ADVANCES IN MATHEMATICS **131**, 279–343 (1997)
ARTICLE NO. AI971669

# Differential Equations and Integral Geometry

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Received April 19, 1997; accepted May 29, 1997

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## 1. INTRODUCTION

### 1. *An Example*

Let $f(x)$ be a smooth function in $\mathbb{R}^m$ and

$$I: f(x) \mapsto If(y; r) := \int_{|\omega| = 1} f(y + \omega \cdot r)\, d\omega$$

be the operator of mean value over a radius $r$ sphere centered at $y \in \mathbb{R}^m$. The integral transform $I$ is clearly injective.

Let $C$ be a compact hypersurface in $\mathbb{R}^m$ isotopic to a sphere.

THEOREM 1.1. *Let $f(x)$ be a smooth function vanishing near $C$. Then one can recover $f$ from its mean values along the spheres tangent to $C$, and the inversion is given by an explicit formula.*

In fact, we will show that this theorem is true for any compact manifold $C$ satisfying a mild condition. The only previously known case is the family

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0001-8708/97 $25.00

of all spheres tangent to a plane (horospheres in the hyperbolic geometry, see [GGV]).

The function $If(y; r)$ satisfies the Darboux differential equation

$$\left(\frac{\partial^2}{\partial r^2} - \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} + \frac{m-1}{r}\frac{\partial}{\partial r}\right) If(y; r) = 0$$

Let $\mathrm{Sol}(\Delta_{\mathscr{D}}, C^\infty(\mathbb{R}^m \times R^*_+))$ be a space of smooth solutions of the Darboux equation $\Delta_{\mathscr{D}}$. We will construct an inverse operator $J$ as a map

$$J\colon \mathrm{Sol}(\Delta_{\mathscr{D}}, C^\infty(\mathbb{R}^m \times \mathbb{R}^*_+)) \to C^\infty(\mathbb{R}^m).$$

Namely, let $\mathscr{A}^m(X)$ be the space of smooth differential $m$-forms on a manifold $X$. We will define a differential operator $\nu\colon C^\infty(\mathbb{R}^m \times \mathbb{R}^*_+) \to \mathscr{A}^m(\mathbb{R}^m \times \mathbb{R}^*_+)$ such that the $m$-form $\nu\varphi$ is closed if (and only if) $\Delta_{\mathscr{D}}\varphi = 0$. For a solution $\varphi(y, r)$ we define $(J\varphi)(x)$ integrating the (closed!) differential $m$-form $\nu\varphi(y, r)$ over a certain $m$-cycle. In particular restricting this form to the $m$-dimensional subvariety of all spheres tangent to $C$ and integrating over it we get the theorem, see Chapter 7 for details and generalizations.

## 2. *General Problem*

Let $X$ be a smooth manifold of dimension $n$ and $\mathscr{M}$ a system of linear partial differential equations on $X$. Denote by $\mathrm{Sol}(\mathscr{M}, C^\infty(X))$ the space of smooth solutions to $\mathscr{M}$.

Let $\mathscr{N}$ be a linear system of PDE on a manifold $Y$. Let $K(x, y)\, dx$ be a $(n, 0)$-form on $X \times Y$ with a compact support along $X$. Assume that it satisfies the system $\mathscr{N}$ along $Y$. Then the kernel $K(x, y)\, dx$ defines a linear map $I_K\colon C^\infty(X) \to \mathrm{Sol}(\mathscr{N}, C^\infty(X))$, $f(x) \to \int_X K(x, y)\, dx$. Its restriction to $\mathrm{Sol}(\mathscr{M}, C^\infty(X))$ gives an operator $\mathrm{Sol}(\mathscr{M}, C^\infty(X)) \to \mathrm{Sol}(\mathscr{N}, C^\infty(X))$. However, if $\mathscr{M}$ is nontrivial the functional dimension of $\mathrm{Sol}(\mathscr{M}, C^\infty(X))$ is less than $n$, so many kernels represent the same operator.

In this paper I address the following:

*Problem.* What is the *natural description* for the linear maps

$$\mathrm{Sol}(\mathscr{M}, C^\infty(X)) \to \mathrm{Sol}(\mathscr{N}, C^\infty(Y)). \tag{1}$$

When $Y$ is a point we come to the question of a *natural description* for linear functionals on the space $\mathrm{Sol}(\mathscr{M}, C^\infty(X))$. On the other hand, the composition of a linear map (1) with the evaluation at a point $y \in Y$ gives a linear functional on $\mathrm{Sol}(\mathscr{M}, C^\infty(X))$. So these questions are closely related.

In Sections 5–6 we suggest a construction of operators between solution spaces of linear PDE called *natural linear maps*. Unlike the operators given by the Schwartz kernels $K(x, y)\,dx$, the natural linear maps are obtained by integration of closed forms over certain cycles in $X$. We apply these ideas to solve some old problems in integral geometry.

In Section 5 a general construction of linear maps between solution spaces is given. The natural linear maps seem to be the most interesting particular case of that construction.

To discuss these questions we need the language of $\mathscr{D}$-modules.

## 3. *Systems of Linear PDE and $\mathscr{D}$-Modules*

Let $\mathscr{D}_X$ (or $\mathscr{D}$) be the sheaf of rings of differential operators on a smooth manifold $X$. Suppose we have a linear system $\mathscr{M}$ of $p$ differential equations on $q$ functions $f_1, ..., f_q$:

$$\mathscr{M} = \left\{ \sum_{j=1}^{q} P_{ij} f_j = 0,\ i = 1, ..., p \right\}$$

Then we can assign to $\mathscr{M}$ a left coherent $\mathscr{D}$-module $\mathscr{M}$ with $q$ generators $e_1, ..., e_q$ and $p$ relations:

$$\mathscr{M} = \frac{\oplus \mathscr{D} \cdot e_j}{(+\mathscr{D}(\sum P_{ij} f_j))} = \text{Coker}(\mathscr{D}^p \to \mathscr{D}^q).$$

On the other hand, a coherent $\mathscr{D}$-module $\mathscr{M} = \text{Coker}(\mathscr{D}^p \to \mathscr{D}^q)$ provides a linear system of $p$ differential equations on $q$ functions.

