}_{m,n}).$$

The group $O(n)$ acts isometrically on $L^2(\mathscr{G}^\circ_{m,n})$ by

$$\sigma \cdot f(\tilde{p}) = f(\sigma^{-1}\tilde{p}).$$

The structure of $L^2(\mathscr{G}^\circ_{m,n})$ as an $O(n)$-module is well-known, and is given for instance in [8, p. 546]. To be precise, if we consider the subset $\mathscr{R}(\mathscr{G}^\circ_{m,n})$ of regular functions on $\mathscr{G}^\circ_{m,n}$ (i.e. the set of functions induced by regular functions on $O(n)$) which is a dense subset of $L^2(\mathscr{G}^\circ_{m,n})$, we have the following decomposition:

$$\mathscr{R}(\mathscr{G}^\circ_{m,n}) = \oplus H^\mu_{m,n}$$

into pairwise orthogonal nonisomorphic irreducible $O(n)$-submodules $H^\mu_{m,n}$, the sum being over partitions $\mu = \mu_1 \geqslant \mu_2 \geqslant \cdots \mu_m \geqslant 0$, of depth at most m, with $\mu_i \equiv \mu_j \bmod 2$ for all (i,j). We call these partitions *m-admissible*, or *admissible* for short. They split into *odd* and *even*, according to the parity of the μ_i.

Remark 1. For a given even partition μ, the admissibility does not depend on m, as long as $\mathrm{depth}(\mu) \leqslant m \leqslant n/2$, whereas for μ odd it does, since in that case the μ_i have to be nonzero for all $1 \leqslant i \leqslant m$.

It turns out that the $O(n)$-isomorphism class of $H^\mu_{m,n}$ is independent of m, provided that $\mathrm{depth}(\mu) \leqslant m \leqslant n/2$ and μ is m-admissible (see [1] for a more detailed description of $H^\mu_{m,n}$). The space $H^\mu_{m,n}$ is isomorphic to the irreducible representation of $O(n)$ canonically associated to the partition μ, and denoted V^μ_n in [8]. We shall denote the dimension of this space d_μ.

The sum over *even*, resp. *odd*, partitions corresponds respectively to $\mathscr{R}(\mathscr{G}_{m,n})$ and its orthogonal complement $\mathscr{R}(\mathscr{G}_{m,n})^\perp$. This also corresponds to the eigenspace decomposition of $\mathscr{R}(\mathscr{G}^\circ_{m,n})$ with respect to the canonical involution σ^* induced by orientation changing, namely

$$\mathscr{R}(\mathscr{G}_{m,n}) = \mathscr{R}(\mathscr{G}^\circ_{m,n})^+ := \{f \in \mathscr{R}(\mathscr{G}^\circ_{m,n}) \mid \sigma^*(f) = f\},$$

and

$$\mathscr{R}(\mathscr{G}_{m,n})^\perp = \mathscr{R}(\mathscr{G}^\circ_{m,n})^- := \{f \in \mathscr{R}(\mathscr{G}^\circ_{m,n}) \mid \sigma^*(f) = -f\}.$$

Let $\mathscr{Z}^\circ$ (resp. $\mathscr{Z}$) be the set of $O(n)$-invariant functions F on $\mathscr{G}^\circ_{m,n} \times \mathscr{G}^\circ_{m,n}$ (resp. $\mathscr{G}_{m,n} \times \mathscr{G}_{m,n}$), such that

$$F(\tilde{p},.) \in \mathscr{R}(\mathscr{G}^\circ_{m,n}), F(.,\tilde{q}) \in \mathscr{R}(\mathscr{G}^\circ_{m,n}) \quad \text{for all } (\tilde{p},\tilde{q}) \in \mathscr{G}^\circ_{m,n} \times \mathscr{G}^\circ_{m,n}, \tag{3}$$

resp.

$$F(p,.) \in \mathscr{R}(\mathscr{G}_{m,n}), F(.,q) \in \mathscr{R}(\mathscr{G}_{m,n}) \quad \text{for all } (p,q) \in \mathscr{G}_{m,n} \times \mathscr{G}_{m,n}. \tag{3'}$$

As usual, we call such functions *zonal*. Alternatively, if a base point $\tilde{p}$ is fixed, one can identify $\mathscr{Z}^o$ with $\mathscr{R}(\mathscr{G}^\circ_{m,n})^{\mathrm{Stab}(\tilde{p})}$, mapping $F \in \mathscr{Z}^o$ on $F(\tilde{p},.) \in \mathscr{R}(\mathscr{G}^\circ_{m,n})^{\mathrm{Stab}(\tilde{p})}$, and similarly $\mathscr{Z}$ identifies with $\mathscr{R}(\mathscr{G}_{m,n})^{\mathrm{Stab}(p)}$.

