The set of geodesics in a graph

Ladislav Nebeský

Faculty of Arts and Philosophy, Charles University, nám. J. Palacha 2, 11638 Praha 1, Czech Republic

Abstract

Nebeský [Math. Bohem. 119 (1994) 15] found a necessary and sufficient condition for a set of paths in a given connected graph \( G \) to be the set of all geodesics in \( G \). A simple proof of an extension of that result will be outlined here. © 2001 Elsevier Science B.V. All rights reserved.

Assume that a (finite) connected nontrivial (undirected) graph \( G \) is given. Let \( V \) and \( d \) denote its vertex set and its distance function, respectively. We denote by \( \Sigma \) the set of all finite sequences over \( V \) including the empty one. Sequences in \( \Sigma \) will be written as formal words.

Let \( u_0 \ldots u_m \), where \( u_0, \ldots, u_m \in V \) and \( m \geq 0 \). Then \( \bar{u} \in \Sigma \). Denote
\[
\bar{u} = u_m \ldots u_0, \quad A\bar{u} = u_0, \quad B\bar{u} = u_m \quad \text{and} \quad ||\bar{u}|| = m.
\]

If \( \bar{t} \) is the empty sequence (the empty word) in \( \Sigma \), we put \( \bar{t} = t \) and \( ||\bar{t}|| = -1 \).

As usual, by a path in \( G \) we mean such a \( \pi \in \Sigma \) that there exist mutually distinct \( v_0, \ldots, v_n \in V \) (\( n \geq 0 \)) with the properties that (a) \( \pi = v_0 \ldots v_n \) and (b) vertices \( v_i \) and \( v_{i+1} \) are adjacent in \( G \) for each \( i, 0 \leq i < n \). Let \( P \) denote the set of paths in \( G \).

Similarly as in [6], by a route system on \( G \) we mean a subset \( R \) of \( P \) such that the following statements hold for all \( u, v \in V \) and all \( \alpha, \beta, \gamma, \delta \in \Sigma \):

- if \( x \neq y \), then there exists \( \zeta \in \Sigma \) such that \( x\zeta y \in R \);
- if \( x \) and \( y \) are adjacent in \( G \), then \( xy \in R \);
- if \( x \in R \), then \( \bar{x} \in R \);
- if \( xxy \in R \), then \( x\gamma y \in R \);
- if \( xxy, \beta\gamma y \delta \in R \), then \( \beta xxy \delta \in R \).

By a geodesic (or a shortest path) in \( G \) we mean such a path \( \lambda \in P \) that \( ||\bar{\lambda}|| = d(A\lambda, B\lambda) \). Let \( \Gamma \) denote the set of all geodesics in \( G \). It is easy to see that \( \Gamma \) is a route system on \( G \).

Theorem 0 (cf. L. Nebeský [4]). Let \( R \) be a route system on \( G \). Then \( R = \Gamma \) if and only if \( R \) satisfies the following conditions:

1. if \( uv\alpha x, \upsilon\beta y, u\beta y x \in R \), then \( v\beta xx y \in R \) (for all \( u, v, x, y \in V \) and all \( \alpha, \beta \in \Sigma \)).
(II) if $xy, uvwx \in \mathbf{R}$ and $vzxy \notin \mathbf{R}$, then there exists $\xi \in \Sigma$ such that either $uv\xi y \in \mathbf{R}$ or $u\xi yx \in \mathbf{R}$ (for all $u, v, x, y \in V$ and all $x \in \Sigma$); (III) if $uv \in \mathbf{R}$, then $uxxy \notin \mathbf{R}$ (for all $u, v, x \in V$ and all $x \in \Sigma$).

The proof of Theorem 0 given in [4] was rather complicated. A more transparent proof of an extension of Theorem 0 will be outlined here. (Note that a proof of Theorem 0 based on a different idea is given in [8].)

