Differential Equations Satisfied by the Components with Respect to the Cyclic Group of Order $n$ of Some Special Functions

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Let $f$ be a complex function of the variable $z$ admitting a Laurent expansion in an annulus $C$ with center in the origin. For an arbitrary positive integer $n$, Ricci’s theorem asserts that the function $f$ can be written as the sum of $n$ functions $f_{[n,k]}$, $k \in \{0, 1, \ldots, n-1\}$, defined by

$$f_{[n,k]}(z) = \frac{1}{n} \sum_{i=0}^{n-1} \exp\left(-\frac{2i\pi kl}{n}\right) f\left(z \exp\left(\frac{2i\pi l}{n}\right)\right), \quad z \in C.$$ 

In this paper, we present a technique which, starting from a suitable differential equation satisfied by the function $f$, provides a differential equation satisfied by the functions $f_{[n,k]}$. Some known special functions are treated for illustration. © 1999 Academic Press

1. STATEMENT OF THE PROBLEM

Throughout this paper, $n$ denotes a positive integer and $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$ designates the set of the first $n$ integers. Let $\mathcal{H}(C) = \mathcal{H}$ be the vector space of holomorphic functions on an annulus $C$ with center in the origin and let $f$ be a function belonging to $\mathcal{H}$. The components with respect to the cyclic group of order $n$ (components, for short) of the function $f$ are given by

$$f_{[n,k]}(z) = \frac{1}{n} \sum_{i=0}^{n-1} \exp\left(-\frac{2i\pi kl}{n}\right) f\left(z \exp\left(\frac{2i\pi l}{n}\right)\right), \quad z \in C, \; k \in \mathbb{N}_n.$$
In fact, we have

\[ f = \sum_{k=0}^{n-1} f_{[n,k]}. \]  

(1.1)

The functions \( f_{[2,0]} \) and \( f_{[2,1]} \) amount, respectively, to the even part and odd part of the function \( f \).

The decomposition (1.1) was used by many authors for various reasons. We cite, for instance, Duran [14], to generalize the Favard theorem; Ismail [21], to express generating functions related to some orthogonal polynomials; Ricci [30], to study Tchebycheff polynomials with several variables; Srivastava and Manocha [35, Chap. 3], as a technique for finding some generating functions; and Liczberski and Polubinski [24], for the investigation of the sets of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings. We also used this tool, in [6], to study harmonic functions associated with the differential operator \( D^n, D = \frac{d}{dz} \), in the complex domain.

Now, let \( f, f \in \mathcal{H} \), be a special function. With the two additional parameters \( n \) and \( k \), the components \( f_{[n,k]} \) can be viewed as generalizations of the function \( f \) (obviously, we have \( f_{[1,0]} = f \)). Recall that in the special functions theory each generalization is associated with a carrying problem of properties which we formulate in our case by

\((P)\) If the special function \( f \) satisfies a property \( \mathcal{P} \), what corresponds to the property \( \mathcal{P} \) for the components \( f_{[n,k]} \)?

Such a property may be related to a hypergeometric representation, an integral representation, an operational representation, a generating function, a differential equation, a recurrence relation, a functional relation, or the orthogonality . . . .

Many authors dealt with this problem for particular functions or for particular properties. For particular functions, Erdélyi et al. [15], Good [17], Ricci [29], Ungar [36], and Zachary [38] treated the problem \((P)\) with \( f(z) = \exp(z) \) and \( f(z) = \exp((-1)^{1/n}z) \). More generally, Ungar [37] and Muldoon and Ungar [26] considered the function \( f_\alpha(z) = \exp(\alpha^{1/n}z) \), \( \alpha \) being a complex parameter, to obtain

\[ f_{\alpha[n,k]}(z) = \alpha^{k/n} F_{\alpha,n,k}(z) = \alpha^{k/n} \sum_{m=0}^{\infty} \frac{\alpha^m}{(nm+k)!} z^{nm+k}, \]

\[ n = 0, 1, \ldots, k \in \mathbb{N}_n. \]

