Exponential stability of singularly perturbed impulsive delay differential equations

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Abstract

In this paper, the exponential stability of singularly perturbed impulsive delay differential equations (SPIDDEs) is concerned. We first establish a delay differential inequality, which is useful to deal with the stability of SPIDDEs, and then by the obtained inequality, a sufficient condition is provided to ensure that any solution of SPIDDEs is exponentially stable for sufficiently small $\varepsilon > 0$. A numerical example and the simulation result show the effectiveness of our theoretical result.

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1. Introduction

Singularly perturbed delay differential equations arise in the study of “optically bistable device” [1] and in a variety models for physiological processes or diseases [2]. Such a problem has also appeared to describe the so-called human pupil-light reflex [3]. For example, Ikeda [4] adopted the model

$$\varepsilon y'(t, \varepsilon) = -y(t, \varepsilon) + A^2 \left[1 + 2B \cos(y(t - 1, \varepsilon))\right]$$
to describe an optically bistable device and showed numerically that instability or chaotic behavior occurs for small $\varepsilon$ and certain values of $A$, $B$. In recent years, in a number of papers [5–11], with or without delays, the stability of singularly perturbed differential equations (SPDEs) are considered.

But, in many systems, in addition to delay effects and singular perturbation, there also have impulse effects [12–15]. For example, some biological systems such as biological neural networks and bursting rhythm models in pathology, as well as frequency-modulated signal processing systems, and flying motions, are characterized by abrupt changes of states at certain time instants. Their study is assuming a greater importance [12,16].

In [17], the exponential stability of one form of singularly perturbed impulse systems without delays is discussed, but as well known, delay is also a factor that make system instability. So it is necessary to study the stability of singularly perturbed impulse delay differential equations (SPIDDEs). In this paper, the exponential stability of another form of singularly perturbed systems [5] with delays and impulses is concerned. First, a delay differential inequality, which is useful to deal with the stability of SPIDDEs, is derived, and then by the obtained inequality, a sufficient condition is provided to ensure that any solution of SPIDDEs is exponentially stable for sufficiently small $\varepsilon > 0$. A numerical example is given to illustrate the effectiveness of the theoretical result.

2. Preliminaries

Let $R^n$ be the space of $n$-dimensional real column vectors and $R^{m \times n}$ denotes the set of $m \times n$ real matrices. For $A, B \in R^{m \times n}$ or $A, B \in R^n$, $A \geq B$ ($A > B$) means that each pair of corresponding elements of $A$ and $B$ satisfies the inequality “$\geq$” (>). Especially, $A$ is called a nonnegative matrix if $A \geq 0$, and $z$ is called a positive vector if $z > 0$.

$C[X, Y]$ denotes the space of continuous mappings from the topological space $X$ to the topological space $Y$. Especially, let $C \triangleq C([−\tau, 0], R^n)$.

$PC[I, R^n] \triangleq \{\phi : I \rightarrow R^n \mid \phi(t) = \phi(t) \text{ for } t \in I, \phi(t−) = \phi(t) \text{ for all but points } tk \in I\}$, where $I \subset R$ is an interval, $\phi(t−)$ and $\phi(t+)$ denote the left limit and right limit of scalar function $\phi(t)$, respectively. Especially, let $PC = PC([−\tau, 0], R^n)$.

For $x \in R^n$, $A \in R^{n \times n}$, we define

$$[x]^+ = ([x_1], \ldots, [x_n])^T,$$

$$[A]^+ = ([a_{ij}])_{n \times n},$$

$$\phi(t) = ([\phi_1(t)]_+, \ldots, [\phi_n(t)]_+)^T,$$

$$[\phi(t)]_+ = [[\phi(t)]_+]_+,$$

where $[\phi_i(t)]_+ = \sup_{-\tau \leq s \leq 0} \phi_i(t + s)$, and introduce the corresponding norm for them as follows

$$\|x\| = \max_{1 \leq i \leq n} \{|x_i|\}, \quad \|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|, \quad \|\phi\| = \max_{1 \leq i \leq n} \{|[\phi_i(t)]_+|\}.$$

