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An Evolution Equation in Phase Space and the Weyl Correspondence

R. CRESSMAN

Department of Mathematics, University of British Columbia, Vancouver, Canada

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The Weyl correspondence that associates a quantum-mechanical operator to a Hamiltonian function on phase space is defined for all tempered distributions on R^2. The resulting Weyl operators are shown to include most Schroedinger operators for a system with one degree of freedom. For each tempered distribution, an evolution equation in phase space is defined that is formally equivalent to the dynamics of the Heisenberg picture. The evolution equation is studied both through a separation of variables technique that expresses the evolution operator as the difference of two Weyl operators and through the geometric properties of the distribution. For real tempered distributions with compact support the evolution equation has a unique solution if and only if the Weyl equation does. The evolution operator has skew-adjoint extensions that solve the evolution equation if the distribution satisfies an orthogonal symmetry condition.

1. INTRODUCTION

A classical dynamical system with one degree of freedom is represented by a function, the Hamiltonian, defined on the plane. The problem of quantization is to associate a quantum-mechanical operator to the Hamiltonian. This paper is concerned with two such associations, both of which arise from representations of the canonical commutation relations [5]: (a) the Weyl correspondence [10, 14] that leads to the Schroedinger equation and the conventional formulation of quantum-mechanical dynamics; (b) the phase-space correspondence that leads to the evolution equation and phase-space quantum mechanics (see Sect. 4 and [2]).

The difficulty with these quantization procedures is that one must decide between two physically-awkward alternatives in order to obtain a nice mathematical theory. On the one hand, if unnatural restrictions are placed on the Hamiltonians, the operators have such desirable properties as self-adjointness or boundedness. Most authors (e.g., [5, 10]) have pursued this alternative. At the other extreme, if the Hamiltonian is in a function space large enough so that the set of operators include most significant quantum-mechanical systems, the corresponding operator generally lacks the desirable properties. The latter point of view is adopted here.

This paper investigates primarily association (b). However, in Sect. 2, it begins by extending the Weyl correspondence to all tempered distributions on the plane as in [11]. These operators give rise to Schroedinger-like processes (called Weyl operators) for each distribution.

Through the main result, Theorem 3.4, the regular representation that is used in (b) is explicitly decomposed into its irreducible components. This decomposition shows that the evolution equation of Section 4 is equivalent to the Schroedinger-like process acting in one direction plus the conjugate process acting in the perpendicular direction. That is, phase-space quantum mechanics can be pictured as the combination of two independent copies of conventional quantum mechanics. The consequences of this equivalence, especially for real distributions written as a polynomial plus a distribution with compact support, are developed in Section 4. Finally, Section 5 disregards the connection with the Weyl operator, studies the evolution operator as an "integral" operator that looks remarkably like the ordinary convolution on the plane, and demonstrates that a skew-adjoint extension exists under quite general circumstances.

Excellent references to background material and related research areas in both mathematics and physics are found in [4, 11]. All the results of the sequel can be suitably generalized to a system with n degrees of freedom.

2. Representations and the Weyl Correspondence

A representation of the canonical commutation relations is a pair of self-adjoint operators X_1, X_2 acting on a Hilbert space that satisfy

$$X_1X_2 - X_2X_1 = icI$$

where I is the identity operator and the positive constant c, included for completeness, should be thought of as Planck's constant divided by 2π. For arbitrary representations there is always the difficulty of domains of operators. In this regard, the two concrete representations described next are well understood [5] and present no difficulties.

In the Schroedinger representation, $X_1 = Q$ and $X_2 = cP$ are the multiplier x and the differential operator $-icd/dx$, respectively, that operate on $L^2(R)$. This representation is irreducible. The regular representation acting on $L^2(R^2)$ is reducible (cf. Remark at end of Sect. 3) and is given by $X_1 = Y$, $X_2 = Z$. Here $Yf(x_1, x_2) = x_1 f(x_1, x_2) + (1/2)\, ic(\partial f/\partial x_2)(x_1, x_2)$ and $Zf(x_1, x_2) = x_2 f(x_1, x_2) - (1/2)\, ic(\partial f/\partial x_1)(x_1, x_2)$.

Henceforth, whenever X_1, X_2 are written, they refer either to one or to the other of the above pair of operators. Throughout the paper, unless otherwise specified, the functions f and g are always in $\mathscr{S}(R^2)$ —the Schwartz class of rapidly decreasing test functions on the plane; h is in $\mathscr{S}'(R^2)$—the space of tempered distributions with strong dual topology; ϕ and ψ are in $\mathscr{S}(R)$.