As explained in [1,9], it follows from the Frobenius reciprocity theorem that ${H^\mu_{m,n}}^{\mathrm{Stab}(\tilde{p})}$ is one-dimensional for any μ (if μ is even, then $H^\mu_{m,n} \subset \mathscr{R}(\mathscr{G}_{m,n}) = \mathscr{R}(\mathscr{G}^\circ_{m,n})^+$, so that ${H^\mu_{m,n}}^{\mathrm{Stab}(\tilde{p})} = {H^\mu_{m,n}}^{\mathrm{Stab}(p)}$). Consequently, to each summand $H^\mu_{m,n}$ is attached a unique (up to scaling) *zonal function* P_μ, which can be computed in the following way: denoting by d_μ the dimension of $H^\mu_{m,n}$, and $\{e_{\mu,i}\}_{1\leqslant i\leqslant d_\mu}$ an orthonormal basis of it, one has

$$P_\mu(\tilde{p},\tilde{p}') := \frac{1}{d_\mu}\sum_{i=1}^{d_\mu} e_{\mu,i}(\tilde{p})\overline{e_{\mu,i}(\tilde{p}')}. \tag{4}$$

The results of the next section rely on the following properties of the P_μ:

Lemma 2. (i) $P_\mu(\tilde{p},\tilde{p})=1$, *for any* μ *and* $\tilde{p}$.

(ii) *For any* λ, μ *and* $\tilde{p}$, $\tilde{p}'$, *one has*

$$\langle P_\mu(\tilde{p},.),P_\lambda(\tilde{p}',.)\rangle = \frac{\delta_{\lambda,\mu}}{d_\mu}P_\mu(\tilde{p},\tilde{p}'), \tag{5}$$

if, for fixed $\tilde{p}$, *we view the map* $\tilde{q}\mapsto P_\mu(\tilde{p},\tilde{q})$ *as a function in* $L^2(\mathscr{G}^\circ_{m,n})$. *In particular, for any finite set* $X\subset\mathscr{G}^\circ_{m,n}$, *the matrix* $(P_\mu(\tilde{p},\tilde{p}'))_{\tilde{p},\tilde{p}'\in X^2}$ *is positive semidefinite.*

(iii) *For any* λ, μ, *one has*

$$P_\lambda P_\mu = \sum_\tau c_{\lambda,\mu}(\tau)P_\tau, \tag{6}$$

with non-negative coefficients $c_{\lambda,\mu}(\tau)$. *In particular*, $c_{\lambda,\mu}(0)=\delta_{\lambda,\mu}/d_\mu$.

Proof. Since $P_\mu(\tilde{p},\tilde{p})$ does not depend on $\tilde{p}$, one has

$$\begin{aligned}P_\mu(\tilde{p},\tilde{p}) &= \frac{1}{2}\int_{\mathscr{G}^\circ_{m,n}} P_\mu(\tilde{p},\tilde{p})\,\mathrm{d}\tilde{p} = \frac{1}{2d_\mu}\sum_{i=1}^{d_\mu}\int_{\mathscr{G}^\circ_{m,n}}|e_{\mu,i}(\tilde{p})|^2\,\mathrm{d}\tilde{p}\\ &= \frac{1}{d_\mu}\sum_{i=1}^{d_\mu}\langle e_{\mu,i},e_{\mu,i}\rangle = 1,\end{aligned}$$

which proves (i). As for (ii), this is clear using (4) and the orthogonality relations between the $e_{\mu,i}$. Finally, assertion (iii) is classical, see [10, Theorem 3.1]. □

The algebraic structure of $\mathscr{Z}$ and $\mathscr{Z}^o$ can be easily deduced from [8]. For lack of reference, we state it in the next proposition.

Proposition 3. (i) *There is an isomorphism*

$$\mathscr{Z}\simeq\mathbb{C}[Y_1,\ldots,Y_m]^{S_m},$$

the ring of symmetric polynomials in m variables, mapping Y_i *to* $y_i=y_i(p,q)$. *Similarly, one has*

$$\mathscr{Z}^o\simeq\mathbb{C}[Y_1,\ldots,Y_m]^{S_m}[\theta],\quad \textit{with } \theta^2=Y_1\cdots Y_m,$$

by mapping θ to $\varepsilon t_1 \cdots t_m$, where $\varepsilon = \varepsilon(\tilde{p},\tilde{q})$, $t_i = t_i(\tilde{p},\tilde{q})$. Moreover, the eigenspace decomposition of $\mathscr{Z}^o$ with respect to the involution σ^* is given by

$$\mathscr{Z}^{o+} = \mathscr{Z} \quad \text{and} \quad \mathscr{Z}^{o-} = \varepsilon t_1 \cdots t_m \mathscr{Z} \simeq \theta\mathbb{C}[Y_1,\ldots,Y_m]^{S_m}.$$