Consider a class of SPIDDEs

$$\left\{ \begin{array}{l}
\varepsilon \dot{x}(t) = A(t)x(t) + f(t, x(t− \tau(t))), \quad t \geq t_0, \ t \neq t_k, \\
x(t_k) = J_k(t_k, x(t_{k−})), \quad k = 1, 2, \ldots,
\end{array} \right.$$

where $0 \leq \tau(t) \leq \tau, \ x(t) = (x_1(t), \ldots, x_n(t)) \in PC[R, R^n], \ A(t) = (a_{ij}(t))_{n \times n} \in PC[R, R^{n \times n}], \ J_k \in C[R \times R^n, R^n], \ f \in PC[R \times PC, R^n]$, and $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, and $t_1 < t_2 < \ldots$ is a strictly increasing sequence such that $\lim_{k \rightarrow \infty} t_k = \infty$. 

A function \( x(t) : [t_0 - \tau, \infty) \to \mathbb{R}^n \) is called a solution of (1) with the initial condition given by
\[
x(t) = \phi(t) \in PC, \quad t \in [t_0 - \tau, t_0],
\]
if \( x(t) \in PC[t_0 - \tau, \infty) \) satisfies (1) under the initial condition (2). We denote a solution through \((t_0, \phi)\) by \( x(t, t_0, \phi) \) or simply by \( x(t) \) if no confusion arisen.

As usual in the theory of impulsive differential equations, at the points of discontinuity \( t_k \) of the solution \( x(t) \) we assume that \( x(t_k) \equiv x(t_k^-) \). It is clear that, in general, the derivatives \( \dot{x}(t_k^-) \) does not exist. On the other hand, from the first equality of (1), there is a limit of \( \dot{x}(t_k^-) \) and we assume \( \dot{x}(t_k) \equiv \dot{x}(t_k^-) \).

The existence and uniqueness of the solution \( x(t) \) of (1) has been given by X.Z. Liu and G. Ballinger [18,19].

**Definition 1.** The solution of (1) is said to be exponentially stable for sufficiently small \( \varepsilon \) if there exist finite constant vectors \( K > 0 \) and \( \sigma > 0 \), which are independent of \( \varepsilon \in (0, \varepsilon_0] \) for some \( \varepsilon_0 \), and a constant \( \lambda > 0 \) such that
\[
[x(t) - y(t)]^+ \leq Ke^{\lambda(t-t_0)} \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \sup_{s \in [t_0-\tau, t_0]} [\phi(s) - \varphi(s)]^+ < \sigma.
\]
Here \( y(t) \) is the solution of (1) corresponding to the initial condition \( \varphi \).

**3. Main results**

In order to prove the main result in this paper, we first need the following technique lemma.

**Lemma 1.** Assume that
\[
\begin{align*}
D^+u(t) &\leq P(t)u(t) + Q(t) [u(t)]_\tau, \quad t \geq t_0, \\
u(t) &\equiv \varphi(t), \quad t \in [t_0 - \tau, t_0], \quad \varphi(t) \in PC,
\end{align*}
\]
where \( P(t) = (p_{ij}(t))_{n \times n} \geq 0 \) for \( t \geq t_0 \) and \( i \neq j \), \( Q(t) = (q_{ij}(t))_{n \times n} \geq 0 \) for \( t \geq t_0 \).

If there exist a positive vector \( z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n \) and two positive diagonal matrices \( L = \text{diag}[L_1, \ldots, L_n], \quad H = \text{diag}[h_1, \ldots, h_n] \) with \( 0 < h_i < 1 \) such that
\[
(Q(t) + HP(t) + L)z \leq 0, \quad t \geq t_0.
\]
Then we have
\[
u(t) \leq ze^{-\lambda(t-t_0)}, \quad t \geq t_0,
\]
where the positive constant \( \lambda \) is defined as
\[
0 < \lambda < \lambda_0 = \min_{1 \leq i \leq n} \left\{ \inf_{t \geq t_0} \lambda_i(t) : \lambda_i(t)z_i + \sum_{j=1}^{n} (p_{ij}(t) + q_{ij}(t)e^{\lambda_j(t)\tau})z_j = 0 \right\},
\]
for the given \( z \).