A solution $f$ to the system $\mathscr{M}$ in some space of functions $\mathscr{F}$ is nothing but a morphism of $\mathscr{D}$-modules $\alpha_f\colon \mathscr{M} \to \mathscr{F}$.

## 4. *Natural Functionals on Solutions to $\mathscr{M}$*

Below $X$ usually will be an algebraic manifold over $\mathbb{R}$ of dimension $n$. Let $\mathscr{D}'(X)$ be the space of distributions on $X$ understood as the space of linear continuous maps on the space of smooth differential forms of top degree with compact support on $X$. Denote by $D^m(X)$ the space of $m$-currents on $X$, i.e., linear continuous functionals on the space of smooth differential $(n-m)$-forms with compact support on $X$.

The de Rham complex $\text{DR}(\mathscr{M})^\bullet$ of a $\mathscr{D}$-module $\mathscr{M}$ is defined as follows:

$$\mathscr{M} \xrightarrow{d} \mathscr{M} \otimes_{\mathscr{O}} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{M} \otimes_{\mathscr{O}} \Omega^{n-1} \xrightarrow{d} \mathscr{M} \otimes_{\mathscr{O}} \Omega^n \qquad (2)$$

where $\mathscr{M} \otimes_{\mathscr{O}} \Omega^n$ is sitting in degree 0, $d$ has degree $+1$ and is given by

$$d(m \otimes \omega) := m \otimes d\omega + \sum \frac{\partial}{\partial x_i} m \otimes dx_i \wedge \omega$$

(it does not depend on coordinates $x_i$), and $\mathcal{O}$ is the structural sheaf of $X$.

Consider the complex

$$\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))^{\bullet} = \mathrm{DR}(\mathcal{M})^{\bullet} \otimes_{\mathcal{O}} D'(X)$$

Note that $\mathrm{DR}(D'(X))^{\bullet}$ coincides with $D^{\bullet}(X)[n]$, the usual de Rham complex of currents on $X$ shifted by $n$ to the left. Therefore any $f \in \mathrm{Sol}(\mathcal{M}, C^{\infty}(X))$ defines a homomorphism of complexes

$$m \circ f\colon \mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))^{\bullet} \to D^{\bullet}(X)[n]$$

given by the composition

$$\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))^{\bullet} \xrightarrow{f} \mathrm{DR}(C^{\infty}(X) \otimes_{\mathcal{O}} D'(X))^{\bullet} \xrightarrow{m} D^{\bullet}(X)[n].$$

Here $m$ is induced by the homomorphism of $\mathcal{D}$-modules $C^{\infty}(X) \otimes_{\mathcal{O}} D'(X) \to D'(X)$ provided by the multiplication. We get a pairing

$$H^m(\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))[-n]) \otimes \mathrm{Sol}(\mathcal{M}, C^{\infty}(X)) \to H^m(X, \mathbb{R})$$
$$(\kappa, f) \to [(m \circ f)(\kappa)]. \qquad (3)$$

Evaluation on a homology class $[\gamma] \in H_m(X, \mathbb{R})$ leads to a functional

$$\mathrm{Sol}(\mathcal{M}, C^{\infty}(X)) \to \mathbb{R}, \qquad f \to \langle [(m \circ f)(\kappa)], [\gamma] \rangle. \qquad (4)$$

Such functionals are called the *natural functionals* on $\mathrm{Sol}(\mathcal{M}, C^{\infty}(X))$.

How does the integer $m$ depend on $\mathcal{M}$? Let $\Sigma_{\mathcal{M}} \subset T^*X$ be the characteristic variety for a $\mathcal{D}$-module $\mathcal{M}$. It is coisotropic, so $d_{\mathcal{M}} := \dim \Sigma_{\mathcal{M}} - \dim X \geqslant 0$. The number $d_{\mathcal{M}}$ can often be viewed as the functional dimension of the solution space to $\mathcal{M}$. We will see in Section 3 that

$$H^m(\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))[-n]) = 0 \qquad \text{for} \quad m > d_{\mathcal{M}}.$$

In particular, natural functionals for a nonzero system of PDE are never given by integration over a fundamental class of $X$.

For a $\mathcal{D}$-module $\mathcal{M}$ one may ask whether the natural functionals described above give all of the duals to $\mathrm{Sol}(\mathcal{M}, C^{\infty}(X))$. Integral geometry (including the cohomological Penrose transform) provides a wide class of examples where the answer is positive.

*Remark.* If $X$ is noncompact, the integration over a (possibly noncompact) $m$-cycle $\gamma_m$ defines a linear functional $f \to \int_{\gamma_m} \kappa(f)$ on an appropriately chosen class of functions with certain decreasing conditions at infinity. In a sense a system of PDEs “changes topology of the space”; see the examples in Sections 2 and 5.6 below. I will not pursue this point further but hope to return to it in the future. (If $\mathcal{M}$ is holonomic the complex of solutions is a constructible complex of sheaves on $X$ "changing" the topology of $X$.)

5. *An Elementary Description of Natural Functionals*

Assume that a $\mathcal{D}$-module $\mathcal{M}$ has $q$ generators. Then an $m$-chain $\kappa$ in the de Rham complex $\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} D'(X))[-n]$ may be written as

$$\kappa = \sum_{j=1}^{q} P_j \otimes \omega_j, \qquad P_j \in \mathcal{D}_X, \quad \omega_j \in D^m(X).$$

So we may think about it as a differential operator

$$\bar{\kappa}\colon C^\infty(X)^q \to D^m(X); \qquad \bar{\kappa}(f_1, ..., f_q) \mapsto \sum_{i=1}^{q} P_j(f_j) \cdot \omega_j.$$

Suppose that $\kappa$ is a cycle in the De Rham complex of $\mathcal{M}$. Then the $m$-current $\bar{\kappa}(f_1, ..., f_q)$ is *closed on solutions of the system* $\mathcal{M}$, i.e., $d\bar{\kappa}(f_1, ..., f_q) = 0$, whenever the functions $(f_1, ..., f_q)$ satisfy the system $\mathcal{M}$. In this case we will say that the differential operator $\bar{\kappa}$ is $\mathcal{M}$-closed.

*Remark.* This definition makes sense for any system of partial differential equations, not necessarily *linear*. It leads to a notion of conservation laws for a system of *nonlinear* PDE.