(ii) *The P_μ corresponding to even partitions may be expressed as*

$$P_\mu(p,q) = p_\mu(y_1(p,q),\ldots,y_m(p,q))$$

with $p_\mu(Y_1,\ldots,Y_m)$ a symmetric polynomial of total degree $|\mu|/2$, and those corresponding to odd partitions as

$$P_\mu(\tilde{p},\tilde{q}) = (\varepsilon t_1,\ldots,t_m)p_\mu(y_1,\ldots,y_m),$$

with $p_\mu(Y_1,\ldots,Y_m)$ a symmetric polynomial of degree $|\mu| - m/2$.

Proof. (i) As explained above, we can identify $\mathscr{Z}$ with $\mathscr{R}(\mathscr{G}_{m,n})^{\mathrm{Stab}(p)}$ (resp. $\mathscr{Z}^o$ with $\mathscr{R}(\mathscr{G}^\circ_{m,n})^{\mathrm{Stab}(\tilde{p})}$), p (resp. $\tilde{p}$) being any fixed base point. From the isomorphism $\mathrm{Stab}(p) \simeq O(m) \times O(n-m)$, it is easily seen, using (2), that an element $F = F(p,\cdot) \in \mathscr{R}(\mathscr{G}_{m,n})^{\mathrm{Stab}(p)}$ is of the form

$$F = P(\cos\theta_1,\ldots,\cos\theta_m,\sin\theta_1,\ldots \sin\theta_m),$$

where $P(T_1,\ldots,T_m,Z_1,\ldots,Z_m)$ is a polynomial, symmetric in $T_1,\ldots,T_m$ and $Z_1,\ldots,Z_m$ respectively. Now the $\mathrm{Stab}(p)$-invariance also implies that all the exponents are even, so that F is indeed a symmetric polynomial in $Y_1 = T_1^2,\ldots,Y_m = T_m^2$, which is the first part of assertion (i). As for the second part of the assertion, one first shows in the same way that a $\mathrm{Stab}(\tilde{p})$-invariant element in $\mathscr{R}(\mathscr{G}^\circ_{m,n})$ is of the form

$$F = F(\tilde{p},\cdot) = P(\varepsilon\cos\theta_1,\ldots,\cos\theta_m,\sin\theta_1,\ldots,\sin\theta_m),$$

where $P(T_1,\ldots,T_m,Z_1,\ldots,Z_m)$ is a polynomial, symmetric in $T_1,\ldots,T_m$ and $Z_1,\ldots,Z_m$ respectively. Since $\mathrm{Stab}(\tilde{p}) \simeq SO_m \times O_{n-m}$, the $\mathrm{Stab}(\tilde{p})$-invariance also implies that the exponents in the last m variables are even, whereas the exponents in the first m ones are only restricted to have the same parity. Consequently, P is the sum of a polynomial in $Y_1 = T_1^2,\ldots,Y_m = T_m^2$ plus $T_1\cdots T_m$ times a polynomial in $Y_1,\ldots,Y_m$, as asserted. The eigenspace decomposition is clear.

As for assertion (ii), we only need to observe that the P_μ belong to $\mathscr{Z}^{o+}$ or $\mathscr{Z}^{o-}$ according to as μ is even or odd, and that the p_μ have total degree $|\mu|$ in $T_1,\ldots,T_m$. □

3. Bounds on codes and designs

Among the various equivalent definitions of a t-design given in [1] we recall the following one (see [1, Proposition 4.2])

Definition 4. A finite subset $\mathscr{D}$ of $\mathscr{G}_{m,n}$ is a $2k$-design if

$$\forall\varphi \in H^+_{2k}, \quad \langle\varphi,1\rangle = \frac{1}{|\mathscr{D}|}\sum_{p\in\mathscr{D}}\varphi(p). \tag{7}$$

As for spherical codes, the natural generalization to our context is as follows:

Definition 5. Let $f(Y_1,\dots,Y_m)$ be a symmetric polynomial, normalized so as $f(1,\dots,1)=1$. A finite subset $\mathscr{D}$ of the Grassmannian space $\mathscr{G}_{m,n}$ is a f-code, if for any pair (p,q) of distinct elements in $\mathscr{D}$ one has

$$f(y_1(p,q),\dots,y_m(p,q))=0.$$

On the other hand, one can associate canonically to a symmetric polynomial $f(Y_1,\dots,Y_m)$ as above, an $O(n)$-invariant function F on $\mathscr{G}_{m,n}\times\mathscr{G}_{m,n}$, satisfying $F(p,p)=1$, by the formula:

$$F(p,q):=f(y_1(p,q),\dots,y_m(p,q)),$$

and the definition of an f-code now reads

$$F(p,q)=\delta_{p,q},\quad (p,q)\in\mathscr{D}^2. \tag{8}$$

The following notion of *type* is consistent with [5, Definition 5.4.]:

Definition 6. The type of an f-code is 1 if $Y_1,\dots,Y_m$ divides the polynomial $f(Y_1,\dots,Y_m)$, and 0 otherwise.

For any integer k, we define

$$H_k=\bigoplus_{\substack{|\mu|\leqslant k\\ \mu \text{ admissible}}} H_{m,n}^{\mu}.$$

It decomposes under σ^* as $H_k=H_k^+\oplus H_k^-$, and we have, for the respective dimensions $d_k^{\pm}$ of $H_k^{\pm}$,

$$d_k^+:=\sum_{\substack{|\mu|\leqslant k\\ \mu\text{ even, admissible}}} d_\mu,\quad \text{resp. } d_k^-:=\sum_{\substack{|\mu|\leqslant k\\ \mu\text{ odd, admissible}}} d_\mu.$$

It's worth noticing, from Remark 1, that for fixed k, and big enough m (namely, $m\geqslant[k/2]$), d_k^+ does not depend on m, while d_k^- does. The next two theorems establish bounds for t-designs and f-codes in terms of these numbers. Some explicit values of d_k^+ and d_k^- are collected in the appendix (the d_μ are computed from the formulas in [7, Section 24.2, pp. 407–410]):

Remark 7. In [1], we considered only non-oriented Grassmanians, and what was denoted H_k there, corresponds to what is denoted H_k^+ here.

Theorem 8. *Let $\mathscr{D}\subset\mathscr{G}_{m,n}$ be a* $2k$-design. *Then*

$$|\mathscr{D}|\geqslant\max\{d_k^+,d_k^-\}. \tag{9}$$

If equality holds in (9), *then* $\mathscr{D}$ *is an f-code for* $f = 1/d_k^+ \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ even}}} d_\mu p_\mu$ *or* $f = (Y_1,\ldots,Y_m)/d_k^- \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ odd}}} d_\mu p_\mu$, *depending on whether* $d_k^+ \geqslant d_k^-$ *or not.*

Proof. Let s be a section of the canonical surjection $\mathscr{G}_{m,n}^\circ \to \mathscr{G}_{m,n}$, and $\tilde{\mathscr{D}} = s(\mathscr{D})$. Fix $\tilde p$ and $\tilde q$ in $\mathscr{G}_{m,n}^\circ$. If μ and λ are two partitions of degree $\leqslant k$, the formula

$$\varphi(\tilde p') = P_\mu(\tilde p, \tilde p')P_\lambda(\tilde q, \tilde p')$$

defines an element in H_{2k}. If moreover μ and λ are both *even* (resp. *odd*), then φ is σ^*-invariant, so it belongs to H_{2k}^+. Consequently (7) applies in both cases and reads:

$$\begin{aligned}
\frac{1}{|\mathscr{D}|}\sum_{\tilde p' \in \tilde{\mathscr{D}}} \varphi(\tilde p') = \frac{1}{|\mathscr{D}|}\sum_{p' \in \mathscr{D}} \varphi(p') \\
&= \langle \varphi, 1\rangle_{L^2(\mathscr{G}_{m,n})} \\
&= \langle P_\mu(p,.)P_\lambda(q,.), 1\rangle_{L^2(\mathscr{G}_{m,n})} \\
&= \langle P_\mu(\tilde p,.), P_\lambda(\tilde q,.)\rangle_{L^2(\mathscr{G}_{m,n}^\circ)} \\
&= \frac{\delta_{\lambda,\mu}}{d_\mu} P_\mu(\tilde p, \tilde q).
\end{aligned}$$