**Proof.** Note that the result is trivial if \( \tau = 0 \). In the following, we assume that \( \tau > 0 \).

Denote
\[
F(\lambda_i(t)) = \lambda_i(t)z_i + \sum_{j=1}^{n} (p_{ij}(t) + q_{ij}(t)e^{\lambda_j(t)\tau})z_j, \quad i = 1, \ldots, n, \quad t \geq t_0,
\]

then for any given $t \geq t_0$, we have

$$F(0) = \sum_{j=1}^{n} (p_{ij}(t) + q_{ij}(t))z_j$$

$$\leq \sum_{j=1}^{n} p_{ij}(t)z_j - h_i \sum_{j=1}^{n} p_{ij}(t)z_j$$

$$= (1 - h_i) \sum_{j=1}^{n} p_{ij}(t)z_j$$

$$\leq -(1 - h_i) \frac{L_i}{h_i} z_i$$

$$< 0,$$  \hfill (8)

the first inequality and the second inequality are because (4), the last inequality is because $0 < h_i < 1$, $L_i > 0$, $z_i > 0$, $i = 1, \ldots, n$.

We also have

$$\lim_{\lambda_i(t) \to \infty} F(\lambda_i(t)) = \infty \quad \text{and} \quad F'(\lambda_i(t)) = z_i + \sum_{j=1}^{n} q_{ij}(t)e^{\lambda_i(t)\tau} > 0.$$  \hfill (9)

So by (8) and (9), for any $t \geq t_0$, there is a unique positive $\lambda_i(t)$ such that

$$\lambda_i(t)z_i + \sum_{j=1}^{n} (p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau})z_j = 0, \quad i = 1, \ldots, n.$$  \hfill (10)

Therefore, from the definition of $\lambda_0$, one can know that $\lambda_0 \geq 0$.

Next, we will show that $\lambda_0 \neq 0$.

If this is not true, fix $v_i$ satisfying $0 < h_i < v_i < 1$, $i = 1, \ldots, n$, there exist a $t^* \geq t_0$ and some integer $l$ such that $\lambda_l(t^*) < \delta$, where $0 < \delta < \min\{(1 - \frac{h_i}{v_i})\frac{L_i}{h_i}, \frac{1}{\tau} \ln \frac{1}{v_l}\}$, such that

$$\lambda_l(t^*)z_l + \sum_{j=1}^{n} (p_{lj}(t) + q_{lj}(t)e^{\lambda_l(t^*)\tau})z_j = 0.$$  \hfill (11)

Then, we have

$$0 = \lambda_l(t^*)z_l + \sum_{j=1}^{n} (p_{lj}(t) + q_{lj}(t)e^{\lambda_l(t^*)\tau})z_j$$

$$< \delta z_l + \sum_{j=1}^{n} (p_{lj}(t) + q_{lj}(t)e^{\delta \tau})z_j$$

$$< \delta z_l + \sum_{j=1}^{n} (p_{lj}(t) + \frac{1}{v_l} q_{lj}(t))z_j$$

$$\leq \delta z_l + \sum_{j=1}^{n} p_{lj}(t)z_j - \frac{h_l}{v_l} \sum_{j=1}^{n} p_{lj}(t)z_j$$
\[
\delta z_l + \left(1 - \frac{h_l}{v_l}\right) \sum_{j=1}^{n} p_{lj}(t) z_j \\
\leq \delta z_l - \left(1 - \frac{h_l}{v_l}\right) \frac{L_l}{h_l} z_l
\]

< 0,

this contradiction shows that \( \lambda_0 > 0 \), so there at least exists a positive constant \( \lambda \) such that \( 0 < \lambda < \lambda_0 \), that is, the definition of \( \lambda \) for (5) is reasonable.