Let $T(X_1, X_2)f$ be the operator defined through the functional calculus of [1, 2]; namely,

$$T(X_1, X_2)f = (1/2\pi) \int_{R^2} \mathscr{F}f(x_1, x_2) \exp(-i(x_1 X_1 + x_2 X_2))\, dx_1\, dx_2 . \quad (2.1)$$

In the integral, $x_1 X_1 + x_2 X_2$ is the essentially self-adjoint operator generating the unitary group $\exp(it(x_1 X_1 + x_2 X_2))$ by Stone's Theorem and $\mathscr{F}$ is the usual Fourier transform given by

$$\mathscr{F}f(y) = (1/2\pi) \int_{R^2} e^{i(y \cdot z)} f(z)\, dz.$$

$T(Q, cP)$ is, in fact, the association between functions and operators suggested by Weyl [14].

Certainly (2.1) could be defined as a Bochner integral [15] for a much wider class of functions. However, to extend (2.1) to all tempered distributions, $\mathscr{S}(R^2)$ is the most convenient function space.

As in Poulsen [11], but for arbitrary c, Definition 2.2 below is the quantization in configuration space related to the Schroedinger representation. The Weyl correspondence does extend $T(Q, cP)$ since [11, Theorem 1] insures that, for $h \in \mathscr{S}(R^2)$, the restriction of $T(Q, cP)h$ to $\mathscr{S}(R)$ is simply the Weyl operator (i.e., the two operators agree on $\mathscr{S}(R)$).

Definition 2.1. The following two operators, defined initially on $\mathscr{S}(R^2)$, extend to homeomorphisms of $\mathscr{S}(R^2)$ and $\mathscr{S}'(R^2)$ and to unitary operators on $L^2(R^2)$.

1. (Partial Fourier Transform)

$$\mathscr{F}_2 f(x_1, x_2) = \frac{1}{(2\pi)^{1/2}} \int_R e^{(izx_2)} f(x_1, z)\, dz.$$

2. (Twisting Operator)

$$S_c f(x_1, x_2) = (c)^{1/2} f(x_1 - \tfrac{1}{2}cx_2, x_1 + \tfrac{1}{2}cx_2),$$

$$S_c^{-1} f(x_1, x_2) = (1/(c)^{1/2}) f((x_1 + x_2)/2, (x_2 - x_1)/c).$$

DEFINITION 2.2 (The Weyl Correspondence). Let $h[f]$ denote the action of h on the test function f and let $\phi \times \psi$ denote the function $\phi \times \psi(x_1, x_2) = \phi(x_1)\psi(x_2)$. Define a continuous linear map $A_c(h)$: $\mathscr{S}(R) \to \mathscr{S}'(R)$ by

$$\begin{aligned}(A_c(h)\phi)[\psi] &= (1/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h[\phi \times \psi], \text{ or equivalently,}\\ (A_c(h)\phi)(x) &= (1/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(\cdot, x)[\phi(\cdot)].\end{aligned} \tag{2.2}$$

To emphasize that $A_c(h)$ is a different operator from (2.1) because of its domain, $A_c(h)$ is called the Weyl operator corresponding to the Hamiltonian h. If the distribution is clear from the context, the dependence of A_c on h is often suppressed.

PROPOSITION 2.3. (a) *If A is a bounded operator on all of $L^2(R)$, then there is a unique tempered distribution whose Weyl operator agrees with A on $\mathscr{S}(R)$.*

(b) *If A is a symmetric operator on $L^2(R)$ (not necessarily bounded) with domain containing $\mathscr{S}(R)$, then there is a unique real tempered distribution whose Weyl operator agrees with A on $\mathscr{S}(R)$.*

Proof. The existence and uniqueness of the distribution is a direct result of the Schwartz Kernel Theorem [13] if the restriction of A to $\mathscr{S}(R)$ produces a continuous map A: $\mathscr{S}(R) \to \mathscr{S}'(R)$. This is obviously satisfied in (a).

(b) By the closed graph theorem [15], the above condition is satisfied by every operator that restricts to a closed operator from $\mathscr{S}(R)$ into $L^2(R)$ since $\mathscr{S}(R)$ is a Fréchet space. The operator in (b) is trivially closed.

All that is left to prove is that the distribution in (b) is real (i.e.,

$h[f]$ is real for all real-valued test functions). If $\overline{}$ denotes the complex conjugate, it is easy to check in general that

$$S_c^{-1}\mathscr{F}_2\bar{h}(x_1, x_2) = \overline{S_c^{-1}\mathscr{F}_2 h(x_2, x_1)}. \tag{2.3}$$

By symmetry, the distribution satisfies

$$S_c^{-1}\mathscr{F}_2 h[\phi \times \psi] = (A_c(h)\phi, \bar{\psi}) = (\phi, A_c(h)\bar{\psi}) = \overline{S_c^{-1}\mathscr{F}_2 h[\bar{\psi} \times \bar{\phi}]}.$$

Thus $S_c^{-1}\mathscr{F}_2 h(x_1, x_2) = \overline{S_c^{-1}\mathscr{F}_2(x_2, x_1)}$ and so h is real by (2.3).

