Embeddings into \(k\)-Efficient Groups

Graham Ellis

Mathematics Department, National University of Ireland, Galway, Galway, Ireland
E-mail: graham.ellis@nuigalway.ie

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We prove a theorem with the following corollary: For each integer \(k \geq 1\), an arbitrary finite group \(G\) embeds into some finite group \(G_k\) for which there exists an Eilenberg–Mac Lane CW-space \(X = K(G_k, 1)\) whose finite \(n\)-skeleton \(X^n\) has Euler–Poincaré characteristic \(\chi(X^n) = 1 + (-1)^n dH_n(G_k)\) for all \(n \leq k\). The theorem can be viewed as a generalisation of a result of J. Harlander [1996, \textit{J. Algebra} 182, 511–521] on the embedding of finite groups into groups with “efficient” presentations.

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1. INTRODUCTION

To each finite group \(G\) one can associate a reduced CW-space \(X\) whose fundamental group \(\pi_1 X\) is isomorphic to \(G\), whose universal cover \(\tilde{X}\) is contractible, and whose \(n\)-skeleton \(X^n\) is a finite CW-space for \(n \geq 0\). The \(n\)-skeleton of such a space is called an \(n\)-\textit{presentation} of the group \(G\). A 2-presentation is thus equivalent to the usual notion of a presentation for \(G\) in terms of generators and relators [11]. A 3-presentation can be viewed as a collection of generators and relators for \(G\) together with a set of “identities between the relators” [2]. This paper is concerned with the “efficiency” of \(n\)-presentations of finite groups and has its origins in the explicit examples of 3-presentations of finite groups listed in [3].

Given an \(n\)-presentation for \(G\), the isomorphisms \(H_i(X^n) \xrightarrow{\cong} H_i(G)\) (\(0 \leq i \leq n - 1\)) and surjection \(H_n(X^n) \twoheadrightarrow H_n(G)\) between the integral singular homology of \(X^n\) and the integral Eilenberg–Mac Lane homology of \(G\), together with the finiteness of each \(H_i(G)\), lead to the inequality

\[ (-1)^n \chi(X^n) \geq dH_n(G) + (-1)^n \]  \hspace{1cm} (1)
between the Euler–Poincaré characteristic of $X^n$ and the minimal number of generators of $H_n(G)$. (Recall that $\chi(X^n) = \sum_{i=0}^{n} (-1)^i |X^i|$ where $|X^i|$ denotes the number of $i$-cells in $X$.) We say that the $n$-presentation $X^n$ is efficient if equality holds in (1). We say that the group $G$ is $k$-efficient if it admits a CW-space $X$ whose $n$-skeleton is an efficient $n$-presentation of $G$ for all $1 \leq n \leq k$.

Any finite cyclic group is $k$-efficient for all $k \geq 1$. (To see this, recall that a finite cyclic group $C_q$ has a classifying space with exactly one cell in each dimension and, furthermore, that $H_{2i+1}(C_q) \cong C_q$ and $H_{2i+2}(C_q) \cong 0$ for $i \geq 0$.) Elementary results on direct products show, more generally, that any finite abelian group is $k$-efficient for all $k \geq 1$. However, some nonabelian finite groups admit no efficient $n$-presentation for certain values of $n$. For example, the alternating group $A_4$ admits no efficient 1-presentation and no efficient 3-presentation, though it does admit an efficient 2-presentation [3].

Adapting ideas of Harlander [5], we show that the situation for finite nonabelian groups is to some extent redeemed by the following theorem.

**Theorem 1.** Let $G$ be an arbitrary finite group. Then there exists an integer $M$ such that, for any $m \geq M$ and any 2-efficient group $Q$ of prime-power order, the direct product $G_k = G \times \prod_{i=1}^{m} Q$ is $k$-efficient.

On taking $Q$ to be a cyclic group of prime-power order, the theorem implies that for each $k \geq 1$ every finite group $G$ embeds into some finite $k$-efficient group $G_k$.