Since \( \phi(t) \in PC \) is bounded and (4) holds, we always can choose a sufficiently large \( z > 0 \) such that

\[
u(t) \leq ze^{-\lambda(t-t_0)}, \quad t_0 - \tau \leq t \leq t_0.
\]

(12)

In order to prove (5), we first prove for any given \( k > 1 \),

\[
u_i(t) < kz_i e^{-\lambda(t-t_0)} \equiv v_i(t), \quad t \geq t_0, \; i = 1, \ldots, n.
\]

(13)

If (13) is not true, then by continuity of \( u(t) \), there must exist some integer \( m \) and \( \hat{t} > t_0 \) such that

\[
u_m(\hat{t}) = v_m(\hat{t}), \quad D^+ u_m(\hat{t}) \geq v_m'(\hat{t}),
\]

(14)

\[
u_i(t) \leq v_i(t), \quad t_0 - \tau \leq t \leq \hat{t}, \; i = 1, \ldots, n.
\]

(15)

So, by (3), the equality of (14), (15) and \( p_{ij}(t) \geq 0 \) for \( t \geq t_0 \) and \( i \neq j \), \( q_{ij}(t) \geq 0 \), \( t \geq t_0 \), and the definition of \( \lambda \), we derive that

\[
D^+ u_m(\hat{t}) \leq \sum_{j=1}^{n} \left( p_{mj}(\hat{t}) u_j(\hat{t}) + q_{mj}(\hat{t}) u_j(\hat{t} - \tau) \right)
\]

\[
\leq \sum_{j=1}^{n} \left( p_{mj}(\hat{t}) kz_j e^{-\lambda(\hat{t}-t_0)} + q_{mj}(\hat{t}) kz_j e^{-\lambda(\hat{t}-t_0)} \right)
\]

\[
= \sum_{j=1}^{n} \left( p_{mj}(\hat{t}) + q_{mj}(\hat{t}) e^{\lambda \tau} \right) kz_j e^{-\lambda(\hat{t}-t_0)}
\]

\[
< \sum_{j=1}^{n} \left( p_{mj}(\hat{t}) + q_{mj}(\hat{t}) e^{\lambda m(\hat{t}) \tau} \right) kz_j e^{-\lambda(\hat{t}-t_0)}
\]

\[
= -\lambda_m(\hat{t}) z_m k e^{-\lambda(\hat{t}-t_0)}
\]

\[
< -\lambda z_m k e^{-\lambda(\hat{t}-t_0)}
\]

\[
= v'_m(\hat{t}),
\]

which contradicts the inequality in (14), and so (13) holds for all \( t \geq t_0 \). Letting \( k \to 1 \), then (5) holds, and the proof is completed. \( \Box \)

**Remark 1.** If \( n = 1 \) in (3), concretely, set \( P(t) = -\alpha(t) \), \( Q(t) = \beta(t) \), where \( \alpha(t) \) and \( \beta(t) \) are continuous with \( \alpha(t) \geq \alpha_0 > 0 \) and \( 0 < \beta(t) \leq q \alpha(t) \) for all \( t \geq t_0 \) with \( 0 \leq q < 1 \), then we can easily obtain Lemma 2.1 in [5] by Lemma 1.
Remark 2. In (3), if \( P(t) \equiv P \) and \( Q(t) \equiv Q \) for \( t \geq t_0 \), where \( P \) and \( Q \) are constant matrices, and satisfy that \( P = (p_{ij})_{n \times n} \geq 0 \) for \( i \neq j \), \( Q = (q_{ij})_{n \times n} \geq 0 \). By a similar argument with Lemma 1, we have the following result.

**Corollary 1.** If \( -(P + Q) \) is a nonsingular \( M \)-matrix, then there must exist a positive vector \( z = (z_1, \ldots, z_n)^T \) such that
\[
u(t) \leq z e^{-\lambda(t-t_0)},
\]
where the positive constant \( \lambda \) is defined as
\[
0 < \lambda < \lambda_0 = \min_{1 \leq i \leq n} \left\{ \lambda_i : \lambda_iz_i + \sum_{j=1}^{n}(p_{ij} + q_{ij}e^{\lambda_i t})z_j = 0 \right\},
\]
for the given \( z \).