6. *Natural Linear Maps between Solution Spaces*: *A Naive Version*

Let $\mathcal{M}$ and $\mathcal{N}$ be systems of linear PDE on manifolds $X$ and $Y$. A natural linear map

$$I\colon \mathrm{Sol}(\mathcal{M}, C^\infty(X)) \to \mathrm{Sol}(\mathcal{N}, D'(Y))$$

is defined as follows. Let $\kappa_y\colon C^\infty(X) \to D^m(X)$ be an $\mathcal{M}$-closed differential operator whose coefficients are distributions on $Y$ satisfying the system $\mathcal{N}$. Then

$$I\colon f \mapsto \int_{\gamma_m} \kappa_y f \in \mathrm{Sol}(\mathcal{N}, D'(Y))$$

where $\gamma_m$ is an $m$-cycle in $X$ and by definition $\int_{\gamma_m} \kappa_y f := \langle [\gamma_m], \kappa_y f \rangle$. The key idea of this paper is the following:

*If there is a* (*continuous*) *linear functional on solutions to a system of linear partial differential equations* $\mathcal{M}$ *or an operator between solution spaces to* $\mathcal{M}$ *and* $\mathcal{N}$, *then one should look for its natural realization.*

7. *Relation with Integral Geometry*

Let $B$ be a manifold of dimension $n$ and let a linear operator

$$I_K\colon C_0^\infty(B) \to C^\infty(\Gamma) \qquad f(x) \mapsto \int_B K(x, \xi)\, dx \tag{5}$$

enjoy the following properties:

*It is injective, transforms functions $f(x)$ to solutions of a linear system of PDE $\mathcal{N}$ on $\Gamma$, and $I_K(C_0^\infty(B))$ is dense in* $\mathrm{Sol}(\mathcal{N}, C^\infty(\Gamma))$.

Usually $K(x, \xi)$ satisfies a holonomic system of differential equations.

Such a situation is typical in integral geometry and appears as follows. Let $\{B_\xi\}$ be a family of submanifolds of $B$ parametrized by a manifold $\Gamma$. Suppose on $\{B_\xi\}$ the densities $\mu_\xi$ (depending smoothly on $\xi$) are given. Then there is an integral operator

$$I\colon C_0^\infty(B) \to C^\infty(\Gamma), \qquad f(x) \mapsto \int_{B_\xi} f(x)\, \mu_\xi. \tag{6}$$

So here $K(x, \xi) = \mu(x, \xi) \cdot \delta(A)\, db$, where $db$ is a volume form on $B$, and

$$A := \{(x, \xi) \mid x \in B_\xi\} \subset B \times \Gamma$$

is the incidence subvariety. The integral transform $I$ often satisfies the list of properties above. This was discovered by F. John [J] for the family of all lines in $\mathbb{R}^3$, and was developed much further by Gelfand, Graev, and Schapiro [GGrS]. Here are some examples:

Example 1. Consider the integral transformation

$$I\colon f(x_1, ..., x_{n+1}) \to \int f\left(t_1, ..., t_n, a \sum_{i=1}^n t_i^2 + \sum_{i=1}^n b_i t_i + x\right) d^n t \tag{7}$$

related to the $(n+2)$-parametrical family of paraboloids in $\mathbb{R}^{n+1}$. Let $S(\mathbb{R}^{n+1})$ be the Schwartz space of functions, in $\mathbb{R}^{n+1}$.

Lemma 1.2. *If $f \in S(\mathbb{R}^{n+1})$ then $((\partial^2/\partial a\, \partial c) - \sum_{i=1}^n (\partial^2/\partial b_i^2))\, If = 0$. The integral transformation $I$ is injective on $S(\mathbb{R}^{n+1})$.*

*Proof.* Applying $\partial^2/\partial a\, \partial c$ to the right-hand side of (7), we get

$$\int \sum_{i=1}^n t_i^2 f''_{t_{n+1}}\left(t_1, ..., t_n, a \sum_{i=1}^n t_i^2 + \sum_{i=1}^n b_i t_i + c\right) d^n t.$$

Applying $\sum_{i=1}^{n} (\partial^2/\partial b_i^2)$ we get the same result. Let $a=0$. Then $I$ is the Radon transform and so the lemma follows from its standard properties.

EXAMPLE 2. Consider the integral transformation

$$I_k: f(x_1, ..., x_n) \to \int f\left(t_1, ..., t_k, \sum_{j=1}^{k} a_1^j t_j + a_1^0, ..., \sum_{j=1}^{k} a_{n-k}^j t_j + a_{n-k}^0\right) \times dt_1 \cdots dt_k$$

related to the family of $k$-planes in $\mathbb{R}^n$: $x_{k+i} = \sum_{j=1}^{k} a_i^j x_j + a_i^0$.

THEOREM 1.3 [GGrS]. (a) *If* $f(x) \in S(\mathbb{R}^n)$ *then*

$$\left(\frac{\partial^2}{\partial a_{i_1}^{j_1}\, \partial a_{i_2}^{j_2}} - \frac{\partial^2}{\partial a_{i_1}^{j_2}\, \partial a_{i_2}^{j_1}}\right) If = 0$$

*where* $k+1 \leqslant i_1,\ i_2 \leqslant n,\ 0 \leqslant j_1,\ j_2 \leqslant k.$

(b) $I_k$ *is injective on* $S(\mathbb{R}^n)$ *and provides an integral formula for solutions of the system of PDEs above.*

Let us return to the integral transform $I_K$ (see (5)). Its properties imply that its inverse would provide a continuous linear map

$$J_K: \mathrm{Sol}(\mathcal{N}, C^\infty(\Gamma)) \to C^\infty(B).$$

DEFINITION 1.4. An integral transform $I_K$ admits a universal inversion formula if the inverse operator $J_K$ is given by a natural linear map.