They referred to \( F_{n,k}^{\alpha} \) as \( \alpha \)-hyperbolic functions of order \( n \) and \( k \)th kind.
Notice that these functions may be expressed by the well-known generalized Mittag-Leffler function

$$E_{s,t}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(sm + t)}, \quad s, t > 0, |z| < \infty,$$

where $\Psi_q$ denotes the Wright generalized hypergeometric series (cf. Erdélyi et al. [15, p. 183] or Srivastava and Karlsson [34, p. 21] for the definition). In fact, we have

$$F_{n,k}^a(z) = z^k E_{n,k+1}(\alpha z^n) = z^k \Psi_{1} \left[ (1, 1), (k + 1, n), \alpha z^n \right].$$

In some earlier works, we treated the problem (P) for the Bessel functions in [3], for the Laguerre polynomials in [5], and for the Boas-Buck polynomials in [4].

Other papers dealt with the problem (P) for particular properties, we mention:

1. The case where the property $\mathcal{P}$ is related to a hypergeometric representation was treated by Srivastava [33] where the author expressed the components of the Wright generalized hypergeometric function $\Psi_q$ in terms of suitable functions $\Psi_q$. This result generalized a corresponding one for the generalized hypergeometric functions $F_q$ already established by Osler [27] and Srivastava [32]. Such decompositions of functions $F_q$ allowed us to express in $[3]$ (resp. $[5]$) the components of the Bessel functions (resp. Laguerre polynomials) by the so-called hyper-Bessel functions (resp. Brafman polynomials).

2. The case where the property $\mathcal{P}$ is related to an integral representation (resp. a generating function, a recurrence relation) was treated in [2] (resp. [4, 7]).

3. The case where the property $\mathcal{P}$ is related to the orthogonality on the real axis, for $n = 2$, was treated by Carlitz [11]. This result was generalized to an arbitrary positive integer $n$ in [8].

In this paper, we consider the case where the property $\mathcal{P}$ is related to a differential equation. We present a technique which, starting from a suitable differential equation satisfied by the function $f$, provides a differential equation for its components. In this context, it should be mentioned that a different approach was used by Ronveaux et al. [31] for homogeneous linear differential equations with polynomial coefficients. Our approach is based on the algebraic properties of linear operators, acting on $\mathcal{H}$, called homogeneous mappings (see definition below).

The outline of the paper is as follows. Section 2 deals with mappings on $\mathcal{H}$ having two homogeneous components. The results obtained there are
applied to certain differential operators in Section 3. Some well-known examples, including generalized hypergeometric functions, \( F \), the \( d \)-symmetric polynomials, and the hyper-Bessel functions will be treated for illustration in Section 4.

2. HOMOGENEOUS MAPPINGS

Denote by \( \mathcal{L}(\mathcal{H}) \) the vector space of all linear mappings from \( \mathcal{H} \) into \( \mathcal{H} \) and consider the following problem:

Let \( \varphi \in \mathcal{L}(\mathcal{H}) \) and \( f \in \ker \varphi \).

Find \( \bar{\varphi} \in \mathcal{L}(\mathcal{H}) \), \( \bar{\varphi} \neq 0 \), such that \( f_{[n, k]} \in \ker \bar{\varphi} \) for all \( k \in \mathbb{N}_n \). (II)

To solve this problem, recall the following:

**DEFINITION** (cf. [2, Sect. III]). Let \( \varphi \) be a linear mapping from \( \mathcal{H} \) into \( \mathcal{H} \), \( \varphi \) is called **homogeneous of degree** \( l \), \( l \in \mathbb{N}_n \), if and only if

\[
\varphi([n, k]) \subseteq [n, k + l] \quad \text{for all} \quad k \in \mathbb{N}_n,
\]

where \( r, r \in \mathbb{N}_n \) denotes the class of \( r \) modulo \( n \). For our purposes, we mention the following examples of homogeneous mappings:

1. For each \( h \in \mathbb{N}_n \), \( \text{II}_{[n, k]} \) is homogeneous of degree 0.
2. The scaling operator \( S_{\alpha} \), \( \alpha \in \mathbb{C}^* \), defined by

\[
(S_{\alpha}f)(z) = f(\alpha z),
\]

is homogeneous of degree 0.

3. The derivative operator \( D = \frac{d}{dz} \) is homogeneous of degree \( (n - 1) \).

4. Given a function \( g \in \mathcal{H}_{[n, l]} \), by the mapping \( g \), we mean the mapping defined on \( \mathcal{H} \) which results from multiplication by \( g \). This mapping is homogeneous of degree \( l \).