**Proof.** Since \( -(P + Q) \) is a nonsingular \( M \)-matrix, then by the property of \( M \)-matrix (see [20]), there must be a positive vector \( z = (z_1, \ldots, z_n)^T \) such that
\[
(P + Q)z < 0.
\]
Then by continuity, there must exist two positive diagonal matrices \( H = \text{diag}\{h_1, \ldots, h_n\} \) with \( 0 < h_i < 1 \) and \( L = \text{diag}\{L_1, \ldots, L_n\} \) such that
\[
(Q + HP + L)z \leq 0.
\]
By \( (P + Q)z < 0 \) and a similar argument with (6), we know there exists a \( \lambda > 0 \), which is determined by
\[
0 < \lambda < \lambda_0 = \min_{1 \leq i \leq n} \left\{ \lambda_i : \lambda_iz_i + \sum_{j=1}^{n}(p_{ij} + q_{ij}e^{\lambda_i t})z_j = 0 \right\},
\]
for the given \( z \).

The proof of the remainder is similar with that of Lemma 1, so be omitted here. □

**Theorem 1.** Assume that \( A(t) = (a_{ij}(t))_{n \times n} \geq 0 \) for \( t \geq t_0 \) and \( i \neq j \), further suppose that:

(H1) For any \( x, y \in R^n \), there exists nonnegative matrix \( U(t) = (u_{ij}(t))_{n \times n}, \ t \geq t_0 \), such that
\[
[f(t, x) - f(t, y)]^+ \leq U(t)[x - y]^+, \ t \geq t_0.
\]
(H2) For any \( x, y \in R^n \), there exist nonnegative constant matrices \( M_k \) such that
\[
[J_k(t, x) - J_k(t, y)]^+ \leq M_k[x - y]^+, \ t \geq t_0.
\]
(H3) There exist a positive vector \( z = (z_1, \ldots, z_n)^T \in R^n \) and two positive diagonal matrices \( W = \text{diag}\{w_1, \ldots, w_n\}, S = \text{diag}\{s_1, \ldots, s_n\} \), with \( 0 < s_i < 1 \), \( i = 1, \ldots, n \) such that
\[
(U(t) + SA(t) + W)z \leq 0, \ t \geq t_0.
\]
(H4) There exists a positive constant \( \eta \) satisfying
\[
\frac{\ln \eta_k}{t_k - t_{k-1}} \leq \eta < \lambda(\varepsilon), \ k = 1, 2, \ldots,
\]
where \( \eta_k \triangleq \max\{\|M_k\|, 1\} \), and \( \lambda(\varepsilon) \) is defined as

\[
0 < \lambda(\varepsilon) < \lambda_0(\varepsilon) = \min_{1 \leq i \leq n} \left\{ \inf_{t \geq t_0} \lambda_i(t,\varepsilon): \lambda_i z_i + \sum_{j=1}^{n} \left( \frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} \right) z_j = 0 \right\},
\]

(25)

for the given \( z \).

Then there exists a small \( \varepsilon_0 > 0 \) such that the solution of (1) is exponentially stable for sufficiently small \( \varepsilon \in (0, \varepsilon_0] \).

Proof. By a similar argument with (6), one can know that the \( \lambda(\varepsilon) \) defined by (25) is reasonable.

For any \( \phi, \psi \in PC \), let \( x(t), y(t) \) be two solutions of (1) through \( (t_0, \phi), (t_0, \psi) \), respectively. Since \( \phi, \psi \in PC \) are bounded and (23) holds, we can always choose a positive vector \( z \) such that

\[
[x(t) - y(t)]_+ \leq ze^{-\lambda(t-t_0)}, \quad t \in [t_0 - \tau, t_0].
\]

(26)