To clarify the meaning of this definition consider the composition $J_b$ of the operator $J$ with the $\delta$-functional at a point $b \in B$. Its natural realization is given by an $\mathcal{N}$-closed differential operator $\nu_b$: $C^\infty(\Gamma) \to D^n(\Gamma)$ and a certain $n$-dimensional cycle $\gamma_b$ in $\Gamma$ such that

$$\int_{\gamma_b} \nu_b(I_K f) = c_{[\gamma_b]} \cdot f(b).$$

Here $c_{[\gamma_b]}$ is a constant depending linearly on the homology class of $\gamma_b$ and $n = \dim B$. We define the left-hand side as $\langle \nu_b(I_K f), [\gamma_b] \rangle$. To compute the integral we may use any cycle $\gamma_b$ transversal to the wave front of the distribution $\kappa_b(I_K f)$. Then the restriction of this distribution to $\gamma_b$ is defined and we can integrate it over the fundamental class of $\gamma_b$. So we can find the value $f(b)$ if we know only the values of $I_K(f)$ at an infinitesimal neighborhood of any such cycle. This explains the name "*universal* inversion formula."

8. *Local and Nonlocal Inversion Formulas in Integral Geometry.*

Let us discuss in more detail the general Radon transform (6).

DEFINITION 1.5. A local universal inversion formula for the Radon transform (6) is given by a differential operator $\kappa_b: C^\infty(\Gamma) \to \mathscr{A}^k(\Gamma_b)$ such that $\kappa_b(If)$ is closed (on $\Gamma_b$) and

$$\int_{\gamma_k} \kappa_b(If) = c_{[\gamma_k]} \cdot f(b)$$

where $c_{[\gamma_k]}$ is a constant (depending linearly on the homology class $[\gamma_k]$).

In particular, the value of any smooth function $f$ on $B$ at any point $b$ can be recovered from its integrals over the submanifolds $B_\xi$ passing through an *infinitesimal* neighborhood of $b$.

Let $\Gamma_b$ be the variety parametrizing all the subvarieties $B_\xi$ passing through a given point $b$. Set $k := \dim B_\xi$. Note that $\dim \Gamma - \dim B = \dim \Gamma_b - \dim B_\xi$. So if $\dim \Gamma > \dim B$ the degree of the form $\kappa_b(If)$ is less than $\dim \Gamma_b$.

A first example of local universal inversion formula was discovered in 1967 by I. M. Gelfand, M. I. Graev, and Z. Ya. Shapiro for integral transformation $I_k^{\mathbb{C}}$ related to the family of all $k$-planes in $\mathbb{C}^n$ ([GGrS]). Here we treat complex planes as real submanifolds and integrate smooth functions along them. Later more generic examples were studied, including local universal inversion formulas for the families of complex curves; see [GGiG, BG, Gi].

However, in integral geometry there are many examples where there are no *local* inversion formulas. This is quite typical in "real" integral geometry (i.e. we integrate over family of real submanifolds). For instance, in Example 2 (resp., 3) the inversion formula is nonlocal if the dimension of hyperboloids (resp., planes) is odd. It is always nonlocal for integral transformations related to any family of real curves.

A very interesting approach to integral geometry on $k$-planes in $\mathbb{R}^n$ was suggested by I. M. Gelfand and S. G. Gindikin [GGi] (see also [GGR]). However, it was based on the Fourier transform in $\mathbb{R}^n$ and so cannot be generalized to families of "curved" submanifolds, as in Examples 1, 2. What is even more important, the differential $k$-form $\kappa$ was replaced by a $k$-density, so the possibility of using the Stokes formula was missed. It seems that this approach to integral geometry was not really understood yet.

As a result, the nature of the form $\kappa_b$ and the inversion of the general integral transformations, especially if they do not admit a local inversion formula, were the key unsolved problems in integral geometry.

The main idea of this paper is that

*Inversion formulas in integral geometry are given by natural linear maps between solution spaces of systems of partial differential equations.*

Let me explain how the local universal inversion formulas fit in this concept. The form $\kappa_b(If)$ is a differential $k$-form on $\Gamma_b$. Since $n-k=\dim \Gamma - \dim B$, a $k$-form on $\Gamma_b$ defines an $n$-current on $\Gamma$. *The $n$-current corresponding to $\kappa_b(If)$ leads to a natural linear map* given by integration of $\kappa_b(If)$ over an $n$-cycle $K \subset \Gamma$. We will demonstrate this for the Radon transform over spheres in $\mathbb{R}^m$.

In general, our approach leads to a universal inversion formula where the functional $J_b$ is represented by a differential $n$-form on $\Gamma$. The fact that this $n$-form does not concentrate on a subvariety $\Gamma_b$ (or a certain bigger subvariety of $\Gamma$) means that we get a nonlocal universal inversion formula. So we treat simultaneously both local and nonlocal inversion formulas.

The form $\kappa_b$ appeared in [GGS] as a construction "ad hoc" and looks like a very special phenomenon. In our approach the universal inversion formula is a very general property of the corresponding system of linear PDE. Its locality, however, is a rather rare phenomena, which generalizes the Huygens principle or, more generally, the notion of lacunas for hyperbolic differential equations.

In particular, in these examples our natural functionals describe the whole dual to the space of solutions of a linear system of PDE.

## 9. *Some General Remarks on Analytic Theory of Overdetermined Systems*

The classical theory of PDE usually studies systems of $p$ linear partial differential equations on $p$ unknown functions on $X$, i.e., the characteristic variety of the system has codimension 1. (Of course, there are some exceptions with extremely reach analytic theory, like the Cauchy–Riemann system.) It seems that one of the reasons is this: a system $P_1 f = P_2 f = 0$ of two *general* differential equations on one unknown function has no solutions because the corresponding $\mathscr{D}$-module is equal to zero (even if $P_i$ are differential operators of order one). This shows that overdetermined systems (i.e., the ones where the codimension of the characteristic subvariety is greater then 1) cannot describe a physical process in a way similar to systems of $p$ equations on $p$ unknown functions (like the Laplas, Schrodinger, etc., equation): a small perturbation of the experimental data leads to a system without solutions. Therefore one should not expect an analytic theory of general overdetermined systems, i.e., a theory stable under a perturbation of a system.

The theory of $\mathscr{D}$-modules is a tool providing nontrivial linear systems of PDE. We think that an interesting overdetermined system of PDE should be a part of a richer data. For example, for a system $\mathscr{N}$ on a variety $\Gamma$

appearing in integral geometry (see Section 6) we should also remember the kernel $K(x, \xi)$ on $B \times \Gamma$. So perturbing such a system we should deform all the data, not only the system $\mathcal{N}$ on $\Gamma$.

We may wonder about the goals of analytic theory for some *special* overdetermined systems. It seems that the problem of natural description of linear maps between solution spaces looks quite promising.