Notice that the composite of two homogeneous mappings of degree respectively \( l \) and \( l' \) is homogeneous of degree \( (l + l') \).

Let \( \mathcal{L}(\mathcal{H})_{[n, h]} \) be the subspace of homogeneous mappings of degree \( l \). The following decomposition in direct sum holds:

**Theorem 2.1** (cf. [2, Theorem III-4]).

\[
\mathcal{L}(\mathcal{H}) = \bigoplus_{l=0}^{n-1} \left( \mathcal{L}(\mathcal{H})_{[n, l]} \right).
\]
This means that for any operator \( \varphi \) belonging to \( \mathcal{L}(\mathcal{F}) \) there exists a unique sequence \( \{ \varphi_{[n,l]} \}_{l \in \mathbb{N}} \), \( \varphi_{[n,l]} \in (\mathcal{L}(\mathcal{F}))_{[n,l]} \), such that

\[
\varphi = \sum_{l=0}^{n-1} \varphi_{[n,l]}.
\]

The operators \( \varphi_{[n,l]} \) are called the homogeneous components with respect to the cyclic group of order \( n \) (components, for short) of the operator \( \varphi \).

Also, we state the following properties:

**Theorem 2.2** (cf. [2, Theorem III-1]). Let \( \varphi \) be a linear mapping in \( \mathcal{F} \) and let \( l \in \mathbb{N}_n \). The following statements are equivalent:

(i) \( \varphi \) is homogeneous of degree \( l \).

(ii) \( \Pi_{[n,k+l]} \circ \varphi = \varphi \circ \Pi_{[n,k]} \) for all \( k \in \mathbb{N}_n \).

We deduce, in particular,

**Corollary 2.3.** The projection operators \( \Pi_{[n,k]} \) commutate with every homogeneous mapping of degree 0.

From this corollary, we obtain

**Theorem 2.4.** Let \( (A, B) \in (\mathcal{L}(\mathcal{F}))_{[n,l]} \times (\mathcal{L}(\mathcal{F}))_{[n,l']} \) and let \( s \) be the smallest positive integer such that \( sl = 0(n) \) and \( sl' = 0(n) \).

We assume that there exists a sequence \( \{ S_r \}_{r \in \mathbb{N}} \), depending on \( B \), in \( (\mathcal{L}(\mathcal{F}))_{[n,l]} \) such that

\[
A'B = S_r A' \quad \text{for all} \quad r \in \mathbb{N}.
\]

Let \( f \in \mathcal{F} \) such that \( Af = Bf \). Then the components with respect to the cyclic group of order \( n \) of \( f \) satisfy the identity

\[
\left( A^s - \prod_{j=0}^{s-1} S_{r-1-j} \right) f_{[n,k]} = 0, \quad k \in \mathbb{N}_n,
\]

where \( \prod_{j=0}^{s-1} T_j = T_0 \circ T_1 \ldots T_{s-1} \) (in this order).

**Proof.** The action of \( A^{s-1} \) on both sides of the identity \( Af = Bf \) gives rise, by virtue of (2.2), to the relation

\[
A^s f = S_{s-1} A^{s-1} f,
\]
which, by reiteration, yields the identity

\[
A' - \prod_{j=0}^{s-1} S_{r-1-j} f = 0. \tag{2.4}
\]

The operator involved in this equation is homogeneous of degree 0 since we have \( sI = 0(n) \) and \( sI' = 0(n) \). So the action of the projection operator \( \Pi_{[n, k]} \) on the two members of the identity (2.4) and the use of Corollary 2.3 lead to the relation (2.3).