In other words, the matrices $S_\mu := (d_\mu P_\mu(\tilde p, \tilde p'))_{\tilde p, \tilde p' \text{ in } \tilde{\mathscr{D}}}$, $|\mu| \leqslant k$, satisfy the relations

$$S_\mu S_\lambda = \delta_{\lambda,\mu}|\tilde{\mathscr{D}}|S_\mu = \delta_{\lambda,\mu}|\mathscr{D}|S_\mu,$$

as long as μ and λ are both *even* (resp. *odd*). Setting $S^+ := \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ even}}} S_\mu$, resp. $S^- := \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ odd}}} S_\mu$, it follows that

$$S^{\pm^2} = |\mathscr{D}|S^\pm. \tag{10}$$

On the other hand, one has $\operatorname{Tr} S_\mu = d_\mu|\mathscr{D}|$, from Lemma 2, so that $\operatorname{Tr} S^+ = \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ even}}} \operatorname{Tr} S_\mu = d_k^+|\mathscr{D}|$, and likewise $\operatorname{Tr} S^- = d_k^-|\mathscr{D}|$. Therefore,

$$d_k^\pm = \frac{\operatorname{Tr} S^\pm}{|\mathscr{D}|} = \operatorname{rank} S^\pm \leqslant |\mathscr{D}|,$$

whence the conclusion.

When equality holds, then (10) implies that $S^+ = |\mathscr{D}|I_{|\mathscr{D}|} = d_k^+ I_{|\mathscr{D}|}$ resp. $S^- = d_k^- I_{|\mathscr{D}|}$, depending on whether $d_k^+ \geqslant d_k^-$ or not, where $I_{|\mathscr{D}|}$ stands for the identity matrix in dimension $|\mathscr{D}|$. This means that $F(p,q) = \delta_{p,q}$, for all $(p,q) \in \mathscr{D}^2$, where $F = 1/d_k^+ \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ even}}} d_\mu P_\mu$, resp. $F = 1/d_k^- \sum_{\substack{|\mu|\leqslant k \\ \mu \text{ odd}}} d_\mu P_\mu$. In the first case, this is clearly equivalent to the assertion that $\mathscr{D}$ be an f-code, according to (8) and the definition of f. This is also true in the second case, since each P_μ, μ odd, is divisible by the product $t_1\cdots t_m$, so that

$$\frac{1}{d_k^-}\sum_{\substack{|\mu|\leqslant k\\ \mu\ \text{odd}}} d_\mu P_\mu(p,q)=\delta_{p,q} \quad\Leftrightarrow\quad \frac{1}{d_k^-}\sum_{\substack{|\mu|\leqslant k\\ \mu\ \text{odd}}}(t_1\cdots t_m)d_\mu P_\mu(p,q)=\delta_{p,q}. \qquad \square$$

Theorem 9. *Any f-code $\mathscr{D}$ in $\mathscr{G}_{m,n}$ satisfies*

$$|\mathscr{D}|\leqslant d_k^+ \tag{11}$$

where $k=2\deg f$. If moreover f is of type 1, then

$$|\mathscr{D}|\leqslant d_k^- \tag{12}$$

where $k=2\deg f-m$. Whenever equality holds in (11), resp. (12), then

$$f=\frac{1}{d_k^+}\sum_{\substack{|\mu|\leqslant k\\ \mu\ even}} d_\mu p_\mu,$$

resp.

$$f=\frac{Y_1\cdots Y_m}{d_k^-}\sum_{\substack{|\mu|\leqslant k\\ \mu\ odd}} d_\mu p_\mu,$$

and $\mathscr{D}$ is a 2k-design.

Proof. Setting $k=2\deg f$, we first see that the functions $F(p,.)$, $p\in\mathscr{D}$ are in H_k^+. We claim that they form a linearly independent system. Indeed, if $\sum_{p\in\mathscr{D}}\lambda_p F(p,.)=0$, then evaluating the left-hand side successively on each $p\in\mathscr{D}$, and using (8), we see that $\lambda_p=0$ for all $p\in\mathscr{D}$. Hence $|\mathscr{D}|\leqslant\dim H_k^+=d_k^+$, which is the first assertion. As for the second one, if f is divisible by $Y_1\cdots Y_m$, we write it as $f(Y_1,\dots,Y_m)=(Y_1\cdots Y_m)g(Y_1,\dots,Y_m)$. Then the functions $t_1(p,.)\cdots t_m(p,.)G(p,.)$, $p\in\mathscr{D}$ are linearly independent elements in H_k^-, with $k=2\deg f-m$, and the inequality $|\mathscr{D}|\leqslant d_k^-$ follows, as in the first case.