Consider arbitrary $\phi, \psi \in \Sigma$ such that $||\phi|| \geq 2$, $||\psi|| \geq 2$, $A\phi = A\psi$ and $B\phi = B\psi$. Then there exist $r, t, w, z \in V$ and $\kappa, \chi \in \Sigma$ such that $\phi = rtkw$ and $\sigma = r\chi zw$. Put

$$C(\phi, \psi) = trzw \quad \text{and} \quad D(\phi, \psi) = tr\chi z.$$ 

Let $\alpha, \beta \in \Sigma$ such that $||\alpha|| \geq 2$, $||\beta|| \geq 2$, $A\alpha = A\beta$ and $B\alpha = B\beta$. Denote $C_0(\alpha, \beta) = \alpha$, $D_0(\alpha, \beta) = \beta$, and

$$C_{k+1}(\alpha, \beta) = C(C_k(\alpha, \beta), D_k(\alpha, \beta))$$

and

$$D_{k+1}(\alpha, \beta) = D(C_k(\alpha, \beta), D_k(\alpha, \beta))$$

for each $k \geq 0$.

**Lemma.** Let $\mathbf{R}$ be a route system on $G$, let $\alpha \in \mathbf{R}$, let $\beta \in \Sigma$, and let $A\alpha = A\beta$, $B\alpha = B\beta$ and $||\alpha|| \geq ||\beta|| \geq 2$. Denote $m = ||\alpha||$ and $n = ||\beta||$. Assume that $\mathbf{R}$ satisfies (I), and either $m \neq n$ or $\beta \notin \mathbf{R}$.

If $\beta \in \mathbf{R}$, then there exists $k$, $0 \leq k < n$, such that $C_k(\alpha, \beta) \in \mathbf{R}$ and $D_{k+1}(\alpha, \beta) \notin \mathbf{R}$. If $\beta \notin \mathbf{R}$, then there exists $k$, $0 \leq k < n$, such that $C_k(\alpha, \beta) \in \mathbf{R}$ and $C_{k+1}(\alpha, \beta)$, $D_k(\alpha, \beta) \notin \mathbf{R}$.

**Proof.** Denote $\delta = C_n(\alpha, \beta)$. We first prove that $\delta \notin \mathbf{R}$. To the contrary, let $\delta \in \mathbf{R}$. Then $\delta \in \mathbf{R}$. We get $\beta \in \mathbf{R}$. Thus $m > n$. There exist $r, s, t \in V$ and $\eta, \pi, \omega \in \Sigma$ such that $\alpha = rt\omega s t\eta\pi\omega$ and $\delta = r\eta t\omega s t\eta\pi\omega$. Since $\alpha \in \mathbf{R}$, we have $r\eta t\omega s t\eta\pi\omega \in \mathbf{R}$ and therefore, $\mathbf{R} - \mathbf{P} \neq \emptyset$; a contradiction. Hence $\delta \notin \mathbf{R}$.

Let $\beta \in \mathbf{R}$. Then $m > n$. Since $\alpha \in \mathbf{R}$ and $\delta \notin \mathbf{R}$, there exists $k$, $0 \leq k < n$, such that $C_k(\alpha, \beta), D_k(\alpha, \beta) \in \mathbf{R}$ and

either $C_{k+1}(\alpha, \beta) \notin \mathbf{R}$ or $D_{k+1}(\alpha, \beta) \notin \mathbf{R}$.

Certainly,

there exist $u, v, x, y \in V$ and $\mu, \sigma \in \Sigma$ such that $C_k(\alpha, \beta) = uv\sigma x$ and $D_k(\alpha, \beta) = u\mu y x$. (1)

Let $uv\sigma y \in \mathbf{R}$. Since $u\mu y x \in \mathbf{R}$, (1) implies that $v\sigma xy \in \mathbf{R}$; a contradiction. Hence $D_{k+1}(\alpha, \beta) \notin \mathbf{R}$.

Let $\beta \notin \mathbf{R}$. Since $\delta \notin \mathbf{R}$, there exists $k$, $0 \leq k < n$, such that $C_k(\alpha, \beta) \in \mathbf{R}$, $D_k(\alpha, \beta) \notin \mathbf{R}$, and

either $C_{k+1}(\alpha, \beta) \notin \mathbf{R}$ or $D_{k+1}(\alpha, \beta) \in \mathbf{R}$.
Accept notation (1). Let \( v\sigma y \in R \). Then \( uu\mu y \in R \). Since \( u\nu \sigma y \in R \), (I) implies that \( uu\nu yk \in R \), a contradiction. Hence \( C_{k+1}(x, \beta) \not\in R \), which completes the proof. □

If \( T \subseteq P \) and \( i \geq 0 \), then we put \( T(i) = \{ \tau \in T; d(A\tau, B\tau) = i \} \).