A sufficient condition to ensure the condition (2.2) is given by the following:

**Lemma 2.5.** Let \( (A, B) \in (\mathcal{D}(\mathcal{F})_{[n, l]} \times (\mathcal{D}(\mathcal{F})_{[n, l']}) \). If there exists a sequence \( (C_t)_{t \in \mathbb{N}} \), depending on \( B \), in \( (\mathcal{D}(\mathcal{F})_{[n, l']}) \) such that

\[
C_0 = B,
AC_t = (C_t + C_{t+1}) A, \quad t = 0, 1, \ldots,
\]

then the sequence

\[
S_r = \sum_{t=0}^{r} \binom{r}{t} C_t, \quad r = 0, 1, \ldots,
\]

satisfies the condition (2.2), which may be proved by induction on \( r \) and by use of the well-known identity

\[
\binom{r}{t} + \binom{r}{t-1} = \binom{r+1}{t}.
\]

An interesting special case of Theorem 2.4, corresponding to \( B = \lambda I \), \( \lambda \in \mathbb{C} \), is worthy of note here:

**Corollary 2.6.** Let \( A \) be a homogeneous mapping of degree \( l \) and let \( s \) be the smallest positive integer such that \( sI = 0(n) \). If \( f \) is an eigenfunction of \( A \) associated with the eigenvalue \( \lambda \), then the components \( f_{[n, k]} \) of \( f \) are eigenfunctions of \( A' \) associated with the eigenvalue \( \lambda' \).

To deduce this result from Theorem 2.4, it is sufficient to take

\[
S_r = \lambda I \quad \text{for all} \ r \in \mathbb{N}.
\]

Now, let us return to problem (II) to point out two cases where the desired mapping is found.

**Case 1.** \( \varphi \) is homogeneous of degree \( l \).
According to Corollary 2.6, the operator $\tilde{\varphi} = \varphi' \in (\mathcal{A}(\mathcal{R}))_{h,0}$, where $s$ is the smallest positive integer such that $s\ell \equiv 0(n)$, is a solution to the problem (II).

Case 2. $\varphi$ has two homogeneous components.

If $\varphi = A - B$, where $A$ and $B$ satisfy the assumption of Theorem 2.4, the operator involved in the identity (2.3) is a solution to the problem (II).

In the following sections, we show how Theorem 2.4 leads to a technique for obtaining differential equations for the components with respect to the cyclic group of order $n$ of some special functions. We illustrate this technique by treating some well-known examples.

3. DIFFERENTIAL EQUATIONS SATISFIED BY THE COMPONENTS

Let $L$ be a differential operator. According to Theorem 2.1, $L$ can be written as

$$L = \sum_{k=0}^{n-1} z^k L_k,$$

where $L_k$ is homogeneous of degree $0$. This means that $L_k$ may be expressed by the operators $z^n$ and $\partial = \frac{d}{dz}$.

In this section, we shall be concerned with the case where $L$ has two homogeneous components: The first one is of degree 0 and the second one is of degree $l$, $l \in \mathbb{N}_n$. Such differential equations are satisfied by various special functions such as the generalized hypergeometric functions (cf. [28]), the classical $d$-orthogonal polynomials generated by $e^{t} \Psi(zt)$ (cf [10]), the classical $d$-orthogonal polynomials generated by $G((d + 1)zt - t^{d+1})$ (cf. [9]), the Faber polynomials (cf. [19]), and the hyper-Bessel functions (cf. [23]). We state

**THEOREM 3.1.** Let $f$ be a function satisfying the differential equation

$$(z A_1(\partial) - B(\partial))y = 0,$$

where $A_1$ and $B$ are polynomials in $\partial$ and $l \in \mathbb{N}_n$.

Then the components of $f$ satisfy the differential equation

$$\left(z^l \sum_{r=0}^{s-1} A_1(\partial + br) - \sum_{r=0}^{s-1} B(\partial - br)\right)y = 0,$$

where $s$ is the smallest positive integer such that $s \ell \equiv 0(n)$. 
Proof. The operators $z$ and $\vartheta$ are not commutative. Indeed, we have

$$\vartheta^m z^{m'} = z^{m'} (\vartheta + m')^m, \quad m, m' = 0, 1, \ldots.$$  \hspace{1cm} (3.4)

From which, we obtain

$$(z^l A_1(\vartheta))^r B(\vartheta) = z^{lr} \prod_{j=0}^{r-1} A_1(\vartheta + j) B(\vartheta)$$

$$= z^{lr} B(\vartheta) \prod_{j=0}^{r-1} A_1(\vartheta + j)$$

$$= B(\vartheta - lr) z^{lr} \prod_{j=0}^{r-1} A_1(\vartheta + j)$$

$$= B(\vartheta - lr)(z^l A_1(\vartheta))'.$$

Now, the application of Theorem 2.4 with $A = z^l A_1(\vartheta)$, $S_r = B(\vartheta - rl)$, and $l' = 0$ leads to (3.3).