To see when equality is achieved, let us assume, for instance that $|\mathscr{D}|=d_k^+$, $k=2\deg f$ (the case $|\mathscr{D}|=d_k^-, k=2\deg f-m$ for $\mathscr{D}$ of type 1 is dealt with similarly). Under this assumption, the family $\{F(p,.),\ p\in\mathscr{D}\}$ is now a basis of H_k^+. Moreover, it is readily checked that the following formula holds for any φ in H_k^+:

$$\varphi=\sum_{p\in\mathscr{D}}\varphi(p)F(p,.). \tag{13}$$

On the other hand, we know from Proposition 3 that F (resp. f) may be written as a linear combination of the P_μ (resp. p_μ),

$$F=\sum_{\substack{|\mu|\leqslant k\\ \mu\ \text{even}}} f_\mu P_\mu\left(\text{resp. } f=\sum_{\substack{|\mu|\leqslant k\\ \mu\ \text{even}}} f_\mu p_\mu\right). \tag{14}$$

Let $P := \sum_{\substack{|\mu|\leqslant k\\ \mu \text{ even}}} d_\mu P_\mu$. What we want to show is that $F = 1/d_k^+ P$, or in other words that

$$f_\mu = \frac{d_\mu}{d_k^+} \quad \text{for } |\mu| \leqslant k, \;\; \mu \text{ even}. \tag{15}$$

To that end, it is sufficient to show that

$$0 \leqslant f_\mu \leqslant \frac{d_\mu}{d_k^+} \quad \text{for } |\mu| \leqslant k, \;\; \mu \text{ even}. \tag{16}$$

since applying (14) to any p in $\mathscr{D}$, we see that $1 = F(p,p) = \sum f_\mu P_\mu(p,p) = \sum f_\mu$, so the right inequality in (16) is an equality. First we note, using (5) and the above decomposition, that

$$\left\langle F(q,.), \sum_{|\mu|\leqslant k} d_\mu P_\mu(p,.) \right\rangle = F(p,q),$$

so that the condition $F(p,q) = \delta_{p,q}$, $(p,q) \in \mathscr{D}^2$ implies that the family $\{\sum_{|\mu|\leqslant k} d_\mu P_\mu(p,.), \;\; p \in \mathscr{D}\}$ is a basis, dual to $\{F(p,.), \;\; p \in \mathscr{D}\}$ with respect to the scalar product $\langle , \rangle$. Consequently, the matrix $S = (P(p,q))_{p,q\in\mathscr{D}^2} = (\langle P(p,.), P(q,.)\rangle)_{p,q\in\mathscr{D}^2}$ is invertible and its inverse is given by

$$S^{-1} = (\langle F(p,.), F(q,.)\rangle)_{p,q\in\mathscr{D}^2}.$$

One easily checks, using Lemma 2 that

$$\langle F(p,.), P_\mu(q,.)\rangle = \frac{f_\mu}{d_\mu} P_\mu(q,p),$$

for p, q in $\mathscr{D}^2$, and $|\mu| \leqslant k$, μ even. But according to (13), this means that the functions $P_\mu(q,.)$, $q \in \mathscr{D}$, $|\mu| \leqslant k$, μ even, are eigenfunctions of the matrix $(\langle F(p,.), F(q,.)\rangle)_{p,q\in\mathscr{D}^2} = S^{-1}$, with corresponding eigenvalue f_μ/d_μ. Thus the f_μ/d_μ are eigenvalues of the Gram matrix of a basis of H_k^+, hence positive. Now, writing (14) for all $(p,q) \in \mathscr{D}^2$, and adding up we obtain

$$|\mathscr{D}| = \sum_{(p,q)\in\mathscr{D}^2} F(p,q) = \sum_{\substack{|\mu|\leqslant k\\ \mu \text{ even}}} f_\mu \sum_{(p,q)\in\mathscr{D}^2} P_\mu(p,q),$$

whence

$$|\mathscr{D}|(1 - f_0|\mathscr{D}|) = \sum_{\substack{|\mu|\leqslant k\\ \mu \text{ even}, \mu\neq 0}} f_\mu \sum_{(p,q)\in\mathscr{D}^2} P_\mu(p,q) \geqslant 0, \tag{17}$$

because of the positivity of the matrix $(P_\mu(p,p'))_{p,p'\in\mathscr{D}^2}$ (Lemma 2)(ii)), so that $f_0 \leqslant 1/|\mathscr{D}| = 1/d_k^+$. If we now consider the annihilator polynomial

$$F_\lambda := P_\lambda F = \sum_{\substack{|\mu|\leqslant k\\ \mu \text{ even}}} g_{\lambda,\mu} P_\mu,$$

we contend that the $g_{\lambda,\mu}$ are nonnegative and that $g_{\lambda,0} = f_\lambda/d_\lambda$: this is an easy consequence of Lemma 2 (6). Consequently, the argument used to get (17) still holds, and we obtain $g_{\lambda,0} = f_\lambda/d_\lambda \leqslant 1/d_k^+$, as desired.