Remark. A few words on what type of operators result from the Weyl correspondence. Note that many Schroedinger operators, which are usually formally self-adjoint, fall into the second part of Proposition 2.3. The fact that the distribution is real is significant because, intuitively, these are the the ones that are physically relevant for measurements. In Sections 4 and 5, real distributions will again be important.

The set of distributions whose Weyl operators are bounded on $L^2(R)$ with domain $\mathscr{S}(R)$ is much larger than $\mathscr{S}(R^2)$. In fact, the Weyl correspondence establishes an isometry (up to the factor $1/(2\pi c)^{1/2}$) between $L^2(R^2)$ and the Hilbert–Schmidt operators on $L^2(R)$ [10, 7]. Also in this set are all finite Radon measures h with total variation $\| h \|_1$. An upper bound for the operator norm, $\| A_c(h)\|_{\mathrm{op}} \leqslant (1/c)(2/\pi)^{1/2} \| h \|_1$, is determined by an elementary application of the Cauchy–Schwarz inequality.

3. The Skew Product

By letting the multiplication (i.e., skew product) of two functions correspond to the product of operators of the form (2.1), $\mathscr{S}(R^2)$ becomes an algebra in Proposition 3.1 (see [2] for the proof). The skew product is precisely the "twisted multiplication" of [5] for functions in $\mathscr{S}(R^2)$. As such, every statement involving the "twisted convolution" of [5, 7, 8] can be translated into one involving the skew product via the isomorphism provided in the first paper.

Proposition 3.1. *There is a unique continuous map* $*_c$: $\mathscr{S}(R^2) \times \mathscr{S}(R^2) \to \mathscr{S}(R^2)$ *called the skew product given by* $(f, g) \to f *_c g$ *that satisfies* $T(X_1, X_2) f T(X_1, X_2) g = T(X_1, X_2)(f *_c g)$ *and* $A_c(f)\, A_c(g) = A_c(f *_c g)$.

Let $(\tau(x)f)(y) = f(x+y)$ *and* $V_c f(x_1, x_2) = f(cx_2/2, -cx_1/2)$. *Then*

$$f *_c g(x_1, x_2)$$
$$= \frac{1}{2\pi}\int_{R^2} (\mathscr{F}_2\tau(x_1, x_2)f)\left(\frac{sc}{2}, t\right)(\mathscr{F}_2\tau(x_1, x_2)g)\left(-\frac{tc}{2}, s\right) ds\, dt$$
$$= \frac{1}{2\pi}\int_{R^2} (V_c\tau(x_1, x_2)f)(y)(\mathscr{F}\tau(x_1, x_2)g)(y)\, dy. \tag{3.1}$$

Notice that

$$\overline{f *_c g} = \bar{g} *_c \bar{f}. \tag{3.2}$$

For the same reason as in formula (2.1) (to effect an efficient extension to tempered distributions), the skew product is originally restricted to $\mathscr{S}(R^2)$. The extension is to the separately continuous map $*_c$: $\mathscr{S}'(R^2) \times \mathscr{S}(R^2) \to \mathscr{S}'(R^2)$ defined by the duality relation $h *_c f[g] = h[f *_c g]$. Properties of this map along with justification of the notation $*_c$ are summarized next.

PROPOSITION 3.2. (a) *If* $h \in \mathscr{S}(R^2)$, *then* $h *_c f$ *of the last paragraph is the same function as the skew product of* h *and* f *in Proposition* 3.1.

(b) *Comparison with* (3.1) *yields*

$$h *_c f(x) = (1/2\pi)(V_c\tau(x)h)[\mathscr{F}\tau(x)f]. \tag{3.3}$$

$h *_c f$ *is a* C^∞ *function with polynomially bounded derivatives of all orders that satisfies*

$$\begin{gathered} h *_c f[g] = h[f *_c g] \ (\textit{duality}), \\ h *_c (f *_c g) = (h *_c f) *_c g \ (\textit{associativity}). \end{gathered} \tag{3.4}$$

Proof. [2, Sect. 3].

Theorem 3.4, the main result, can now be stated after a preparatory lemma.

LEMMA 3.3. *If* $h \in \mathscr{S}(R^2)$, *then*

$$S_c^{-1}\mathscr{F}_2(h *_c \mathscr{F}_2^{-1}S_c f)(x_1, x_2) = (1/(2\pi c)^{1/2})\int_R S_c^{-1}\mathscr{F}_2 h(y, x_2) f(x_1, y)\, dy.$$

In other words, $U^{-1}(Uh *_c Uf) = (1/(2\pi c)^{1/2})\, h \circ f$ *where* $U = \mathscr{F}_2^{-1}S_c$ *and* $h \circ f(x_1, x_2) = \int_R h(y, x_2) f(x_1, y)\, dy$.

