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Letter to the Editor

Comments on some recent methods for the simultaneous determination of polynomial zeros

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Abstract

In this note we give some comments on the recent results concerning a simultaneous method of the fourth-order for finding complex zeros in circular interval arithmetic. The main discussion is directed to a rediscovered iterative formula and its modification, presented recently in (Sun and Kosmol (J. Comput. Appl. Math. 130 (2001) 293)). The presented comments include some critical parts of the papers (Jpn. J. Indust. Appl. Math. 15 (1998) 295) and (Sun and Kosmol, 2001) which treat the same subject. © 2002 Elsevier Science B.V. All rights reserved.

The aim of this note is to give some comments on the recent results concerning iterative methods for the simultaneous determination of all zeros of a polynomial. The simultaneous zero-finding methods are often the subject of investigation of scientific papers and monographs (see [10]) for their interesting properties and convenient application when they are implemented on parallel computers. For this reason, independent discovery of the same simultaneous iterative method by several authors within a short period is not so surprising. So we have today the Weierstrass–Durand–Dochev–Kerner method [21,5,4,8] the Maehly–Ehrlich–Aberth method [9,6,1], the Börsch–Supan–Nouren method [2,12], the Prešić–Milovanović–Tanabe method [18,11,20], and so on, named after the authors who (re)discovered the mentioned simultaneous methods. Let us note that the rediscovery of the same formula is not a seldom event in mathematics. Same results appear sometimes independently almost at the same time, in other situations they are rediscovered after a certain period of time, quite frequently derived using different approaches. For example, in the recent paper [13] it was shown that even seven classes of iterative methods for solving nonlinear equations,

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constructed in various ways and expressed in different forms during the last 35 years, are mutually equivalent.

Let P be a monic polynomial of degree $n \geqslant 3$ with simple complex zeros $\zeta_1, \ldots, \zeta_n$, that is,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n} (z - \zeta_j) \quad (a_i \in \mathscr{C}).$$

In our consideration we will use the notation from the paper [19] and, according to this, we introduce

$$u_i = \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)}, \quad s_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_j}{z_i - z_j},$$

where $z_1, \ldots, z_n$ are some distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$. In the recent paper [19] the following fixed point relation has been derived:

$$\zeta_i = z_i - \frac{2u_i}{1 + s_i + [(1 + s_i)^2 + 4u_i \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j/(z_i - z_j)(\zeta_i - z_j)]_*^{1/2}} \quad (i = 1, \ldots, n), \tag{1}$$

where the symbol $*$ denotes a proper value of the square root.

Let $Z = \{c; r\} = \{z \mid |z - c| \leqslant r\}$ denote a disk Z with center c and radius r. Assume that an array of disjoint initial disks $Z_1^{(0)} = \{z_1^{(0)}; r_1^{(0)}\}, \ldots, Z_n^{(0)} = \{z_n^{(0)}; r_n^{(0)}\}$ was found such that $\zeta_i \in Z_i^{(0)}$ $(i = 1, \ldots, n)$. Then, starting from the fixed point relation (1) and using inclusion isotonicity property of circular interval arithmetic, the following interval method for the simultaneous inclusion of the distinct zeros was derived in [19]:

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{2u_i^{(m)}}{1 + s_i^{(m)} + [(1 + s_i^{(m)})^2 + 4u_i^{(m)} \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j^{(m)}/(z_i^{(m)} - z_j^{(m)})(Z_i^{(m)} - z_j^{(m)})]_*^{1/2}} \tag{2}$$

for $i = 1, \ldots, n$ and $m = 0, 1, \ldots$. The symbol $*$ denotes a proper value of the square root of a disk.

Let

$$\rho^{(0)} = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \{|z_i^{(0)} - z_j^{(0)}| - r_j^{(0)}\}, \quad r^{(0)} = \max_{1 \leqslant j \leqslant n} r_j^{(0)}.$$

Assuming that the initial condition

$$\frac{r^{(0)}}{\rho^{(0)}} \left(1 + \frac{r^{(0)}}{\rho^{(0)}}\right)^n < \frac{1}{3} \quad (n \geqslant 3) \tag{3}$$

holds, it was proved in [19] that the interval method (2) is convergent with the order of convergence *four*, and $\zeta_i \in Z_i^{(m)}$ for each $i = 1, \ldots, n$ and $m = 0, 1, 2, \ldots$. Furthermore, with the abbreviations

$$t_i^* = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_j}{(z_i - z_j)(\zeta_i - z_j)}, \quad T_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_j}{(z_i - z_j)(Z_i - z_j)},$$

under the condition

$$u_i t_i^* - s_i u_i t_i^* + \mathrm{O}(r^4) \subset u_i T_i \tag{4}$$

the following interval method was derived from the interval method (2):

$$\hat{Z}_i = z_i - \frac{u_i}{1 + s_i + u_i T_i} \quad (i = 1, \ldots, n). \tag{5}$$

The purpose of this note is to give some comments on the previously given results from [19]. In particular, a special attention is devoted to the fixed point relation (1), the corresponding interval method (2) and the modified interval method (5).