## 10. *The Structure of the Paper*

Section 2 contains examples of functionals and natural functionals on solution spaces of systems of PDE. In Section 3 we recall some general information about $\mathcal{D}$-modules, including the duality on the derived category of $\mathcal{D}$-modules, needed for applications to integral geometry. In Section 4 our key tool appears: the Green class of a $\mathcal{D}$-module. It generalizes the classical Green formula for a single differential operator. Chapter 5 contains a definition and properties of general linear maps between solution spaces of (complexes of) $\mathcal{D}$-modules. Then we define natural linear maps as a quite special case of general linear maps. The definitions use the language of derived categories. This is necessary for many reasons, including:

(1) Even nice systems like $\mathcal{M} = \{x_1 \cdot f = \cdots = x_k \cdot f = 0\}$ may have no smooth solutions, so one should consider the spaces of "higher" solutions. (In the example above only $\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}, C^\infty(\mathbb{R}^n)) \neq 0$; it is isomorphic to $C^\infty(\mathbb{R}^{n-k})$).

(2) The duality may send a $\mathcal{D}$-module to a complex of $\mathcal{D}$-modules.

In applications the dual complex for a $\mathcal{D}$-module $\mathcal{M}$ is often concentrated in just one degree. Such $\mathcal{M}$'s will be called *excellent* $\mathcal{D}$-modules. In Section 6 we define natural linear maps between solution spaces for excellent $\mathcal{D}$-modules. This allows us to eliminate derived categories and makes the story more elementary. I made this independent of Section 5, so those who are interested only in applications to "nice" systems of PDE could go directly to Section 6.

In Section 7 we demonstrate how the general method works for the family of all spheres in $\mathbb{R}^m$ (see Section 1.1 above). Our approach leads to *universal* inversion formulas which are *nonlocal* when $m$ is odd and local when $m$ is even. The corresponding problem of integral geometry was unsolved even for the family of circles in the plane.

In fact, we study in Section 7 integral operators $I_\lambda$ more general then the Radon transform over the family of spheres. They are intertwiners for the group $O(m+1, 1)$ acting from the space of sections of a line bundle over $S^m$ to the space of sections of a line bundle over the manifold $X_{m+1}$ of oriented hyperplane sections of $S^m$. (The hyperplane sections of $S^m$ can be identified with spheres in $\mathbb{R}^m$ by a stereographic projection.) The image of $I_\lambda$ is described by differential equations. So the inverse operators give examples of intertwiners which are well defined only on a subrepresentation.

The next problem after the definition of natural linear maps would be the development of a "calculas of natural linear maps." In particular, there are the following problems:

(a) How to compose natural linear maps.

(b) How to compute their composition, for instance, when the composition of two natural linear maps is equal to a given natural linear map.

A universal inversion formula for the integral transform $I_K$ is a natural linear map $J_K$: $\mathrm{Sol}(\mathcal{N}, C^\infty(\Gamma)) \to C^\infty(B)$ such that the composition $J_K \circ I_K$ equals the identity map, so this is a very special case of the problem (b).

The development of this program should include a version of the theory of Fourier integral operators as special case when $\mathcal{M} = \mathcal{D}_X$, $\mathcal{N} = \mathcal{D}_Y$.

In Section 8 we study an algebraic version of the problem of composition of natural linear maps. It turns out that one can organize neatly the algebraic structures responsible for that by introducing a *bicategory of $\mathcal{D}$-modules*.

The objects of this bicategory are pairs $(X, \mathcal{M})$, where $\mathcal{M}$ is a complex of $\mathcal{D}$-modules on a variety $X$. A 1-morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ is the algebraic part of the data needed to construct a linear map $R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, C^\infty(X)) \to R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{N}, D'(Y))$. The composition of 1-morphisms mirrors the composition of linear maps. A 2-morphism between two 1-morphisms reflects coincidence of the corresponding maps on functions.

At the end of Section 8 we consider the simplest examples of the composition of 1-morphisms and 2-morphisms relevant to integral geometry.

The main results of this paper were announced in [G1]. Another approach to integral geometry via $\mathcal{D}$-modules was independently developed by A. D'Agnolo and P. Schapira [A, AS1, AS2]. Inversion formulas for real quadrics were also considered by S. G. Gindikin [Gi2].

## 2. EXAMPLES

### 1. *Analytic Functionals* [*GS*]

Let $X = \mathbb{C}$ and $\mathcal{M}$ be the Cauchy–Riemann equation $(\partial/\partial\bar{z})\, f(z, \bar{z}) = 0$. Let $g(z)$ be a holomorphic function. Then

$$f(z, \bar{z}) \to g(z)\, f(z, \bar{z})\, dz$$

is an $\mathscr{M}$-closed operator of order 0. The corresponding linear functional should be

$$f(z) \to \int_{\gamma_1} g(z)\, f(z)\, dz. \tag{8}$$

To make sense out of this consider the space $Z_1$ of holomorphic functions $f(z)$ such that $|z|^q \cdot |f(z)| \leqslant C_q \cdot e^{a \cdot \operatorname{Im} z}$ for any $q > 0$ (the constants $a$ and $C_q$ may depend on $f$). Let $\bar{\mathbb{C}} = \{\mathbb{C} \cup S^1\}$ be a compactification of the complex plane by a circle such that each line compactifies by endpoints at $+$ and $-$ infinity and two lines have the same endpoints if and only if they are parallel. Let $x_-$ and $x_+$ be the endpoints of the $x$-axis ($z = x + iy$). Let $\gamma_1$ be a cycle representing the nontrivial homology class in $H_1(\bar{\mathbb{C}}, x_- \cup x_+)$. Then the right side of (8) is convergent for $g(z) \in Z_1$ and defines a continuous linear functional on $Z_1$.

One can also take $g(z)$ to be a meromorphic function and integrate along compact 1-cycles in $\mathbb{C} \backslash \{\text{poles of } g(z)\}$. For example, if $g(z) = 1/(z - z_0)$ we get the Cauchy formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}.$$

It can be interpreted as the *natural realization* for the $\delta$-functional $f(z) \to f(z_0)$.