Proof. By (3.1),

$$h *_c f(x_1, x_2)$$

$$= \frac{1}{2\pi} \int_{R^2} e^{-(i(s+t)x_2)} \mathscr{F}_2 h\left(x_1 + \frac{sc}{2}, t\right) \mathscr{F}_2 f\left(x_1 - \frac{tc}{2}, s\right) ds\, dt$$

$$= \frac{c}{2\pi} \int_{R^2} e^{-(i(s+t)x_2)} S_c^{-1} \mathscr{F}_2 h(x_1 + (s-t)c/2, x_1 + (s+t)c/2)$$

$$\times \{S_c^{-1} \mathscr{F}_2 f(x_1 - (s+t)c/2, x_1 + (s-t)c/2)\}\, ds\, dt$$

$$= \frac{1}{\pi c} \int_{R^2} e^{-i(2vx_2/c)} U^{-1}h(x_1 + u, x_1 + v)\, U^{-1}f(x_1 - v, x_1 + u)\, du\, dv$$

on replacing $(s-t)c/2$ and $(s+t)c/2$ by u and v respectively. Therefore, $\mathscr{F}_2(h *_c Uf)(x_1, x_2)$ is equal to

$$\frac{1}{\pi c(2\pi)^{1/2}} \int_R e^{i(x_2 z)} \int_{R^2} e^{-i(2vz/c)} U^{-1}h(u, x_1 + v)\, f(x_1 - v, u)\, du\, dv\, dz$$

$$= \frac{1}{(2\pi)^{1/2}} \int_R U^{-1}h(u, x_1 + cx_2/2)\, f(x_1 - cx_2/2, u)\, du.$$

The lemma now follows immediately.

THEOREM 3.4. *If $h \in \mathscr{S}'(R^2)$, the map $\mathscr{S}(R^2) \to \mathscr{S}'(R^2)$ given by $f \to S_c^{-1}\mathscr{F}_2(h *_c \mathscr{F}_2^{-1} S_c f)$ is the unique extension of $I \otimes A_c(h)$ to a continuous linear map from $\mathscr{S}(R^2)$ into $\mathscr{S}'(R^2)$. In the distributional sense, two equivalent formulas are*

$$\begin{aligned} &S_c^{-1}\mathscr{F}_2(h *_c \mathscr{F}_2^{-1} S_c f)(x_1, x_2) \\ &\qquad = (1/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(\cdot, x_2)[f(x_1, \cdot)] \\ &S_c^{-1}\mathscr{F}_2(h *_c \mathscr{F}_2^{-1} S_c f)[g] \\ &\qquad = (1/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(y_1, y_2) \left[\int_R f(z, y_1)\, g(z, y_2)\, dz\right]. \end{aligned} \tag{3.5}$$

Proof. The linear span of $\mathscr{S}(R) \times \mathscr{S}(R)$ is dense in $\mathscr{S}(R^2)$. By the definition of the tensor product [13], all that has to be shown for the first part is that for arbitrary ϕ, ψ

$$U^{-1}(h *_c U(\phi \times \psi)) = \phi \times A_c(h)\psi.$$

By Lemma 3.3, the following diagram is commutative

$$\begin{array}{ccc}
\mathscr{S}(R^2) \times \mathscr{S}(R^2) & \xrightarrow{\quad *_c \quad} & \mathscr{S}(R^2) \\
\Big\downarrow S_c^{-1}\mathscr{F}_2 \qquad \Big\downarrow S_c^{-1}\mathscr{F}_2 & & \Big\downarrow S_c^{-1}\mathscr{F}_2 \\
\mathscr{S}(R^2) \times \mathscr{S}(R^2) & \xrightarrow{(1/(2\pi c))^{1/2} \circ} & \mathscr{S}(R^2)
\end{array}$$

Duality (3.4) and the above diagram immediately imply (3.5). Thus, by (2.2)

$$\begin{aligned}
U^{-1}(h *_c U(\phi \times \psi))(x_1, x_2) &= (1/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(\cdot, x_2)[\phi(x_1)\, \psi(\cdot)] \\
&= \phi(x_1)\, A_c(h)\, \psi(x_2) \\
&= (\phi \times A_c(h)\psi)(x_1, x_2).
\end{aligned}$$