There is a substantial literature on the 2-efficiency of groups, a good survey of which is given in [7]. A finite group $G$ is 2-efficient if and only if it can be presented by a set $x$ of generators and a set $r$ of relators with $|x| = dH_1(G)$ and $|r| - |x| = dH_2(G)$. Many finite groups are known not to admit such presentations [9, 12]. However, Harlander [5] has proved that for any finite group $G$ and any prime $p$ coprime to the order of $G$ there is an integer $m$ such that the direct product $G \times \prod_{i=1}^{m} C_p$ admits an efficient 2-presentation. The above theorem shows that Harlander’s result holds with $C_p$ replaced by an arbitrary 2-efficient prime-power group. Harlander’s result is very much related to results in [4] on the proficiency gap of groups.

2. PROOF OF THE THEOREM

Let $G$ be a finite group. We let $d(G)$ denote the minimal number of elements needed to generate $G$. We define the deficiency and efficiency of an $n$-presentation $X^n$ of $G$, $n \geq 1$, to be the non-negative integers

$$def(X^n) = (-1)^n (\chi(X^n) - 1),$$

$$eff(X^n) = def(X^n) - dH_n(G).$$

Thus an $n$-presentation $X^n$ is efficient precisely when it has zero efficiency.
We prove the theorem by induction on $k$. When $k = 1$ it suffices to show that for any finite $p$-group $Q$, and for sufficiently large $m \geq 0$, the direct product $G_1 = G \times \prod_{i=1}^m Q$ satisfies $d(G_1) = d(G^{ab})$. This was proved in [5] for $Q$ a cyclic group, and the same arguments extend to an arbitrary $p$-group $Q$ as follows. First observe that on replacing $G$ with some direct product $G \times Q \times \cdots \times Q$ we may assume that $G^{ab} = \prod_{i=1}^d C_{n_i}$ is a direct product of $d$ cyclic groups each with order divisible by $p$. Then we can choose elements $g_1, \ldots, g_d$ of $G$ and commutators $[h_i, h'_i]$, $1 \leq i \leq m$, of $G$ that generate $G$. Set $G_1 = G \times \prod_{i=1}^m (Q \times Q)$ and set $q = d(Q) = d(G^{ab})$. Then $d(G_1) \geq d(G^{ab}) = d + 2qm$. But $G_1$ can be generated by $d + 2qm$ elements. To see this let $a_{ij}, b_{ij}, 1 \leq j \leq q$, be the generators of the $i$th factor in the product $\prod_{i=1}^m (Q \times Q)$. Then $G_1$ is generated by the $d + 2qm$ elements $g_1, \ldots, g_d, h_ia_{ij}, h'_ib_{ij}, 1 \leq i \leq m, 1 \leq j \leq q$ (since $[h_ia_{ij}, h'_ib_{ij}] = [h_i, h'_i]$). This proves that $d(G_1) = d(G^{ab})$.

When $k = 2$ it suffices to show that for any 2-efficient finite $p$-group $Q$, and for sufficiently large $m \geq 0$, the direct product $G_2 = G \times \prod_{i=1}^m Q$ admits a presentation on $d(G_2)$ generators and $d(G_2) + dH_2(G_2)$ relators. The Künneth theorem [10] states that

$$H_n(G \times Q) \cong \sum_{i+j=n} H_i(G) \otimes H_j(Q) \oplus \sum_{i+j=n-1} \text{Tor}(H_i(G), H_j(Q))$$

for all $n \geq 0$. This formula for $n = 1, 2$ implies that, on replacing $G$ with some direct product $G \times Q \times \cdots \times Q$, we can assume that both of the homology groups $H_i(G) = G^{ab}$, $H_2(G)$ are direct products of cyclic groups whose orders are multiples of $p$. Furthermore, replacing $G$ with $G_1$ if necessary, we can assume that $d(G) = d(G^{ab})$.

Choose a presentation $\langle x \mid r \rangle$ of $G$ with $|x| = d(G)$. Let $F$ be the free group on $x$ and let $R$ denote the kernel of the canonical surjection $F \to G$. One can derive a non-canonical isomorphism [8]

$$R/[R, F] \cong H_2(G) \oplus R/(R \cap [F, F]).$$

(2)

Since $G^{ab}$ is finite the rank of the free abelian group $R/[R \cap [F, F])$ must equal $|x|$. We can thus choose $d = |x| + dH_2(G)$ elements $s_1, \ldots, s_d$ in $R$ whose images generate $R/[R, F]$. Now the commutators $[r, x]$ with $r \in R, x \in x$ normally generate $[R, F]$. So we obtain a presentation $\langle x \mid s_i (1 \leq i \leq d), [r, x] (r \in R, x \in x) \rangle$ of $G$ with $d + |x| |r|$ relators. For $m \geq |x| |r|$ the following lemma implies that $G_2 = G \times \prod_{i=1}^m Q$ is 2-efficient.