It remains to prove that $\mathscr{D}$ is a $2k$-design. From [1, Proposition 4.2], it amounts to prove that

$$\forall \varphi \in H_{2k}^+, \quad \langle \varphi, 1\rangle = \frac{1}{|\mathscr{D}|}\sum_{p\in\mathscr{D}} \varphi(p) = \frac{1}{d_k^+}\sum_{p\in\mathscr{D}} \varphi(p).$$

It's enough to check this for functions of the form gh, with g, h in H_k^+, since they generate H_{2k}^+. Using expansion (13) of g and h, we see that

$$\begin{aligned}\langle 1, gh\rangle &= \langle g, \bar{h}\rangle\\ &= \sum_{p,q\in\mathscr{D}^2} g(p)h(q)\langle F(p,.), F(q,.)\rangle\\ &= \frac{1}{|\mathscr{D}|}\sum_{p\in\mathscr{D}} g(p)h(p),\end{aligned}$$

whence the conclusion. □

4. Examples

4.1. The case $m = 1$

This is the case of the projective space over the real numbers, the codes of which are studied in [5]. The $2k$-designs in the real projective space can be viewed as *antipodal* $(2k+1)$-designs on the unit sphere of the Euclidean space for which absolute bounds are given in [6]. We recover here these bounds, since for $\mu = \mu_1 \geqslant 0$ the space $H_{1,n}^\mu$ is isomorphic to the space of harmonic polynomials in n variables of degree μ_1. One has $d_k^+ = \binom{n+k-1}{n-1}$ and $d_k^- = \binom{n+k-2}{n-1}$ if k is even, and *vice versa* if k is odd.

A t-design is called tight if its cardinality attains this lower bound. Tight t-designs are only known for $(n,t) = (7,4), (8,6), (23,4), (23,6), (24,10)$. Moreover, it is known that tight t-designs cannot exist when $t \geqslant 8$, apart from the $(24,10)$ given by the lines supporting the minimal vectors of the Leech lattice (see [2]).

4.2. The case $k=2$

In [4,3], packings in the Grassmannian spaces are considered, with respect to the so-called *chordal distance*, given in our notations by

$$d^2(p,q)=m-\sum_{i=1}^{m} y_i.$$

In [4], a *simplex bound* is settled for the sets $\mathscr{D}$ for which $d(p,q)\geqslant d$ (using an isometric embedding into the Euclidean sphere of $\mathbb{R}^{(n-1)(n+2)/2}$). Equality holds if and only if $|\mathscr{D}|=n(n+1)/2$ and $d(p,q)$ is constant.

In [3, Section 5], an infinite family of packings in $\mathscr{G}_{(p-1)/2,\,p}$ meeting this bound, is constructed. Here p is a prime, which is either equal to 3 or congruent to -1 modulo 8. Let us denote it by $\mathscr{D}_p$. Then one has:

Proposition 10. *$\mathscr{D}_p$ is a tight 4-design in $\mathscr{G}_{(p-1)/2,\,p}$.*

Proof. According to [3, Theorem 3], $\mathscr{D}_p$ consists on $p(p+1)/2=d_{[0,\dots,0]}+d_{[2,0,\dots,0]}=d_2^+$ subspaces with same pairwise chordal distance $d^2=(p+1)^2/4(p+2)$. Since $d^2=\sum\sin^2\theta_i=(p-1)/2-\sum y_i$, the conclusion follows, applying Theorem 9 to the polynomial $f=4(p+2)(\sum Y_i)-(p^2-5)/p^2-5$. □

4.3. The case $k=3$

From the definitions, one has $d_3^+=d_2^+=(n(n+1))/2$, and d_3^- equals 0 unless $m=1$ in which case $d_3^-=\binom{n+2}{3}$, or $m=3$ in which case $d_3^-=d_{(1,1,1)}=\binom{n}{3}$. Therefore, it is very unlikely that tight 6-designs exist for $m\neq 1,3$.

A family of packings in the Grassmannian $\mathscr{G}_{2^k,2^m}$ is constructed in [3, Theorem 1], each of them are orbits under the Clifford group $\mathscr{C}_m$. We have checked that, for $m=2,3,4$, and for $(m,k)=(5,4)$, these packings are 6-designs. For each m, the smallest of these sets corresponds to $k=m-1$ and its cardinality equals $2^{2m}+2^m-2=2(d_2^+-1)$.