Remark. I would like to explain the significance of the last theorem in the context of [5, Sect. 5]. Section 3 could be completely developed with $L^2(R^2)$ in place of both $\mathscr{S}(R^2)$ and $\mathscr{S}'(R^2)$. The regular representation, $f \to T(Y, Z)f$, of the algebra $L^2(R^2)$ under the skew product is, by Von Neumann's theorem [5], a direct sum of irreducible representations that are all unitarily equivalent to the Schroedinger representation, $f \to T(Q, cP)f$. By the unitary operator of Theorem 3.4, it is easy to exhibit invariant subspaces associated to an irreducible subrepresentation. An arbitrary subspace of the form $u \times L^2(R)$, $u \in L^2(R)$, is clearly invariant under $f \to S_c^{-1}\mathscr{F}_2(T(Y, Z)(\mathscr{F}_1^{-1}S_c f))$ and is actually minimal since the Schroedinger representation is irreducible. Thus a direct sum decomposition for this new representation is of the form $\bigoplus_{j=0}^{\infty} (u_j \times L^2(R))$ where $\{u_j\}$ is an orthonormal basis for $L^2(R)$. A decomposition for the regular representation is then $\bigoplus_{j=0}^{\infty} \mathscr{F}_2^{-1}S_c(u_j \times L^2(R))$. By an adept choice of basis, $\mathscr{F}_2^{-1}S_c(u_0 \times L^2(R))$ becomes the ideal considered in [5]. The ideal is generated by $\mathscr{F}_2^{-1}S_c(u_0 \times u_0)$ with $u_0(x) = e^{-x^2/2c}$. If $c = 1$, the basis corresponding to the Hermite polynomials [8] would be a natural choice as u_0 is the first element in this set.

4. The Evolution Operator: Relation to Weyl Operator

Either from the viewpoint of being formally equivalent to the passive form of the Schroedinger equation [2, Sect. 3] or from the statistical viewpoint of the Wigner–Moyal formalism ([7, Sect. III] and [9]), the correct equation to consider in phase space in order to describe the dynamics of the system with fixed Hamiltonian h is of the form

$$df/dt = i\{h *_c f - f *_c h\} \tag{4.1}$$

rather than only involving the operator $T(Y, Z)h$ on the right. The two points of view differ on what class of functions the solutions should be sought. For the latter, the functions must have corresponding positive definite Weyl operators.

I will adopt the former view of seeking solutions on $L^2(R^2)$. To do this, it is important to regard the evolution operator of Definition 4.1 as an operator on $L^2(R^2)$ with domain $\mathscr{D}(H_c) = \{f \in \mathscr{S}(R^2)\colon H_c f \in L^2(R^2)$ in the distributional sense$\}$ (and also $A_c(h)$ as an operator on $L^2(R)$ with $\mathscr{D}(A_c(h)) = \{\phi \in \mathscr{S}(R)\colon A_c(h)\phi \in L^2(R)\}$).

The evolution operator is closely related to the Weyl operator (see H_c' in Theorem 4.2). In order to compare them as Hilbert space operators, it is first necessary to know such aspects as; denseness of domains, closures of operators, and boundedness. After these preliminaries are considered in Theorem 4.2 and Proposition 4.3, Theorem 4.4 combines the results to obtain a concise statement of the relationship.

DEFINITION 4.1 (The Evolution Operator). Define the evolution operator $H_c\colon \mathscr{S}(R^2) \to \mathscr{S}'(R^2)$ as

$$H_c f = i(h *_c f - \overline{\bar{h} *_c \bar{f}}). \tag{4.2}$$

The dependence of H_c on h is suppressed. This operator is the quantization in phase space related to the regular representation.

The evolution operator is of the desired form (4.1) via the relation (3.2). By Anderson [2], if h is real, then H_c is a real operator. Furthermore, for general h,

$$H_c f(x) = (i/2\pi)\{V_c\tau(x)\, h[\mathscr{F}\tau(x) f] - V_{-c}\tau(x)\, h[\mathscr{F}\tau(x) f]\}. \tag{4.3}$$

THEOREM 4.2. (a) *H_c is bounded with domain $\mathscr{S}(R^2)$ if and only if A_c is bounded with domain $\mathscr{S}(R)$.*

(b) *If h is a real tempered distribution, then A_c is densely defined if and only if H_c is densely defined. Moreover, iA_c and H_c are both skew-symmetric when densely defined.*

Proof. Let $H_c' = S_c^{-1}\mathscr{F}_2 H_c \mathscr{F}_2^{-1} S_c$. It suffices to prove the statements with H_c' in place of H_c because they are unitarily equivalent. A similar argument as in the proof of Theorem 3.4 applied to the evolution operator demonstrates that $H_c'\colon \mathscr{S}(R^2) \to \mathscr{S}'(R^2)$ is the extension of $i\{I \otimes A_c(h) - \overline{A_c(\bar{h})} \otimes I\}$ where $\bar{A}$ denotes the complex conjugate operator $\bar{A}f = \overline{A\bar{f}}$. Combining (2.3) and (3.5),

$$H_c'f[g] = \frac{i}{(2\pi c)^{1/2}} S_c^{-1}\mathscr{F}_2 h(y_1, y_2) \times \left[\int_R f(z, y_1)\, g(z, y_2)\, dz - \int_R f(y_2, z)\, g(y_1, z)\, dz\right]. \quad (4.4)$$

(b) If $\mathscr{D}(A_c)$ is dense in $L^2(R)$ and h is real, then $\mathscr{D}(H_c')$ includes all functions in $\overline{\mathscr{D}(A_c)} \times \mathscr{D}(A_c)$ whose span is dense in $L^2(R^2)$.