**Lemma 2.** Let $G$ be a finite group with presentation $\mathcal{P} = \langle x \mid s, [r, x] \rangle$ where $x_1 \in x$, $r$ lies in the normal closure of the set $x \cup \{[r, x_1]\}$, and $|x| = d(G^{ab})$. Let $Q$ be a 2-efficient $p$-group, and suppose that $p$ divides all torsion coefficients of $H_1(G)$ and $H_2(G)$. Then $G \times Q$ admits a presentation of the form

$$\mathcal{P}' = \langle x, a \mid s, t, tr^{-1}, [a, x] \mid (a \in a, x \in x) \rangle$$
with \[
\text{eff}(\mathcal{P}) = \text{eff}(\mathcal{P}) - 1.
\]

**Proof.** Let \(\mathcal{P}' = \langle a \mid t' \rangle\) be an efficient presentation of \(Q\) with \(|a| = d(Q)\). Fix some relator \(t \in t'\) and set \(t = t' \setminus \{t\}\). The above presentation \(\mathcal{P}'\) then presents \(G \times Q\) since, modulo the relations of \(\mathcal{P}'\), we have \([x_1, r] = [x_1, t] = 1\) and hence \(t = r = 1\). By the Künneth formula for \(n = 2\),

\[
\begin{align*}
\text{eff}(\mathcal{P}'') &= \text{def}(\mathcal{P}') - dH_2(G \times Q) \\
&= |g| + |l| + |x| |a| - |x| - |a| \\
&\quad - dH_2(G) - dH_2(Q) - d(H_1(G) \otimes H_1(Q)) \\
&= |g| - |x| - dH_2(G) + |l| - |a| - dH_2(Q) \\
&= \text{eff}(\mathcal{P}) - 1 + \text{eff}(\mathcal{P}'') \\
&= \text{eff}(\mathcal{P}) - 1.
\end{align*}
\]

To prove the theorem for \(k \geq 3\) let us now fix some value of \(k (\geq 3)\) and, as an inductive hypothesis, suppose that the theorem has been proved for all values of \(k\) less than this fixed value. Using the inductive hypothesis to replace \(G\) with \(G_{k-1}\) if necessary, we can assume the existence of an Eilenberg–Mac Lane space \(X = K(G, 1)\) whose \(n\)-skeleton \(X^n\) is an efficient \(n\)-presentation of \(G\) for \(1 \leq n \leq k - 1\). We shall extend \(X^{k-1}\) to a \(k\)-presentation of a \(k\)-efficient direct product \(G \times Q \times \cdots \times Q\).

The homotopy group \(\pi_{k-1}X^{k-1}\) is a finitely generated abelian group. This can be seen from the isomorphisms

\[
\pi_{k-1}X^{k-1} \cong \pi_{k-1}\tilde{X}^{k-1} \cong H_{k-1}\tilde{X}^{k-1} \cong \ker(\oplus_{e \in E^{k-1}} ZG \to \oplus_{e \in E^{k-1}} ZG),
\]

where \(\tilde{X}^{k-1}\) is the universal cover of \(X^{k-1}\), \(E^n\) denotes the finite set of \(n\)-cells in \(X^{k-1}\), and \(ZG\) is the integral group ring. The second isomorphism is the Hurewicz isomorphism. As an abelian group \(\oplus_{e \in E^{k-1}} ZG\) is finitely generated, and therefore so too is the subgroup \(\pi_{k-1}X^{k-1}\).