Denote

\[
\text{Sgn}(x) = \text{diag}\{\text{sgn}(x_1), \ldots, \text{sgn}(x_n)\},
\]

where \( \text{sgn}(\cdot) \) is the sign function. Calculating the upper right derivative \( D^+[x(t) - y(t)]_+ \) along the solution of (1), by condition (H1), we have

\[
D^+[x(t) - y(t)]_+ = \text{Sgn}(x(t) - y(t))(x(t) - y(t))' \\
\leq \text{Sgn}(x(t) - y(t)) \frac{A(t)}{\varepsilon} (x(t) - y(t)) + \frac{1}{\varepsilon} [ f(t, x(t-\tau(t))) - f(t, y(t-\tau(t))) ]_+ \\
\leq \frac{A(t)}{\varepsilon} [x(t) - y(t)]_+ + \frac{U(t)}{\varepsilon} [x(t) - y(t)]_{t_1}^+, \quad t_{k-1} \leq t < t_k, \quad t \geq t_0, \quad k = 1, 2, \ldots.
\]

(27)

From condition (H3), we have

\[
\left( \frac{U(t)}{\varepsilon} + S \frac{A(t)}{\varepsilon} + \frac{W}{\varepsilon} \right) z \leq 0, \quad t \geq t_0.
\]

(28)

Therefore, (27) and (28) imply that all the assumptions of Lemma 1 are true. So we have

\[
[x(t) - y(t)]_+ \leq ze^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_0, t_1],
\]

(29)

where \( \lambda(\varepsilon) \) is determined by (25) and the positive constant vector \( z \) is determined by (23).

Using the discrete part of (1), condition (H2), (29), the definition of \( \eta_k \) and \( M_1 z \leq \|M_1\|z \), we can obtain that

\[
[x(t_1) - y(t_1)]_+ = [J_1(t_1, x(t_1^-)) - J_1(t_1, y(t_1^-))]_+ \\
\leq M_1[x(t_1^-) - y(t_1^-)]_+ \\
\leq M_1ze^{-\lambda(\varepsilon)(t_1-t_0)} \\
\leq \|M_1\|ze^{-\lambda(\varepsilon)(t_1-t_0)} \\
\leq \eta_1 ze^{-\lambda(\varepsilon)(t_1-t_0)},
\]

(30)
and so, we have
\[ x(t) - y(t) \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_1 - \tau, t_1]. \] (31)

By a similar argument with (29), we can use (31) derive that
\[ x(t) - y(t) \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_1, t_2). \] (32)

Therefore, by simple induction, we have
\[ x(t) - y(t) \leq \eta_1 \cdots \eta_k - 1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_k - 1, t_k), k = 1, 2, \ldots. \] (33)

In term of (24), we have \( \eta_k \leq e^{\eta(t_k-t_k-1)}, k = 1, 2, \ldots, \) and then
\[ \eta_1 \cdots \eta_k - 1 \leq e^{\eta(t_k-t_0)} \leq e^{\eta(t-t_0)}, \quad t \in [t_k - 1, t_k), k = 1, 2, \ldots. \] (34)

Therefore, combining (33) and (34), we obtain
\[ x(t) - y(t) \leq z e^{-\lambda(\varepsilon)-\eta}(t-t_0), \quad t \in [t_k - 1, t_k), k = 1, 2, \ldots, \forall \varepsilon > 0. \] (35)

For any \( t \geq t_0, \) let \( \lambda_i(t, \varepsilon) \) be defined as the unique positive zero of
\[ \lambda_i z_i + \sum_{j=1}^{n} \left( \frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} \right) z_j = 0. \] (36)

Differentiate both sides of (36) with respect to the variable \( \varepsilon, \) we have
\[ \frac{d}{d\varepsilon} \lambda_i(t, \varepsilon) = -\frac{\lambda_i z_i}{\varepsilon z_i + \sum_{j=1}^{n} u_{ij}(t) e^{\lambda_i \tau} z_j} < 0, \] (37)

so \( \lambda_i(t, \varepsilon) \) is monotonically decreasing with respect to the variable \( \varepsilon, \) which implies that \( \lambda_0(\varepsilon) \) is also monotonically decreasing with respect to the variable \( \varepsilon. \) So we can choose the \( \lambda(\varepsilon) \) in (25) satisfying the same monotonicity with \( \lambda_0(\varepsilon), \) for example, \( \lambda(\varepsilon) = \lambda_0(\varepsilon) - \delta, \) where \( 0 < \delta < \lambda_0(\varepsilon) - \lambda(\varepsilon). \) Hence we can deduce that there exists a small \( \varepsilon_0 > 0 \) such that the solution of (1) is exponentially stable for sufficiently small \( \varepsilon \in (0, \varepsilon_0]. \) The proof is completed. \( \square \)