Now assume that H_c' is densely defined. Let $\phi_0 \in \mathscr{S}(R)$ and $f_0 \in \mathscr{D}(H_c')$ be arbitrary. It will be shown that $\mathscr{D}(A_c)$ includes

$$\int_R f_0(y, x)\, \phi_0(y)\, dy.$$

Note that, by (2.2) and (4.4),

$$\begin{aligned} iA_c &\left(\int_R f_0(y, x)\, \phi_0(y)\, dy\right) [\psi] \\ &= (i/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(x_1, x_2) \left[\int_R f_0(y, x_1)\, \phi_0(y)\, dy\, \psi(x_2)\right] \\ &= H_c'f_0[\phi_0 \times \psi] \\ &\quad + (i/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(x_1, x_2) \left[\int f_0(x_2, y)\, \psi(y)\, dy\, \phi_0(x_1)\right]. \end{aligned}$$

An easy calculation gives the distributional equality;

$$\begin{aligned} S_c^{-1}\mathscr{F}_2 h(x_1, x_2) &\left[\int_R f_0(x_2, y)\, \psi(y)\, \phi_0(x_1)\, dy\right] \\ &= \int_R S_c^{-1}\mathscr{F}_2 h(x_1, x_2)[f_0(x_2, y)\, \phi_0(x_1)]\, \psi(y)\, dy. \end{aligned}$$

Since $S_c^{-1}\mathscr{F}_2 h(x_1, x_2)[f_0(x_2, y)\, \phi_0(x_1)]$ is clearly in $\mathscr{S}(R)$ as a function of y,

$$\begin{aligned} \left| iA_c \left(\int_R f_0(y, x)\, \phi_0(y)\, dy\right) [\psi] \right| &\leqslant \| H_c'f_0 \|_2 \| \phi_0 \|_2 \| \psi \|_2 + b \| \psi \|_2 \\ &= b' \| \psi \|_2 \end{aligned}$$

for some constants b and b' independent of ψ. Therefore, the domain of A_c contains all such functions. It is an easy exercise to show these are dense in $L^2(R)$.

As the proofs that iA_c and H_c are skew-symmetric both use a similar argument involving (2.3), only one is shown here.

$$
\begin{aligned}
(H_c'f, g) &= H_c'f[\bar{g}] \\
&= (i/(2\pi c)^{1/2})\, S_c^{-1}\mathscr{F}_2 h(x_1, x_2) \\
&\quad \times \left[\int_R f(z, x_1)\,\bar{g}(z, x_2)\,dz - \int_R f(x_2, z)\,\bar{g}(x_1, z)\,dz\right] \\
&= \Bigg(\frac{-i}{(2\pi c)^{1/2}}\, S_c^{-1}\mathscr{F}_2 h(x_2, x_1) \\
&\quad \times \left[\int_R g(z, x_2)\,\bar{f}(z, x_1)\,dz - \int_R g(x_1, z)\,\bar{f}(x_2, z)\,dz\right]\Bigg)^{-} \\
&= -\overline{H_c'g[\bar{f}]} \\
&= -(f, H_c'g).
\end{aligned}
$$

(a) If H_c is bounded with domain $\mathscr{S}(R^2)$, then $A_c(h)$ and $A_c(\bar{h})$ will have domain $\mathscr{S}(R)$ by part (b). If $A_c(h)$ were not bounded, there would be a sequence $\phi_n \in \mathscr{S}(R)$ such that $\| A_c(h)\phi_n \|_2 \to \infty$ but $\| \phi_n \|_2 = 1$. A contradiction occurs for this sequence; namely,

$$
\begin{aligned}
\| H_c'(\phi_1 \times \phi_n)\|_2 &= \| \phi_1 \times A_c(h)\,\phi_n - \overline{A_c(\bar{h})\,\phi_1} \times \phi_n \|_2 \\
&\geqslant \| \phi_1 \|\, \| A_c(h)\,\phi_n \| - \| A_c(\bar{h})\,\bar{\phi}_1 \|\, \| \phi_n \| \\
&\to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

If $A_c(h)$ is bounded with domain $\mathscr{S}(R)$, then the proof of Proposition 2.3 shows that $A_c(\bar{h})$ is the restriction of the adjoint of $A_c(h)$. Thus, $A_c(\bar{h})$ is bounded with domain $\mathscr{S}(R)$. The theorem follows. It should be noted that, while $\| H_c \|_{\text{op}} \leqslant 2 \| A_c \|_{\text{op}}$, there is no estimate in the other direction. Indeed, when $A_c = I$, the evolution operator is the zero operator.

