Global Approximations to the Principal Real-Valued Branch of the Lambert $W$-function

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(Received October 1997; revised and accepted November 1997)

Abstract—$W(z)$ is defined implicitly as the root of $W \exp(W) = z$. It is shown that a simple analytic approximation has a relative error of less than 5% over the whole domain $z \in [-\exp(-1), \infty]$ of the principle branch—sufficiently accurate so that four Newton iterations refine this approximation to a relative error smaller than $1.E-12$. As a second form of global approximation, the $W$-function is expanded as a series of rational Chebyshev functions $T_B$, in a shifted, logarithmic coordinate with an error that decreases exponentially fast with the series truncation. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Lambert $W$-function, Global approximations, Rational Chebyshev function series.

1. INTRODUCTION: THE LAMBERT $W$-FUNCTION

The Lambert $W$-function has been intensively studied beginning with Lambert in 1758 and Euler in 1779. Corless et al. [1] provide a comprehensive review of both applications and theory with 77 references. In this article, we shall concentrate only on the principal real-valued branch.

Fritsch, Shafer and Crowley [2] ($z \geq 0$), Corless et al. [1], and Barry et al. [3,4] have provided numerical algorithms. Because the Lambert function has very different behavior for large and small $z$, these authors initialized Newton (or higher-order) iterations using piecewise approximations, each accurate only on a limited subdomain. Here, we derive good global approximations by two different strategies.

2. ANALYTIC APPROXIMATIONS

The principal real-valued branch of the $W$-function has values $W \in [-1, \infty]$ for $z \in [-1/e, \infty]$. It is convenient to define a new function and new parameter such that both the domain and range are the nonnegative real axis

$$w \equiv W + 1 \leftrightarrow W \equiv w - 1,$$

$$y = 1 + \exp(1)z \leftrightarrow z = \exp(-1)(y - 1),$$

where $w$ is a root of $(w - 1) \exp(w) = y - 1$.

For large $y$,

$$w \sim \log(y) - \log(\log(y)) + \frac{1 + \log(\log(y))}{\log(y)} + \frac{[\log(\log(y))]^2 - 1}{2 \log^2(y)} + \cdots, \quad y \to \infty. \quad (3)$$

This work was supported by NSF Grant OCE9521133.
For small \( y \), near the limit point at \( y = 0 \),

\[
    w \approx \sqrt{2} \sqrt{y} - \left( \frac{2}{3} \right) y + \left( \frac{11}{36} \right) y^{3/2} + \cdots, \quad y \ll 1.
\]

Both lowest-order asymptotics are approximated by

\[
    w_0 \equiv \{ \log(y + 10) - \log(\log(y + 10)) \} \tanh(\lambda \sqrt{y}),
\]

\[
    \lambda = \frac{\sqrt{2}}{\log(10) - \log(\log(10))}.
\]

![Figure 1. Ratios of the analytical approximations to the shifted Lambert function, \( w(y)/w_0 \) and (thick curve) \( w(y)/w_0(y) \).](image1)

![Figure 2. Errors in Newton's iteration, beginning from the "improved" first guess, \( \tilde{w}_0 \). The fourth iteration is unseen because its errors are less than \( 1.E-12 \) for all \( y \).](image2)
The shift in the argument of the logarithms, if greater than one, insures that the logarithms are real for all positive real $y$; the final choice of 10 was made by experimentation. The tanh function asymptotes to unity for large positive $y$ so that the logarithms control the function, but for small $y$, the tanh is approximately equal to its argument, and thus enforces the $\sqrt{y}$ behavior. The maximum relative error of $w_0$ on $y \in [0, \infty]$ is only 15.4\%.

The ratio of $w_0/w(y)$ has a two-lobed structure resembling the first degree Hermite function, which suggests the improved analytical approximation

$$w_0 \equiv w_0 \left\{1 + \left(\frac{1}{10}\right)(\log(y) - 1.4) \exp (-0.075(\log(y) - 1.4)^2)\right\}.$$  \hfill (7)

This has a maximum relative error of only 4.7\% (Figure 1).

When the improved approximation $\tilde{w}_0$ is used as the first guess for Newton’s iteration $w \leftarrow w - ((w - 1) - \exp(-w)(y-1))/w$, only four iterations reduce the relative error to less than $10^{-12}$ over the entire domain (Figure 2).

3. RATIONAL CHERBYSEV EXPANSIONS

The error curves in Figure 1 decay to zero at both ends because the analytic approximations are exact in both asymptotic limits.

This suggests that it might be possible to expand the error curves in terms of a generalized Fourier series using basis functions appropriate for an infinite interval in the same $\log(y)$ coordinate as used in the graph, but with a shift-by-a-constant so that the origin in the expansion variable is centered between the two lobes of the error curves. Many choices of spectral expansion functions are possible. Our choice of rational Chebyshev functions $TB_j(x; L)$ \cite{5,6} has the advantage that these are just Fourier cosines with a change-of-coordinate, that is, $TB_j(x; L) \equiv \cos(jt(x; L))$ where the trigonometric coordinate $t(x)$ is defined below.