Now let $\mathscr{M}$ be the Cauchy–Riemann system in $\mathbb{C}^n$. Let $g(z)$ be a holomorphic function. Then

$$f(z, \bar{z}) \to g(z)\, f(z, \bar{z})\, dz_1 \wedge \cdots \wedge dz_n$$

is an $\mathscr{M}$-closed operator of order 0. The corresponding natural functional is

$$f(z) \to \int_{\gamma_n} g(z)\, f(z)\, dz_1 \wedge \cdots \wedge dz_n$$

where $f(z)$ belongs to the space $Z_n$ of homomorphic functions satisfying some growth condition ([GS]). So any $g(z) \in Z_n$ defines an element of $D\,\mathrm{Sol}(\mathscr{M})_n$.

However, for $n > 1$ there are another $\mathscr{M}$-closed operators. Namely, let us look at the classical Bochner–Martinelly formula

$$f(z_0) = \int_{s_{2n-1}} f(z) \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}} \tag{9}$$

where $\omega(z) = dz_1 \wedge \cdots \wedge dz_n$ and $\omega^*(z) = \sum_{i=1}^{n} (-1)^i z_i \, dz_1 \wedge \cdots \hat{dz}_i \cdots \wedge dz_n$ and $[s_{2n-1}]$ is a generator in $H_{2n-1}(\mathbb{C}^n \backslash z_0)$. The zero order operator

$$f(z, \bar{z}) \to f(z) \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}}$$

represents a non-zero element of $D \operatorname{Sol}(\mathscr{M})_{2n-1}$ for $X = \mathbb{C}^n \backslash z_0$.

In fact, all "integral formulas" in complex analysis (like the Cauchy formula in a polydisc, the Weil formula, ...) are examples of elements in $D \operatorname{Sol}(\mathscr{M})_m$, where $n \leqslant m \leqslant 2n-1$, given by zero order operators.

## 2. *The Green Function of a Differential Operator and a Natural Realization of the δ-Functional*

Let $P = \sum a_I(x) \, \partial_x^I$ be a differential operator and $P^t = \sum \partial_x^I a_I(x)$ the transposed one ($I$ is the multiindex). The classical Green formula is

$$(Pu \cdot v - u \cdot P^t v) \, dx_1 \wedge \cdots \wedge dx_n = d\omega_{n-1}(u; P; v) \tag{10}$$

where $\omega_{n-1}(u; P; v)$ is an $(n-1)$-form depending linearly on $u$ and $v$. For example, if $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ then

$$\omega_{n-1}(\Delta; u, v) = \sum_{i=1}^{n} (u'_{x_i} v - u v'_{x_i}) \, dx_1 \wedge \cdots \hat{dx}_i \cdots \wedge dx_n.$$

A Green function $g(x, y)$ for $P$ is a generalized function on $X \times X$ satisfying

$$P_x^t g(x, y) = P_y \, g(x, y) = \delta(x - y).$$

Let us put in (10) $u := g(x, y)$ and suppose $Pv = 0$. Then

$$v(x) = \int_{s_{n-1}} \omega_{n-1}(g(x, y); P; v)$$

where $s_{n-1}$ is a small $(n-1)$-sphere around $x$. Therefore $\omega_{n-1}(g(x, y); P; v)$ provides a natural realization of the $\delta$-functional $f \to f(x)$.

There are differential operators that do not have a Green function, for example the Lewy operator $1/2(\partial_{x_1} + i\partial_{x_2}) - (x_1 + ix_2) \, \partial_{x_3}$ or the operator $\partial_x - ix\partial_y$. A natural realization of a functional does not always exist.

## 3. *A Green Form for an Arbitrary System* $\mathscr{M}$

DEFINITION 2.1. A Green form for a system $\mathscr{M}$ is an element $g_y \in \operatorname{Sol}(\mathscr{M})_{n-1}$ such that

$$dg_y \overset{\mathscr{M}}{=} \delta(x - y) \, dx_1 \wedge \cdots \wedge dx_n.$$

Here $\overset{\mathscr{M}}{=}$ means $\mathscr{M}$-equivalence, i.e.,

$$dg_y(f) = f(y)\, dx_1 \wedge \cdots \wedge dx_n \qquad f \in \operatorname{Sol}(\mathscr{M}, C^\infty(X)).$$

If $g_y$ is a Green form for $\mathscr{M}$ then for any $f \in \operatorname{Sol}(\mathscr{M}, C^\infty(X))$ one has

$$f(y) = \int_{s_{n-1}} g_y(f).$$

Here $s_{n-1}$ is a small sphere around $y$. This follows from the Stokes formula.

EXAMPLE 2.2. $g_y: f \to \omega_{n-1}(P; g(x, y), f)$ (see Section 2.2 above) is the Green form for a differential equation $Pf = 0$.

EXAMPLE 2.3. The Bochner–Martinelly form (9) provides a Green form

$$f \to f\, \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}} \tag{11}$$

for the Cauchy–Riemann system in $\mathbb{C}^n$.

4. *A Universal Solution of a Boundary Value Problem*

Let $\mathscr{M}$ be a system of PDE on $X$ and $m := d_{\mathscr{M}}$. Let

$$\operatorname{Id}: \operatorname{Sol}(\mathscr{M}, C^\infty(X)) \to \operatorname{Sol}(\mathscr{M}, C^\infty(X))$$

be the identity map. Its natural realization should be given by an $\mathscr{M}$-closed operator $G_x: C^\infty(X) \to D^m(X)$ depending on a parameter $x \in X$ whose coefficients considered as functions on $x$ are also solutions to $\mathscr{M}$. For a given solution $\phi \in \operatorname{Sol}(\mathscr{M}, C^\infty(X))$ one has that $G_x(\phi)$ is an $m$-form on $X$ such that for any closed $m$-dimensional manifold $Y$ one has

$$\int_Y G_x(\phi) = c_{[Y]} \phi(x), \tag{12}$$

where $c_{[Y]}$ is a constant depending linearly on the homology class of $Y$.

According to the definition, to compute $G_x(\phi)$ at a point $y \in Y$ one has to know the restriction and a finite number of transversal derivatives of $\phi$ at $y$. So formula (12) is a universal solution to the Cauchy problem for $\mathscr{M}$ on $Y$. The fact that $d_{\mathscr{M}}$ can often be viewed as a "functional dimension" of the space of solutions to $\mathscr{M}$ looks very natural from this point of view. $G_x$ will be referred to as the boundary value problem Green form.