**COROLLARY 3.2.** If, moreover, $B(j) = 0$, $j = 0, 1, \ldots, l - 1$, that is,

$$B(\vartheta) = \prod_{j=0}^{l-1} (\vartheta - j) B_1(\vartheta) = z^l D^l B_1(\vartheta),$$  \hspace{1cm} (3.5)

then the differential equation (3.3) takes the form

$$\left(\sum_{r=0}^{s-1} A_1(\vartheta + lr) - \prod_{r=0}^{s-1} B_1(\vartheta + (r + 1)l)\right) D^{l'} y = 0.$$  \hspace{1cm} (3.6)

This result follows from the identity (3.4) and the relation

$$\prod_{r=0}^{s-1} \prod_{j=0}^{l-1} (\vartheta - rl - j) = \prod_{i=0}^{s-1} (\vartheta - t) = z^l D^l.$$  \hspace{1cm} (3.7)

4. APPLICATIONS

4.1. Generalized Hypergeometric Functions

The generalized hypergeometric functions are defined by (see, e.g., [25, p. 136, Eq. (1)])

$$_pF_q \left( \begin{array}{c} (a_p) \cr (b_q) \end{array} \right), \quad z \right) = \sum_{m=0}^{+\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!},$$  \hspace{1cm} (4.1)
where

- \( p \) and \( q \) are positive integers or 0 (interpreting an empty product as 1);
- \( z \) is the complex variable;
- \((a_p)\) abbreviates the set of \( p \) complex parameters \( a_1, a_2, \ldots, a_p\);
- \((a)_m\) is the Pochhammer symbol given by

\[
(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)} = \begin{cases} 
1 & \text{if } m = 0, \\
(a(a + 1) \cdots (a + m - 1) & \text{if } m = 1, 2, 3, \
& \quad a \neq 0, -1, -2, \ldots;
\end{cases}
\]

- the numerator parameters \((a_p)\) and the denominator parameters \((b_q)\) take on complex values provided that \(b_j \neq 0, -1, -2, \ldots, j = 1, \ldots, q\).

Denote by

\[
\mathcal{F}((a_p), (b_q), n, k, z) = \Pi_{n,k} \left[ z \rightarrow _pF_q(z) = _pF_q\left( (a_p), (b_q), z \right) \right]
\]

the components with respect to the cyclic group of order \( n \) of the function \( _pF_q \). The Osler–Srivastava identity provides an explicit expression of these components. In fact, we have (cf. [27, p. 890, Eq. (5); 32, p. 194, Eq. (12); 35, p. 213])

\[
\mathcal{F}((a_p), (b_q), n, k, z) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}
\]

\[
\vphantom{\frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}}_pF_{q+n-1} \left( \Delta(n, a_1 + k), \ldots, \Delta(n, a_p + k), \Delta^*(n, k + 1), \Delta(n, b_1 + k), \ldots, \Delta(n, b_q + k), \frac{z^n}{n!^{(1-p+q)n}} \right),
\]

where \( \Delta(n, \lambda) \) abbreviates the set of in parameters

\[
\frac{\lambda}{n}, \frac{\lambda + 1}{n}, \ldots, \frac{\lambda + n - 1}{n}
\]

and the asterisk in \( \Delta^*(n, k + 1) \) represents the fact that the denominator parameter \( \frac{\lambda}{n} \) is always omitted, so that the set \( \Delta^*(n, k + 1) \) obviously contains only \((n - 1)\) parameters.
To establish a differential equation satisfied by the components of the function \( {}_pF_q \), recall that the generalized hypergeometric functions defined by (4.1) satisfy the following differential equation (cf. [28, p. 75, Eq. (3)])

\[
(zA_i(\vartheta) - \vartheta B_i(\vartheta))y = 0,
\]

where

\[
A_i = \prod_{i=1}^{p}(\vartheta + a_i) \quad \text{and} \quad B_i = \prod_{j=1}^{q}(\vartheta + b_j - 1).
\]