PROPOSITION 4.3. *Let $h = h_1 + \mathscr{F} h_2$ be a tempered distribution where h_1 and h_2 are distributions with compact support. Then $\mathscr{D}(H_c) = \mathscr{S}(R^2)$, $\mathscr{D}(A_c(h)) = \mathscr{S}(R)$, and H_c' is in the closure of $i\{I \otimes A_c(h) - \overline{A_c(\bar{h})} \otimes I\}$.*

Proof. Suppose that h has compact support. By the local structure theory of distributions [13], there is a positive integer k and an r such that

$$h[f] \leqslant b \sup\{| D^m f(y)| : 0 \leqslant | m | \leqslant k, | y | \leqslant r\}$$

where b is some constant depending only on h and the notation $m = (m_1, m_2)$, $|m| = m_1 + m_2$ is the usual multiindex notation for derivatives. If $D^m_{(y)}$ means the derivatives with respect to the y variables, then the above inequality together with (3.1) and (3.3) yield

$$\begin{aligned} &| h *_c f(x_1, x_2)| \\ &= \left| \frac{1}{2\pi} h(y_1, y_2) \left[e^{-i(x_1,x_2)\cdot(-2y_2/c, 2y_1/c)} \mathscr{F}f\left(\frac{2(x_2 - y_2)}{c}, \frac{2(y_1 - x_1)}{c}\right)\right]\right| \\ &\leqslant b' \sup | D^m_{(y)}\{e^{-i(x_1,x_2)\cdot(-2y_2/c,2y_1/c)} \mathscr{F}f(2(x_2 - y_2)/c, 2(y_1 - x_1)/c)\}|. \end{aligned}$$

The supremum is taken over the same set as in the local structure of h.

The derivatives in the last expression will produce polynomials in x multiplied by derivatives of $\mathscr{F}f$ evaluated at a translated point. Because $\mathscr{F}f \in \mathscr{S}(R^2)$, $| h *_c f(x)|$ will decrease faster than any negative power of $|x|$ as $|x| \to \infty$. Hence, not only does $h *_c f \in L^2(R^2)$, but $h *_c f$ will approach 0 in $L^2(R^2)$ as f approaches 0 in $\mathscr{S}(R^2)$. The same method applied to $\bar{h} *_c f$ demonstrates $\mathscr{D}(H_c) = \mathscr{S}(R^2)$ and the proof of Theorem 4.2 implies $\mathscr{D}(A_c(h)) = \mathscr{D}(A_c(\bar{h})) = \mathscr{S}(R)$. A standard argument proves the last statement of the theorem.

The Fourier transformed result is true by the following. Let δ_0 be the Dirac delta distribution on the plane (i.e., $\delta_0[f] = f(0)$). By (3.3), one concludes $(2\pi\delta_0 *_c f)(x) = (\mathscr{F}V_c f)(x)$. Through the associativity in (3.4), $\mathscr{F}V_c(f *_c g) = (\mathscr{F}V_c f) *_c g$ and this extends to

$$\mathscr{F}V_c(h *_c f) = (\mathscr{F}V_c h) *_c f.$$

If $\mathscr{F}h$ has compact support, the above method shows $(\mathscr{F}V_c h) *_c f$ is in $L^2(R^2)$ et cetera.

Through comparison with the usual convolution on the plane [15], one can show

$$\frac{\partial}{\partial x_j}(h *_c f) = \frac{\partial h}{\partial x_j} *_c f + h *_c \frac{\partial f}{\partial x_j} \qquad \text{for} \quad j = 1, 2.$$

By means of this derivation, H_c: $\mathscr{S}(R^2) \to \mathscr{S}(R^2)$ is actually a continuous map when h has compact support.

A few examples to show that the results of this section are by no means true for general h. For instance, in Theorem 4.2(b), if $S_c^{-1}\mathscr{F}_2 h = u_1 \times u_2$ where u_2 belongs to $L^2(R)$ but u_1 does not, then $\mathscr{D}(A_c(h))$ is all of $\mathscr{S}(R)$ but $\mathscr{D}(A_c(\bar{h}))$ is not even dense in $L^2(R)$. A trivial example that shows h must also be restricted in Theorem 4.4 is to look at the constant function $h(x) = \lambda$, λ an arbitrary complex number off the real axis. Then $A_c(h) = \lambda I$ is not essentially self-adjoint while H_c is the zero operator.