*Remark* 2.5. There are two different realizations for the identity map given in Section 2.3 and 2.4. I emphasize the following differences between them. The realization given in Section 2.3 is not a *natural* one because the form is not $\mathscr{M}$-closed. However, one may interpret it as a natural realization for a modification of $\mathscr{M}$ at $x$. Further, in general the cycles for $g_y$ and $G_y$ are of different dimension and in fact of different nature. Namely, for a Green form $g_y$ the cycle always exists and represents a class in $H_{n-1}(X\backslash x)$, while for the boundary value problem Green form $G_y$ the cycle in (12) represents a homology class of $X$ of dimension $m$ and its existence is a nontrivial problem.

## 3. BASIC FACTS ABOUT $\mathscr{D}$-MODULES

For the convenience of the reader I will recall some material about $\mathscr{D}$-modules (see [Be, Bo]).

### 1. *The Bimodule* $\mathscr{D}_X^{\Omega}$

I will assume that $X$ is an algebraic manifold, $\mathscr{D}=\mathscr{D}_X$ is the sheaf of regular differential operators on $X$, and $\Omega_X$ is the $\mathscr{O}_X$-sheaf of regular differential forms of highest degree on $X$. $\Omega_X$ has a right $\mathscr{D}_X$-module structure given by

$$\omega \cdot f = f\omega, \qquad \omega \cdot \xi := -L_\xi \omega,$$

where $f \in \mathscr{O}_X$ and $\xi$ is a vector field. Here $L_\xi \omega := di_\xi \omega$. Set

$$\mathscr{D}_X^{\Omega} := \mathscr{D}_X \otimes_{\mathscr{O}} \Omega_X^{-1} = \mathrm{Hom}_{\mathscr{O}_X}(\Omega_X, \mathscr{D}_X^r) \tag{13}$$

where $\mathscr{D}_X^r$ is $\mathscr{D}_X$ viewed as a right $\mathscr{D}$-module via right multiplication. Then (13) carries two commuting left $\mathscr{D}_X$-modules structures. The first is provided by the left multiplication on $\mathscr{D}_X$, and the second, "$\circ$", is given by

$$\xi \circ (\lambda)(\omega) = \lambda(\omega \cdot \xi) - \lambda(\omega) \cdot \xi, \tag{14}$$

where $\xi$ is a vector field and $\lambda \in \mathrm{Hom}_{\mathscr{D}_X}(\Omega_X, \mathscr{D}_X^r)$.

The two natural commuting left $\mathscr{D}_X$-module structures on $\mathscr{D}_X^{\Omega}$ lead us to consider $\mathscr{D}_X^{\Omega}$ as a $\mathscr{D}$-module on $X \times X$. It is canonically isomorphic to the $\mathscr{D}_{X\times X}$-modules $\delta_\Delta$ of $\delta$-functions on the diagonal $\Delta \subset X \times X$. There exists a canonical involution on $\mathscr{D}_X^{\Omega}$ interchanging the two left $\mathscr{D}_X$-module structures. For $\delta_\Delta$ it is induced by the switch of the factors of $X \times X$.

### 2. *The Duality Functor*

Let $D_{\mathrm{coh}}^b(\mathscr{D}_X)$ be the derived category of bounded complexes of $\mathscr{D}_X$-modules whose cohomology groups are coherent $\mathscr{D}_X$-modules. Let us define a duality $\star\colon D_{\mathrm{coh}}^b(\mathscr{D}_X)^0 \to D_{\mathrm{coh}}^b(\mathscr{D}_X)$ by

$$\star\mathscr{M} := R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega})[\dim X].$$

The second $\mathscr{D}_X$-structure on $\mathscr{D}_X^{\Omega}$ provides a left $\mathscr{D}_X$-module structure on the sheaves $\mathrm{Ext}^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega})$.

To compute $\star\mathscr{M}$ we should find a bounded complex $\mathscr{P} = \{\to \mathscr{P}^{-1} \to \mathscr{P}^0 \to \mathscr{P}^1 \to \cdots\}$ of locally projective coherent $\mathscr{D}$-modules quasiisomorphic to $\mathscr{M}$ and set $\star\mathscr{M} = \star\mathscr{P}$ where $(\star\mathscr{P})^i = \star(\mathscr{P}^{-\dim X - i}) := \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{P}^{-\dim X - i}, \mathscr{D}_X^{\Omega})$. It is easy to see that $\star\star\mathscr{P}$ is isomorphic to $\mathscr{P}$. Therefore $\star\star = \mathrm{Id}$.

The object $\star\mathscr{M}$ represents the functor

$$\mathscr{N} \to R\,\mathrm{Hom}_{\mathscr{D}_{X\times X}}(\mathscr{N} \boxtimes \mathscr{M}, \delta_\Delta)[\dim X].$$

i.e., one has

$$R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{N}, \star\mathscr{M}) = R\,\mathrm{Hom}_{\mathscr{D}_{X\times X}}(\mathscr{N} \boxtimes \mathscr{M}, \delta_\Delta)[\dim X]. \qquad (15)$$

Indeed, there is canonical morphism

$$R\,\mathrm{Hom}_{\mathscr{D}_{X\times X}}(\mathscr{N} \boxtimes \mathscr{M}, \mathscr{D}_X^{\Omega}) \to R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{N}, R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega})).$$

It is obviously an isomorphism when $\mathscr{N} = \mathscr{M} = \mathscr{D}_X$, and so using locally free resolutions we see that it is an isomorphism in general.

Let $SS\mathscr{M}$ be the singular support of a $\mathscr{D}$-module $\mathscr{M}$. The following important result was proved by Roose

THEOREM 3.1. (a) *$\mathscr{M}$ has a finite resolution by locally projective $\mathscr{D}_X$-modules.*

(b) $\operatorname{codim} SS(\mathrm{Ext}^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega}) \geqslant i$

(c) *If* $\operatorname{codim} SS(\mathscr{M}) = k$, *then* $\mathrm{Ext}^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega}) = 0$ *for* $i < k$.

Note that $H^i(\star\mathscr{M}) = \mathrm{Ext}^{\dim X + i}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X^{\Omega})$. The Roos theorem implies that $\star\mathscr{M}$ is concentrated in degrees $[-\dim X, -\operatorname{codim} SS\mathscr{M}]$.