Now, the application of Corollary 3.2, with \( l = 1 \) and \( s = n \), provides the following:

**Proposition 4.1.** The components \( \mathcal{A}(a_{ij}, b_{ij}, n, k, z) \), \( n \in \mathbb{N}^* \) and \( k \in \mathbb{N} \), satisfy the differential equation

\[
\left( \prod_{i=1}^{p}(\vartheta + a_i) - \prod_{j=1}^{q}(\vartheta + b_j) \right) D^n y = 0, \tag{4.3}
\]

where

\[
(\vartheta + a)_n = \prod_{r=0}^{n-1}(\vartheta + a + r). \tag{4.4}
\]

As an illustration, consider two examples corresponding to an exponential function and confluent hypergeometric functions:

**Example 1.** The generalized hyperbolic functions \( h_{n,k} \) defined by (cf. [15, p. 213, Eq. (8)])

\[
h_{n,k}(z) = \sum_{m=0}^{\infty} \frac{z^{nm+k}}{(nm+k)!}
\]

are the components of the exponential function

\[
f(z) = e^z = {}_0F_0\left(\begin{smallmatrix}
\cdot \\
\cdot
\end{smallmatrix}; z\right). \tag{4.5}
\]

If we apply Proposition 4.1, we obtain the well-known result (cf. [15, p. 214, Eq. (12)])

\[
(1 - D^n)h_{n,k}(z) = 0.
\]
EXAMPLE 2. The confluent hypergeometric functions are defined by
\[ f(z) = {}_1F_1\left(\frac{a}{b}, z\right), \quad b \not\in \mathbb{N}. \]
As a particular case, consider the Laguerre polynomials
\[ L_m^{(a)}(z) = \frac{(a + 1)_m}{m!} {}_1F_1\left(-m, \frac{-m}{a+1}, z\right). \]
From Proposition 4.1, we deduce that the components of the polynomials \( L_m^{(a)} \) satisfy the differential equation
\[ ((zD + a + 1)_a D^a - (zD - m)_m) y = 0, \]
which, for \( n = 2 \), reduces to
\[ (z^2D^4 + 2(a + 2)zD^3 + ((a + 1)(a + 2) + z^2)D^2 + 2(m - 1)zD - m(m - 1)) y = 0. \]
The following two cases may be treated in the same way:

(i) The hypergeometric functions
\[ f(z) = {}_1F_1\left(a, b, c, z\right), \quad c \not\in \mathbb{N}. \]
As examples of these types of functions, we cite the Fibonacci and Lucas polynomials (cf. [16, p. 160])
\[ F_{m+1}(z) = {}_2F_1\left(-m, -m + 1, z, -4z, z\right), \]
\[ L_m(z) = {}_2F_1\left(-m, -m + 1, z, -4z, 1 - m\right). \]

(ii) The generalized Bessel polynomials introduced by Krall and Frink (cf. [22, p. 108, Eq. (34)])
\[ y_m(z, a, b) = {}_2F_0\left(-m, -m + a - 1, -\frac{z}{b}, -\frac{z}{b}\right) = m! \left(\frac{-z}{b}\right)^m \frac{(1-a-2m)!(b)}{m!(z)}. \]
4.2. \textit{d-Symmetric Polynomials}

Let \(d\) be a positive integer. By a \(d\)-symmetric polynomial set, we mean a polynomial family \(\{P_{m}\}_{m \in \mathbb{N}}\) satisfying (cf. [13])
\[
\deg P_m = m,
\]
\[
P_m(\omega_{d+1}z) = \omega_{d+1}^m P_m(z).
\]
In [9], it was proved that the classical \(d\)-orthogonal polynomials generated by \(G((d+1)x t - t^{d+1})\) are \(d\)-symmetric and satisfy a differential equation of the type
\[
(D^{d+1} - A_1(\vartheta)) y = 0, \tag{4.6}
\]
where \(A_1(\vartheta)\) is a polynomial in \(\vartheta\).

Also, He and Saff showed that the Faber polynomials are \(d\)-symmetric and satisfy a differential equation of this type (cf. [19, p. 421, Eq. (2.12)]).

The following proposition states a differential equation satisfied by the components of the solutions to (4.6).