The following theorem is an extension of Theorem 0.

**Theorem 1.** Let \( R \) be a route system on \( G \). Then the following statements (A)–(C) are equivalent:

(A) \( R = \Gamma \);
(B) \( \Gamma \subseteq R \) and \( R \) satisfies (I) and (III);
(C) \( R \) satisfies (I)–(III).

**Proof (outlined).** It is not difficult to prove that (A) \( \Rightarrow \) (C). We will only prove that ((B) \( \lor \) (C)) \( \Rightarrow \) (A). Let ((B) \( \lor \) (C)) hold. Suppose, to the contrary, that there exists \( n \geq 0 \) such that

\[
R(j) = \Gamma(j) \quad \text{for all } j, \quad 0 \leq j < n,
\]

and \( R(n) \neq \Gamma(n) \). Clearly, \( R(0) = \Gamma(0) \). As follows from (III), \( R(1) = \Gamma(1) \). Thus \( n \geq 2 \).

**Case 1:** Let \( \Gamma(n) \subseteq R \). Then \( R(n) - \Gamma \not\subseteq \emptyset \). There exist \( x \in R(n) - \Gamma \) and \( \beta \in \Gamma \) such that \( Ax = A\beta \) and \( Bx = B\beta \). We have \( ||x|| > n \) and \( \beta \in R \). By the lemma, there exists \( k \), \( 0 \leq k < n \), such that \( C_k(x, \beta) \in R \) and \( D_{k+1}(x, \beta) \not\in R \). Using notation (1), we have \( u\nu \sigma x \in R \) and \( uu\mu y \not\in R \). Therefore, \( d(v, y) < n \). Clearly, \( v\sigma x \in R \). By virtue of (2), \( d(v, x) \geq n \). This implies that there exists \( \lambda \in \Sigma \) such that \( v\lambda yx \in \Gamma(n) \). Thus \( v\lambda yx \in R \). Clearly, \( d(u, y) < n \). As follows from (2),

\[
u\nu\lambda y \not\in R.
\]

Since \( u\nu \sigma x, v\lambda yx \in R \), we get \( uu\lambda yx \in R \) and therefore \( vv\lambda y \in R \); a contradiction.

**Case 2:** Let \( R(n) - \Gamma \not= \emptyset \). Then \( R \) satisfies (II). There exist \( \beta \in \Gamma(n) - R \) and \( x \in R \) such that \( Ax = A\beta \) and \( Bx = B\beta \). By the lemma, there exists \( k \), \( 0 \leq k < n \), such that \( C_k(x, \beta) \in R \) and \( C_{k+1}(x, \beta) \not\in R \). Using notation (1), we get \( uu\sigma x \in R \) and \( uu\sigma y, uu\sigma y \not\in R \).

If \( d(u, x) < n \), then (2) implies that \( uu\sigma x \in \Gamma \) and thus \( ||x|| < n \); a contradiction. We get \( uu\mu y \in \Gamma(n) \). By virtue of (2), \( uu\mu y \in R \). If there exists \( \tau \in \Sigma \) such that \( uu\tau yx \in R \), then \( uu\mu y \in R \); a contradiction. Thus,

\[
\nu\xi yx \not\in R \quad \text{for all } \xi \in \Sigma.
\]

As follows from (II), there exists \( \theta \in \Sigma \) such that \( uu\theta y \in R \). By (2), \( uu\theta y \in \Gamma(n-1) \).

This implies that \( vv\lambda y \in \Gamma(n-1) \). By (2), \( vv\lambda y \in R \). Since \( uu\sigma x \in R \), we have \( uu\theta yx \in R \), which contradicts (3).

Thus \( R = \Gamma \), which completes the proof. □

The main notion of this paper is closely related to the notion of the interval function of a connected graph in the sense of Mulder [3] (i.e. to the notion of a finite graphic
interval space in the sense of Bandelt et al. [2] and Bandelt and Chepoi [1]). The present author found a characterization of the interval function of a connected graph in [5]. A new proof of that characterization is given in [7].

References