**Remark 3.** If \( A(t) \equiv A \) and \( U(t) \equiv U \) for \( t \geq t_0, \) where \( A \) and \( U \) are constant matrices, and satisfy that \( A = (a_{ij})_{n \times n} \geq 0 \) for \( i \neq j, \) \( U = (u_{ij})_{n \times n} \geq 0. \) Then, using Corollary 1 and Theorem 1, we can easily obtain the following result.

**Corollary 2.** In addition to (H1) and (H2) hold, further assume that:

(H5) \( -(A + U) \) is a nonsingular \( M \)-matrix.

(H6) There exists a positive constant \( \eta \) satisfying
\[ \frac{\ln \eta_k}{t_k - t_{k-1}} \leq \eta < \lambda(\varepsilon), \quad k = 1, 2, \ldots, \] (38)

where \( \eta_k = \max\{\|M_k\|, 1\}, k = 1, 2, \ldots, \) the positive constant \( \lambda(\varepsilon) \) is determined by (25), in which \( a_{ij}(t) \equiv a_{ij}, u_{ij}(t) \equiv u_{ij}, i, j = 1, \ldots, n, t \geq t_0. \)

Then there exists a small \( \varepsilon_0 > 0 \) such that the solution of (1) is exponentially stable for sufficiently small \( \varepsilon \in (0, \varepsilon_0]. \)
Remark 4. From Lemma 1 and the proof of Theorem 1, it is obvious that the results obtained in this paper still hold for $\varepsilon = 1$. So this type of exponential stability can obviously be applied to general impulsive delay differential equations.

Remark 5. If $J_k(t,x) = x$, $t \geq t_0$, that is there have no impulses in (1), then by Theorem 1, we can obtain the following result.

Corollary 3. Assume that $A(t) = (a_{ij}(t))_{n \times n} \geq 0$ for $t \geq t_0$ and $i \neq j$, further suppose that (H1) and (H3) hold. Then there exists a small $\varepsilon_0 > 0$ such that the solution of (1) is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$.

By Corollary 3 and Remark 1, we can easily obtain Theorem 3.2 in [5] for the exponential stability of the following singularly perturbed delay differential equation:

$$\varepsilon \dot{x}(t) = f\left(t, x(t), x\left(t - \tau(t)\right)\right), \quad t \geq t_0. \quad (39)$$

Corollary 4. [5, Theorem 3.2] Assume that:

(H7) For any $x_1, x_2, u, u_1, u_2 \in \mathbb{C}^n$, where $\mathbb{C}^n$ is the $n$-dimensional complex vector space, $\| \cdot \|$ is the induced norm of the inner product $(u, v) = v^T u$, $f(t, x_1, u) - f(t, x_2, u) \leq \eta(t)\|x_1 - x_2\|^2, \quad t \geq t_0,$

and

$$f(t, x, u_1) - f(t, x, u_2) \leq \zeta(t)\|u_1 - u_2\|^2, \quad t \geq t_0. \quad (41)$$

(H8) There exist positive constants $\eta_0, 0 < q < 1$ such that $\eta(t) \leq \eta_0 < 0, \quad 0 < \zeta(t) \leq -q\eta(t), \quad t \geq t_0.$

Then there exists a small $\varepsilon_0 > 0$ such that the solution of (39) is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$.

Proof. Denote $z(t) = \|x(t) - y(t)\|^2$. By condition (H7), definition of the norm on $\mathbb{C}^n$, Schwartzs inequality, we have (Ref. [5])

$$D^+ z(t) \leq \frac{\eta(t)}{2\varepsilon} z(t) + \frac{\zeta(t)}{2\varepsilon} [z(t)]_\tau. \quad (43)$$

So by condition (H8), the conclusion follows from Corollary 3 and Remark 1. \quad \Box

Remark 6. Therefore, Corollary 3 extends the exponential stability results of singularly perturbed delays differential equations [5] to SPIDDEs.