\textbf{Proposition 4.2.} Let \(f\) be a complex function satisfying the differential equation (4.6) with \(2 \leq d + 1 \leq n\). Then the components of \(f\) satisfy the differential equation
\[
\left(D^{s(d+1)} - \prod_{r=0}^{s-1} A_i(\vartheta + (d+1)r)\right) y = 0, \tag{4.7}
\]
where \(s\) is the smallest positive integer such that \(s(d + 1) = \theta(n)\).

\textbf{Proof.} If \(d + 1 = n\), the differential operator involved in (4.6) is homogeneous of degree 0. Then, in view of Corollary 2.3, we obtain the desired result with \(s = 1\).

Now, assume that \(2 \leq d + 1 < n\). From (4.6), we deduce that \(f\) satisfies the differential equation
\[
(z^{d+1} A_1(\vartheta) - z^{d+1} D^{d+1}) y = 0.
\]
So the application of Corollary 3.2 with \(l = d + 1\) and \(B_1(\vartheta) = 1\) leads to (4.7).

As an illustration, we consider the case of the Gould–Hopper polynomials \(g_m^{d+1}(z, h)\) defined by means of the generating function (cf. [18, Sect. 6])
\[
\exp(tz + h t^{d+1}) = \sum_{m=0}^{\infty} g_m^{d+1}(z, h) \frac{t^m}{m!}.
\]
Douak [12] gave a differential equation satisfied by the polynomials
\[
\tilde{H}_m(z, d) = g_m^{d+1}\left(z, \frac{-1}{d!(d+1)^2}\right), \quad d \in \mathbb{N}^*,
\]
namely,

\[(D^{d+1} - (d + 1)!zD + (d + 1)!m)y = 0.\]

For \(d = 1\), we have the case of Hermite polynomials.

From Proposition 4.2, we deduce that the components of the polynomials \(\tilde{H}_m(z, d)\) satisfy the differential equation

\[
\left(D^{s(d+1)} - ((d + 1)!)^{s} \prod_{r=0}^{s-1} (zD + (d + 1)r - m)\right)y = 0.
\]

The following two cases may be treated in the same way:

(i) The Gegenbauer polynomials \(C_{n}^{(\alpha)}, m \in \mathbb{N}\), that satisfy the differential equation (see, e.g., [1, p. 781])

\[(1 - z^2)y'' - (2\alpha + 1)zy' + m(m + 2\alpha)y = 0.\]

(ii) The Humbert polynomials \(P_{n}^{(\nu)}\) that are defined by means of the generating function (cf. [20])

\[(1 - 3tz + t^3)^{-\nu} = \sum_{m=0}^{\infty} P_{n}^{(\nu)}(z)t^m\]

and satisfy the differential equation of third order

\[Ly = 0,\]

with

\[
L = (4z^3 - 1)D^3 + 6z^2(2\nu + 3)D^2
- z\left[3m^2 + 3m(2\nu + 1) - (3\nu + 2)(2\nu + 5)\right]D
- m(m + 3\nu)(m + 3\nu + 3).
\]

4.3. Hyper-Bessel Functions

Klyuchantsev [23] studied the function

\[j_{\nu}(z) = {}_{0}F_{r-1}\left(\nu_1, \ldots, \nu_{r-1}, -\left(\frac{z}{r}\right)^{\nu}\right),\]

where \(\nu = (\nu_1, \ldots, \nu_{r-1}), r \in \mathbb{N}^*,\) is a vector index.

This function satisfies the following system

\[
Lu = D^{r}u + \frac{b_1}{z}D^{r-1}u + \cdots + \frac{b_{r-1}}{z^{r-1}}Du = -\lambda' u,
\]

\[u(0) = 1, \quad u'(0) = \cdots = u^{(r-1)}(0) = 0.\]
Here, we assume that $n \geq r$, so the operator $L$ is homogeneous of degree $n - r$. From Corollary 2.6, we deduce that the components of the function $j_y$ are eigenfunctions of the operator $L$ associated with the eigenvalue $(-\lambda)^s$, where $s$ is the smallest positive integer such that $s(n - r) \equiv 0(n)$.

CONCLUSION

Finally, let us mention that further studies on homogeneous operators are needed in order to treat, along a similar line, differential operators having more than two homogeneous components.

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