THEOREM 4.4. *Let $h = h_1 + \mathscr{F}h_2 + h_3$ be a real tempered distribution where h_1 and h_2 have compact support and h_3 corresponds to a bounded Weyl operator with domain $\mathscr{S}(R)$ (cf. Remark at end of Section 2). Then H_c is essentially skew-adjoint if and only if A_c is essentially self-adjoint.*

Let $-iA_c$ generate the strongly continuous unitary group $U(t)$ on $L^2(R)$ and H_c generate $W(t)$ on $L^2(R^2)$. The groups, $U(t)$ and $W(t)$, solve the Weyl equation $d\phi/dt = -iA_c\phi$ and the evolution equation $df/dt = H_c f$ respectively. Explicitly, $W(-t)$ is the closure of $\mathscr{F}_2^{-1} S_c(\overline{U(t)} \otimes U(t))\, S_c^{-1} \mathscr{F}_2$.

Proof. Clearly, Proposition 4.3 remains valid with the addition of h_3. Suppose A_c is essentially self-adjoint. Since h is real, $\overline{A_c}$ is also essentially self-adjoint. Using the resolutions of the identity for the two operators, Ju. Berezanskii [3] proves that $I \otimes A_c - \overline{A_c} \otimes I$ is essentially self-adjoint. As $-iH_c'$ is an extension of this last operator, H_c' and H_c are essentially skew-adjoint. For similar results that concern tempered distributions that are not real, the reader is encouraged to see [6, 12].

Conversely, assume H_c is essentially skew-adjoint while iA_c is not. Without loss of generality, by the theory of deficiency indices [15], there is a nonzero element u_0 in $L^2(R)$ that is perpendicular to the range $\mathscr{R}(iA_c + I)$ (i.e., $((iA_c + I)\phi, u_0) = 0$ for all $\phi \in \mathscr{D}(iA_c)$). Then $\bar{u}_0 \times u_0$ is perpendicular to $\mathscr{R}(H_c' + 2I)$ because H_c' is in the closure of $I \otimes iA_c + \overline{iA_c} \otimes I$. This is a contradiction.

The relation between the two unitary groups is a consequence of the unitary equivalence between H_c and H_c'.

5. EVOLUTION OPERATOR: GEOMETRIC PROPERTIES

The simplicity of the evolution operator given in (4.3) points out why it is worth considering in its own right rather than through the unitary equivalence of (4.4). Theorem 5.2 is a striking justification for its study. Quite possibly, new properties of Weyl (Schroedinger) operators can be learned indirectly by means of the evolution operator.

LEMMA 5.1. *If a skew-symmetric operator K is unitarily equivalent to $-K$, then K has a skew-adjoint extension.*

Proof. Let $UKU^{-1} = -K$ be the assumed unitary equivalence. One can easily show that u_0 is perpendicular to $\mathscr{R}(K+I)$ if and only if $U^{-1}u_0$ is perpendicular to $\mathscr{R}(K-I)$. Thus K has equal deficiency indices and so has a skew-adjoint extension.

THEOREM 5.2. *Let h be a real tempered distribution and H_c be densely defined. Let U be an orthogonal transformation on the plane (with determinant* $\det U$) *and define a unitary operator using the same letter by $Uf(x) = f(Ux)$. If $Uh = -(\det U)h$ where $Uh[f] \equiv h[U^{-1}f]$, then H_c has a skew-adjoint extension.*

Proof. It is an exercise to check that $UH_cU^{-1} = -H_c$ using (4.3) and the following permutation relations.

$$\text{(i)}\quad \tau(Ux)\,U^{-1}f = U^{-1}\tau(x)f, \qquad \text{(ii)}\quad \mathscr{F}U^{-1}f = U^{-1}\mathscr{F}f,$$

$$\text{(iii)}\quad U\tau(Ux)h = \tau(x)\,Uh, \qquad \text{(iv)}\quad UV_{\pm c}h = V_{\pm(\det U)c}Uh.$$

Remark 1. It is rather misleading that the origin appears to play a central role in the above theorem. Orthogonal symmetry relations about any other point would do equally as well since the evolution operators corresponding to h and $\tau(x_0)h$ are unitarily equivalent under $Uf(x) = f(x + x_0)$. Notice that odd distributions ($h(-x) = -h(x)$) and distributions that depend only on distance satisfy the hypothesis of Theorem 5.2.

Remark 2. One purpose of the theorem is to suggest cases when the two operators H_c and iA_c are not equivalent. (In particular, when one has a skew-adjoint extension and the other does not.) In the case $\det U = -1$ both terms $h *_c f$ and $\overline{h *_c \bar{f}}$ of the evolution operator are required to insure a skew-adjoint extension. However, as the Weyl operator is determined by only the first term, there is no immediate reason why iA_c should have a skew-adjoint extension. Unfortunately, no concrete example of such a situation is readily available.

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