4. An illustrative example

In this section, we will give an example to illustrate the exponential stability of (1).
Example. Consider the following SPIDDEs:

\[
\begin{align*}
\varepsilon \dot{x}_1(t) &= (-5 + \sin t)x_1(t) + (2 - \sin t) \arctan x_1(t - \tau(t)) \\
&\quad + (1 + \cos t) \arctan x_2(t - \tau(t)), \quad t \neq t_k, \\
\varepsilon \dot{x}_2(t) &= (-6 + 2\cos t)x_2(t) + \sin t \arctan x_1(t - \tau(t)) \\
&\quad - 2\cos t \arctan x_2(t - \tau(t)), \quad t \neq t_k, \\
x_1(t_k) &= -0.2e^{0.2k}x_1(t_k^-) + 0.4e^{0.2k}x_2(t_k^-), \\
x_2(t_k) &= -0.4e^{0.2k}x_1(t_k^-) + 0.6e^{0.2k}x_2(t_k^-),
\end{align*}
\]

where \(\tau(t) = e^{-t} \leq 1, t \geq t_0 = 0, t_k = t_{k-1} + k, k = 1, 2, \ldots\).

We can easily find that conditions (H1) and (H2) are satisfied with

\[
A(t) = \begin{pmatrix} -5 + \sin t & 0 \\ 0 & -6 + 2\cos t \end{pmatrix}, \quad U(t) = \begin{pmatrix} 2 - \sin t & 1 + \cos t \\ \sin t & -2\cos t \end{pmatrix},
\]

\[
M_k = e^{0.2k} \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}.
\]

So there exist \(z = (1, 1)^T, W = \text{diag}(0.2, 1)\) and \(S = \text{diag}(0.9, 0.5)\) such that

\[
(U(t) + SA(t) + W)z = (-0.2950, -0.5858)^T \leq 0, \quad t \geq t_0,
\]

and \(\eta_k = e^{0.2k} = \max\{1, \|M_k\|\}\), then we obtain that there exists an \(\eta = 0.2 > 0\) such that

\[
\frac{\ln \eta_k}{t_k - t_{k-1}} = \frac{0.2k}{k} = 0.2 \leq \eta, \quad k = 1, 2, \ldots.
\]

And for any \(\varepsilon > 0\), the positive constant \(\lambda(\varepsilon)\) is determined by the following equations:

\[
\begin{align*}
\lambda_1(t) + \frac{1}{\varepsilon}(-5 + \sin t + (3 - \sin t + \cos t)e^{\lambda_1(t)}) &= 0, \\
\lambda_2(t) + \frac{1}{\varepsilon}(-6 + 2\cos t + (\sin t - 2\cos t)e^{\lambda_2(t)}) &= 0,
\end{align*}
\]

(47)

Fig. 1. The simulation result of system (44) with the initial condition \(x(s) = [\sin(s), \cos(s)]^T\) and \(\varepsilon = 0.1, -1 \leq s \leq 0.\)
so for a given \( \varepsilon \), we can obtain the corresponding \( \lambda \) by (47). For example, by computation, if \( \varepsilon = 0.1 \), then \( \lambda = 0.2143 > 0.2 = \eta \); if \( \varepsilon = 0.01 \), then \( \lambda = 0.2180 > 0.2 = \eta \), and by the proof of Theorem 1, we know that \( \lambda \) is monotonically decreasing with respect to the variable \( \varepsilon \), then there exists an \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), we have \( \lambda > \eta \). Therefore, all the conditions of Theorem 1 are satisfied, we conclude that the solution of (44) is exponentially stable for sufficiently small \( \varepsilon > 0 \).

In what follows, the simulation result is illustrated in Figs. 1 and 2 with different values of \( \varepsilon > 0 \).

References