Recall that the fundamental group \(G = \pi_1X^{k-1}\) acts on the homotopy group \(\pi_{k-1}X^{k-1}\). The quotient group \(\pi_{k-1}X^{k-1}/[\pi_{k-1}X^{k-1}, G]\) obtained by killing the \(G\)-action on \(\pi_{k-1}X^{k-1}\) can be expressed as a (non-canonical) direct sum of abelian groups,

\[
\pi_{k-1}X^{k-1}/[\pi_{k-1}X^{k-1}, G] \cong A \oplus H_k(G),
\]

where \(d(A) = dH_{k-1}(X^{k-1})\) since the homology of a finite group is finite. Let \(d = dH_{k-1}(X^{k-1}) + dH_k(G) = (-1)^{k-1}(\chi(X^{k-1}) - 1) + dH_k(G)\) and choose the elements \(s_1, \ldots, s_d\) in \(\pi_{k-1}X^{k-1}\) whose images generate
\[ \pi_{k-1}X^{k-1}/[\pi_{k-1}X^{k-1}, G] \]. Suppose that \( \pi \) is a minimal set of generators for \( G \). Choose a finite set \( r \) of generators for \( \pi_{k-1}X^{k-1} \) as a \( \mathbb{Z}G \)-module. The \( |x||r| \) “commutator” elements \( x.r - r (x \in \pi, r \in r) \) generate \( [\pi_{k-1}X^{k-1}, G] \) as a \( \mathbb{Z}G \)-module. Set \( m = |x||r| \).

Let \( X' \) be the \( k \)-dimensional CW-space obtained from \( X^{k-1} \) by attaching \( d + m \) \( k \)-dimensional cells, one cell via each of the \( d \) generators \( s_i \) and one cell via each of the \( m \) commutators \( x.r - r \). (An element \( s \in \pi_{k-1}X^{k-1} \) is represented by a map \( \phi_s : S^{k-1} \to X^{k-1} \). We say that a \( k \)-cell is attached via \( s \) to mean that its attaching map is \( \phi_s \).) Then \( X' \) is a \( k \)-presentation for \( G \) with \( d + m \) \( k \)-dimensional cells. Set \( G_k = G \times \prod_{i=1}^{m} Q_i \). We shall prove that \( G_k \) is \( k \)-efficient.

Results of Swan [12] imply that any \( 2 \)-efficient prime-power group \( Q \) is in fact \( k \)-efficient for all \( k \geq 1 \). So let \( Y \) be a CW-space whose \( n \)-skeleton is an efficient \( n \)-presentation of \( Q \) for \( n \leq k \). Let \( W = X' \times \prod_{i=1}^{m} Y \) be the direct product endowed with the canonical CW-cell structure. The following lemma implies that \( W^n \) is an efficient \( n \)-presentation of \( G_k \) for \( n \leq k - 1 \) and that \( W^k \) is a \( k \)-presentation with \( \text{eff}(W^k) = \text{eff}(X') \).

**Lemma 3.** Let \( U \) and \( V \) be CW-spaces whose \( n \)-skeletons are \( n \)-presentations of \( G \) and \( H \) for each \( n \leq k \) and, moreover, efficient \( n \)-presentations for \( n \leq k - 1 \). Suppose that the prime \( p \) divides all torsion coefficients of \( H_n(G) \) and \( H_n(H) \) for \( n \leq k - 1 \). Then the \( n \)-skeleton of \( U \times V \) is an \( n \)-presentation of \( G \times H \) for \( n \leq k \); for \( n \leq k - 1 \) it is efficient, and for \( n = k \) it satisfies

\[ \text{eff}((U \times V)^k) = \text{eff}(U^k) + \text{eff}(V^k). \]

**Proof.** The isomorphisms \( \pi_i(U \times V) \cong \pi_i U \times \pi_i V, \ i \geq 1 \), imply that \( U \times V \) is an \( n \)-presentation of \( G \times H \) for \( n \leq k \). Using the Künneth formula and the efficiency of \( U \) and \( V \), we find that

\[ \text{eff}((U \times V)^n) = \text{def}((U \times V)^n) - dH_n(G \times H) \]
\[ = \sum_{i=1}^{n} (-1)^{n-i} \left( \sum_{i+j=l} |U^l| \times |V^j| \right) \]
\[ - \sum_{i+j=n} d(H_i(G) \otimes H_j(H)) \]
\[ - \sum_{i+j=n-1} d\text{Tor}(H_i(G), H_j(H)) \]
\[ = \text{eff}(U^n) + \text{eff}(V^n) + 0 \]

for each \( n \leq k \).
The $k$-presentation $W^k$ is not (in general) efficient since
\[
\text{eff}(W^k) = \text{eff}(X') = (-1)^k \{(−1)^k|X^k| + \chi(X^{k−1}) − 1\} − dH_k(G) = m.
\]
However, by repeated use of the following lemma, we can reduce by $m$ the number of $k$-cells in $W^k$ to produce an efficient $k$-presentation of $G_k$.