Comment 1. The fixed point relation (1), which serves as the base for constructing the interval method (2), was obtained in [19] using rather complicated and extensive manipulations. However, three years before, Petković et al. [14] gave a very short derivation of (1) dealing with the auxiliary function

$$h_i(z) := \frac{P(z)}{\prod_{j \neq i}(z - z_j)} = u_i + (z - z_i)\left(1 + \sum_{j \neq i} \frac{u_j}{z - z_j}\right) \quad \text{(by Lagrangean interpolation)} \tag{6}$$

which has the same zeros as the polynomial P. For any zero ζ_i of P we obtain from (6) (taking $z := \zeta_i$)

$$\zeta_i = z_i - \frac{u_i}{1 + \sum_{j \neq i} u_j/(\zeta_i - z_j)} \quad (i = 1, \ldots, n), \tag{7}$$

which is the well-known fixed point relation derived in [2,3]. After rearranging (7) in the form

$$\frac{u_i}{(\zeta_i - z_i)^2} + \frac{1 + s_i}{\zeta_i - z_i} - \frac{1}{\zeta_i - z_i}\left(s_i - \sum_{j \neq i} \frac{u_j}{\zeta_i - z_j}\right) = 0,$$

we obtain the quadratic equation in $1/(\zeta_i - z_i)$:

$$\frac{u_i}{(\zeta_i - z_i)^2} + \frac{1 + s_i}{\zeta_i - z_i} - \sum_{j \neq i} \frac{u_j}{(z_i - z_j)(\zeta_i - z_j)} = 0.$$

Solving this equation we get (1).

Comment 2. The fixed point relation (1) is very similar to the iterative formula of the fourth order

$$\hat{z}_i = z_i - \frac{2u_i}{1 + s_i + [(1 + s_i)^2 + 4u_i \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j/(z_i - z_j)(z_i - z_j)]_*^{1/2}} \quad (i = 1, \ldots, n), \tag{8}$$

which was constructed in [14] by applying the classical Euler method [7] [1]

$$\hat{z} = z - \frac{2f(z)}{f'(z) \pm \sqrt{f'(z)^2 - 2f(z)f''(z)}}$$

to the function $h_i(z)$ given by (6). For this reason, all iterative methods in ordinary complex arithmetic as well as circular complex interval arithmetic, derived in [14], were referred to as Euler-like methods. Let us note that two another methods of the form (8), with the order of convergence *five* and *six*, were constructed in [14] with negligible number of additional numerical operations.

Comment 3. In 1998 Petković, Tričković and Herceg, in a voluminous study published in *Japan Journal of Industrial and Applied Mathematics* [14], derived the circular interval method (2) directly from (1) (using inclusion isotonicity property) and gave the convergence theorem (without proof). A detailed convergence analysis of (2), including initial conditions for the guaranteed convergence, was given in [15] using complex interval arithmetic. Therefore, the interval method (2), derived and studied in the latter paper [19], can be regarded as a rediscovered method.

Comment 4. The interval method (2) was further improved in [14]. Using Weierstrass' correction $u_i = P(z_i)/\prod_{j\neq i}(z_i - z_j)$, the following interval method with accelerated convergence was stated:

$$\hat{Z}_i = z_i - \frac{2u_i}{1 + s_i + [(1+s_i)^2 + 4u_i \sum_{j\neq i} u_j/(z_i - z_j)(Z_i - u_i - z_j)]_*^{1/2}} \quad (i = 1,\ldots,n). \tag{9}$$

The R-order of convergence of this method is at least $2+\sqrt{7} \approx 4.646$ or 5, depending on the type of the applied inversion of a disk in the denominator of (9) [14, Theorem 4]. A detailed convergence analysis of the improved method (9) was given in [16,17].

Comment 5. Condition (4) [19, (33)], necessary for the implementation of the simplified method (5) [19, formula 34], is not computationally verifiable since it deals with the (unknown) zero ζ_i. Even worse, the iterative formula (5) produces only circular approximations which does not necessarily possess inclusion property (that is, $\zeta_i \in Z_i$), which is the essence of all inclusion methods. The basic and natural property of any interval method $\hat{Z}_i = \phi_i(Z_i)$ is the validity of the fixed point relation $\zeta_i = \phi_i(\zeta_i)$ that arises after the substitution of the inclusion disk Z_i (containing the zero ζ_i) by the "point disk" $\{\zeta_i; 0\}$. However, in the case of the "interval method" (5) we have $\phi_i(\zeta_i) := z_i - u_i/(1 + s_i + u_i t_i^*) \neq \zeta_i$.

The following simple example illustrates the above comment. Let us consider the cubic equation $z^3 - z^2 + 4z - 4 = 0$, and choose the initial disks

$$Z_1^{(0)} = \{0.1 - 2.2\mathrm{i}; 0.3\}, \quad Z_2^{(0)} = \{0.1 + 2.2\mathrm{i}; 0.3\}, \quad Z_3^{(0)} = \{1.2; 0.3\}$$

[1] We note that Euler's method can be derived by expanding f in Taylor series about the origin, dropping third and higher order terms, and solving the obtained quadratic equation.

containing the exact zeros $\zeta_1 = -2i$, $\zeta_2 = 2i$, $\zeta_3 = 1$, respectively. Applying Equation (5) we obtain in the first iteration the following disks:

$$Z_1^{(0)} = \{-1.71 \times 10^{-6} - 2.0002716i; 0.00022\},$$

$$Z_2^{(0)} = \{-1.71 \times 10^{-6} + 2.0002716i; 0.00022\},$$

$$Z_3^{(0)} = \{1.00034; 0.0003366\}.$$

None of these disk contain the zeros of the considered equation. Note that the initial condition of the form (3) is satisfied, namely

$$\frac{r^{(0)}}{\rho^{(0)}}\left(1 + \frac{r^{(0)}}{\rho^{(0)}}\right)^3 \approx 0.205 < 1/3 \quad (r^{(0)} = 0.3,\ \rho^{(0)} \approx 2.16),$$

but the enclosure of the zeros is not attained.

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