The approximation, using the lowest-order analytical approximation, is

$$w(y) \approx w_0(y) \sum_{j=0}^{\infty} a_j \cos(jt(y)).$$  \hfill (8)

$$t(y) = \arccot \left\{ \frac{\log(y) - 1.4}{L} \right\},$$  \hfill (9)

where $L$ is a user-choosable map parameter, here chosen as $L = 6$ by experimentation. To obtain the first $N$ coefficients, one can use interpolation, which requires evaluating $w(y)/w_0(y)$ at a set of $N$ discrete interpolation points and then multiplying this vector of numbers by an $N \times N$ matrix whose elements are trigonometric functions as explained in \cite{5,6}.

Theory shows that if the function being expanded decays exponentially fast to a constant as $|x| \to \infty$, where the expansion variable here is $x \equiv \log(y) - 1.4$, then the spectral coefficients $a_j$ (Table 1) will decay at an exponential but “subgeometric” rate. That is, $|a_j| < p \exp(-qj^r)$ in the limit $j \to \infty$, where $p$, $q$, and $r$ are constants with $0 < r < 1$. Unfortunately, the large $y$ asymptotics (3) show that $w_0(y)$ is decaying only algebraically (as $O(1/\pi)$). This produces a “plateau” in a log-linear plot of the spectral coefficients, Figure 3.

| $TB$ coefficients for $L = 6$ for $\{W[\exp(-1)(y - 1)] + 1\}/w_0(y)$. |
|------------------|------------------|------------------|------------------|------------------|
| 1.01234199977    | 0.005756122776   | -0.0044391365    | -0.0912763312    | -0.0071959854    |
| 0.0453773700     | 0.0001990337     | -0.0142035682    | -0.0020714822    | 0.0037811454     |
| 0.0017158691     | -0.0013578673    | -0.0006974248    | 0.0002115413     | 0.003110781      |
| -0.0000505894    | -0.0001347673    | -0.000207107     | 0.000257019      | 0.000125423      |
| -0.0000216802    | -0.0000109464    | -0.0000057183    | 0.0000010900     | -0.0000045590    |
Figure 3. Rational Chebyshev coefficients for \( w/w_0 \) (solid) and \( w/v \) (dashed).

This problem can be rectified to any order desired by simply dividing \( w(y) \) by a more complicated function that includes additional terms of the large \( y \) asymptotics, and then computing the rational Chebyshev expansion of the ratio. Replacing \( w_0(y) \) in (8) by

\[
v(y) = \left\{ \log(y + 10) - \log(\log(y + 10)) + \frac{1 + \log(\log(y + 10))}{\log(y + 10)} \right\} \tanh(\mu \sqrt{y}),
\]

\[
\mu = \frac{\sqrt{2}}{\log(10) - \log(\log(10)) + (1 + \log(\log(y + 10)))/\log(y + 10)}
\]

lowered the "plateau" in the spectral coefficients by roughly two orders of magnitude as illustrated in Figure 3.

Equation (3) is just the leading order in the doubly-infinite expansion [1],

\[
W = \log(z) - \log \log(z) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \left( \frac{\log(\log(z))}{\log(z)} \right)^{m+k}.
\]

Such series in both the log and log-of-log of a parameter, or a doubly infinite series in powers of both the parameter itself and the logarithm of the parameter, are very common in perturbation theory [7,8].

It is remarkable that the rational Chebyshev series, a single rather than double expansion, furnishes a global approximation. In some sense which is still both subtle and mysterious, the double infinity of powers of the log and the log-of-the-log contains a complicated analytic structure which is almost irrelevant to obtaining a good numerical approximation.

4. SUMMARY

Although our approximations are efficient, we have not attempted head-to-head timing comparisons. Rather, our goal is to illustrate some broader themes as follows.

1. Global, rather than piecewise, approximations are possible even for transcendentals which behave very differently in different parts of its domain.
2. Ad hoc approximations can be very powerful, especially as initializations for a Newton's iteration or for engineering applications where moderate accuracy is acceptable.
3. Newton's method is a simple and efficient algorithm for evaluating functions which are defined by nonlinear equations, if provided with an adequate initialization.


One open question is to generalize *ad hoc*, asymptotic-fitting approximations to other transcendental functions. Another is to understand the great gulf in performance between the log/log-of-log series (local approximation, and slowly convergent unless the small parameter is very small) and the rational Chebyshev expansion, which is rapidly convergent for all parameter values. Will it someday be possible to bridge this gulf with more clever perturbative or nonperturbative analytical methods?

REFERENCES