**Lemma 4.** Let $U$ and $V$ be as in Lemma 3. Suppose that $\pi_{k−1}U^{k−1}$ is generated as a $\mathbb{Z}\pi_1$-module by a set $s \cup \{x.r − r\}$ with $s \in \pi_1U$ and $r \in \pi_{k−1}U^{k−1}$, and suppose that $\pi_{k−1}V^{k−1}$ is generated by a nonempty set $t$. Then $k$-dimensional cells can be attached to $(U \times V)^{k−1}$ to produce a $k$-presentation $W'$ with
\[
\text{eff}(W') = \text{eff}(U^k) + \text{eff}(V^k) − 1.
\]

**Proof.** The attaching maps of the $k$-cells of $(U \times V)^k$ yield a canonical set $w$ of generators for the $\mathbb{Z}\pi_1$-module $\pi_{k−1}(U \times V)^{k−1}$. We must modify $w$ to produce a generating set $w'$ with one less generator. The required CW-space $W'$ is then obtained by attaching to $(U \times V)^{k−1}$ one $k$-dimensional cell via each element in $w'$.

There are canonical inclusions $\pi_{k−1}U^{k−1} \hookrightarrow \pi_{k−1}(U \times V)^{k−1}$, $\pi_{k−1}V^{k−1} \hookrightarrow \pi_{k−1}(U \times V)^{k−1}$ via which we consider $s \cup \{x.r − r\}$ and $t$ to be subsets of $w$. In fact, $w$ is the disjoint union of the set $s \cup \{x.r − r\}$, the set $t$, and a set of generators which we denote by $w^\circ$. A precise description of $w^\circ$ follows from the tensor product of crossed chain complexes, the definition of which can be found on page 127 of [1]. The important point to note is that for any $t \in t$ the element $x.t − t$ lies in the submodule of $\pi_{k−1}(U \times V)^{k−1}$ generated by $w^\circ$.

Choose some $t \in t$ and let $w'$ be obtained from $w$ by deleting the element $x.r − r$, deleting the element $t$, and inserting the element $t − r$. Then $|w'| = |w| − 1$ and $w'$ generates $\pi_{k−1}(U \times V)^{k−1}$, since $x.r − r = (x.t − t) − x.(t − r) + (t − r)$ lies in the submodule generated by $w'$, which implies that $r$ lies in this submodule, which in turn implies that $t = r + (t − r)$ lies in this submodule. $\blacksquare$

3. A REMARK ON INFINITE GROUPS

Let $G$ be a finitely presented group with presentation $\mathcal{P} = \langle x | r \rangle$. Since isomorphism (2) holds even when $G$ is infinite, we see that the deficiency
\[
def(\mathcal{P}) = |x| − |x|\]
satisfies
\[
def(\mathcal{P}) \geq dH_2(G) − \text{rank}(G^{ab}).
\]
The presentation $\mathcal{P}$ is said to be efficient if equality holds in (3). We can define the efficiency of $\mathcal{P}$ to be $\text{eff}(\mathcal{P}) = \text{def}(\mathcal{P}) − dH_2(G) + \text{rank}(G^{ab})$.
and say that $G$ is $2$-efficient if it admits a presentation with zero efficiency on $d(G_{ab})$ generators. For finite $G$ these definitions agree with those above since $\text{rank}(G_{ab}) = 0$. Moreover, one can check that the proof of the cases $k = 1$ and 2 of Theorem 1 also applies to non-finite $G$. We thus have the following result, which strengthens Corollary 4.3 in [5].

**Theorem 4.** Let $G$ be a finitely presented group. Then there exists an integer $M$ such that, for any 2-efficient finite group $Q$ of prime-power order and any $m \geq M$, the direct product $G \times \prod_{i=1}^{m} Q$ is 2-efficient.

A graph of groups construction was used in [6] to provide an alternative proof of the fact that any finitely presented group $G$ embeds into a finitely presented efficient group.

**REFERENCES**