Remark 11. It is known that the orbits of the Clifford group *on the first Grassmannian* provide 6-designs, because the first nontrivial invariant polynomial of this group has degree 8 (and corresponds to the Hamming code, see [11] and the earlier work of B. Runge). We conjecture that the orbits of the Clifford group on all the Grassmannians provide 6-designs. This, according to [1, Theorem 4.5, Remark 4.6], is equivalent to the fact that the $\mathscr{C}_m$-invariants of the $(n=2^m)$ Gl_n-irreducible modules canonically associated to the partitions $(4,2)$ and $(2,2,2)$ (denoted F_n^{μ} in [8]) have dimension 1.

Appendix A.

We list in Table 1 some values of d_k^+ and d_k^- for $m=2,$ 3 and 4.

Table 1

$m=2$				$m=3$				$m=4$			
n	k	d_k^+	d_k^-	n	k	d_k^+	d_k^-	n	k	d_k^+	d_k^-
4	1	1	0	6	1	1	0	8	1	1	0
4	2	10	3	6	2	21	0	8	2	36	0
4	3	10	3	6	3	21	10	8	3	36	0
4	4	40	18	6	4	210	10	8	4	630	35
4	5	40	18	6	5	210	136	8	5	630	35
5	1	1	0	7	1	1	0	9	1	1	0
5	2	15	10	7	2	28	0	9	2	45	0
5	3	15	10	7	3	28	35	9	3	45	0
5	4	105	91	7	4	378	35	9	4	990	126
5	5	105	91	7	5	378	651	9	5	990	126
6	1	1	0	8	1	1	0	10	1	1	0
6	2	21	15	8	2	36	0	10	2	55	0
6	3	21	15	8	3	36	56	10	3	55	0
6	4	210	190	8	4	630	56	10	4	1485	210
6	5	210	190	8	5	630	1352	10	5	1485	210
7	1	1	0	9	1	1	0	11	1	1	0
7	2	28	21	9	2	45	0	11	2	66	0
7	3	28	21	9	3	45	84	11	3	66	0
7	4	378	351	9	4	990	84	11	4	2145	330
7	5	378	351	9	5	990	2541	11	5	2145	330
8	1	1	0	10	1	1	0	12	1	1	0
8	2	36	28	10	2	55	0	12	2	78	0
8	3	36	28	10	3	55	120	12	3	78	0
8	4	630	595	10	4	1485	120	12	4	3003	495
8	5	630	595	10	5	1485	4432	12	5	3003	495
9	1	1	0	11	1	1	0	13	1	1	0
9	2	45	36	11	2	66	0	13	2	91	0
9	3	45	36	11	3	66	165	13	3	91	0
9	4	990	946	11	4	2145	165	13	4	4095	715
9	5	990	946	11	5	2145	7293	13	5	4095	715

References

[1] C. Bachoc, R. Coulangeon, G. Nebe, Designs in Grassmanian spaces and lattices, J. Algebraic Combin. 16 (1) (2002) 5–19.

[2] E. Bannai, R.M. Damerell, Tight spherical designs II, J. London Math. Soc. 21 (3) (1980) 13–30.

[3] A.R. Calderbank, R.H. Hardin, E.M. Rains, P.W. Shor, N.J.A. Sloane, A group-theoretic framework for the construction of packings in Grassmannian spaces, J. Algebraic Combin. 9 (2) (1999) 129–140.

[4] J.H. Conway, R.H. Hardin, N.J.A. Sloane, Packing lines, planes, etc., packings in Grassmannian spaces, Exper. Math. 5 (1996) 139–159.

[5] Ph. Delsarte, J.M. Goethals, J.J. Seidel, Bounds for systems of lines, and Jacobi polynomials, Philips Res. Rep. 30 (1975) 91*–105* (Bouwkamp volume).
[6] Ph. Delsarte, J.M. Goethals, J.J. Seidel, Spherical codes and designs, Geom. Dedicata (6) (1977) 363–388.
[7] W. Fulton, J. Harris, Representation Theory, a first course, in: GTM, Vol. 129, Springer, Berlin, 1991.
[8] R. Goodman, N.R. Wallach, Representations and invariants of the classical groups, in: Encyclopedia of Mathematics and its Applications, Vol. 68, Cambridge University Press, Cambridge, 1998.
[9] A.T. James, A.G. Constantine, Generalized Jacobi polynomials as spherical functions of the Grassmann manifold, Proc. London Math. Soc. 29 (3) (1974) 174–192.
[10] T. Koornwinder, Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula, J. London Math. Soc. 18 (2) (1978) 101–114.
[11] G. Nebe, E. Rains, N.J.A. Sloane, The invariants of the Clifford groups, Des. Codes Cryptogr. 24 (1) (2001) 99–122.