LEMMA 3.2. $\mathrm{DR}(\mathscr{M}) = \Omega_X \overset{L}{\otimes}_{\mathscr{D}_X} \mathscr{M}$.

*Proof.* Using the Koszul complex we see that $\mathrm{DR}(\mathscr{D}_X)$ is a locally free resolution for the right $\mathscr{D}_X$-module $\Omega_X$. One has $\mathrm{DR}(\mathscr{M}) = \mathrm{DR}(\mathscr{D}) \otimes_{\mathscr{O}} \mathscr{M}$.

Let $D^b(Sh_X)$ be the bounded derived category of sheaves on $X$.

THEOREM 3.3. *Let $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathscr{N} \in D^b(\mathscr{D}_X)$. Then there is an isomorphism in $D^b(\mathrm{Sh}_X)$ functorial with respect to $\mathscr{M}$ and $\mathscr{N}$:*

$$\mathrm{DR}(\star\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N})[-\dim X] = R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}). \qquad (16)$$

The nature of this isomorphism and the fact that $\mathcal{N}$ may not be coherent play a crucial role, so we will sketch its proof following [Bo, Chap. 6]. Let us replace $\mathcal{M}$ and $\mathcal{N}$ by bounded locally projective resolutions $\mathcal{P}^{\bullet}_{\mathcal{M}}$ and $\mathcal{P}^{\bullet}_{\mathcal{N}}$. One can suppose $\mathcal{P}^{\bullet}_{\mathcal{M}}$ to be locally free from a certain low degree on. Therefore according to Lemma 3.2 to prove the theorem it is sufficient to construct for given coherent $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ a natural morphism (functorial with respect to $\mathcal{M}$ and $\mathcal{N}$)

$$\alpha\colon \Omega_X \otimes_{\mathcal{D}_X} (\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}^r_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N}) \to \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \qquad (17)$$

which will be an isomorphism if $\mathcal{M} = \mathcal{D}_X$.

The functors $\Omega_X \otimes_{\mathcal{D}_X}$ and $\mathrm{Hom}_{\mathcal{D}_X}$ in the left-hand side of (17) are defined using different commuting left $\mathcal{D}_X$-structures on $\mathcal{D}_X \otimes \Omega_X^{-1}$. So we can interchange them and get the canonical morphism from the left-hand side of (17) to

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Omega_X \otimes_{\mathcal{D}_X}(\mathcal{D}_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N})). \qquad (18)$$

There is canonical isomorphism of $\mathcal{D}_X$-modules

$$\Omega_X \otimes_{\mathcal{D}_X}(\mathcal{D}_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N}) = \mathcal{N}. \qquad (19)$$

Indeed, the left structure on $(\mathcal{D}_X \otimes \Omega_X^{-1})$ we used to define $\Omega_X \otimes_{\mathcal{D}_X}(\mathcal{D}_X \otimes \Omega_X^{-1})$ is provided by the left multiplication in $\mathcal{D}_X$, therefore $\Omega_X \otimes_{\mathcal{D}_X}(\mathcal{D}_X \otimes \Omega_X^{-1}) = \mathcal{O}$. So (18) is canonically isomorphic to $\mathcal{N}$ as an $\mathcal{O}$-module. This isomorphism commutes with the action of vector fields. So (19) is canonically isomorphic to $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$. Theorem 3.3 is proved.

Replacing in (16) $\mathcal{N} = C^\infty(X)$ and using $\star\star\mathcal{M} = \mathcal{M}$, we get

Corollary 3.4.

$$\mathrm{DR}(\mathcal{M} \otimes_{\mathcal{O}} C^\infty(X))[-\dim X] = R\,\mathrm{Hom}_{\mathcal{D}_X}(\star\mathcal{M}, C^\infty(X)). \qquad (20)$$

In particular,

$$\mathrm{Hom}_{\mathcal{D}_X}(\star\mathcal{D}_X, D'(X)) \to \mathrm{DR}(D^\Omega_X \otimes_{\mathcal{O}} D'(X)) = D^n(X).$$

Corollary 3.5. *For any $\mathcal{A}, \mathcal{B} \in D^b_{\mathrm{coh}}(X)$, $\mathcal{C} \in D^b(X)$, one has a canonical functorial isomorphism in $D^b(\mathrm{Sh}_X)$:*

$$R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{A}, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C}) = R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \star\mathcal{B}, \mathcal{C}).$$

*Proof.* By the theorem above both parts are isomorphic to

$$R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \star\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C}).$$

In particular, for $\mathcal{M} \in D^b_{\mathrm{coh}}(X)$ there is a canonical morphism of $\mathcal{D}_X$-modules

$$i_{\mathcal{M}}: \mathcal{O}_X \to \bigstar\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}. \tag{21}$$

More precisely, there exists a canonical section over $X$ of the sheaf $H^0\, R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \bigstar\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$, or, what is the same, a canonical morphism $\mathbb{C}_X \to R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \bigstar\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$ in $D^b(\mathrm{Sh}_X)$.

3. *Functors between the Derived Categories of $\mathcal{D}$-modules*

Let $Y \to X$ be a morphism of varieties and $d_{Y,X} := \dim Y - \dim X$. Let $p^+$ be the naive inverse image functor on $\mathcal{D}$-modules. Then $p^! := Lp^+[d_{Y,X}]$. If $p: Y \to X$ is smooth then $p_*$ can be computed via relative DeRham complex $p_*\mathcal{M} = Rp_\bullet(\mathrm{DR}_{Y|X}(\mathcal{M}))$, where $\mathrm{DR}_{Y|X}(\mathcal{M}) := \Omega^\bullet_{Y|X} \otimes \mathcal{M}[d_{Y,X}]$.

LEMMA 3.6. *Suppose p is smooth. Then there is a canonical isomorphism of functors on $D^b_{\mathrm{coh}}(\mathcal{D})$,*

$$p^* := \bigstar p^! \bigstar = p^![-2d_{Y,X}].$$

*Proof.* See the proof of Proposition 9.13 in [Bo].

Let $p_! := \bigstar p_* \bigstar$.

THEOREM 3.7. *Suppose p is proper. Then $p_! = p_*$ on $D^b_{\mathrm{coh}}(\mathcal{D})$ and the functor $p_!$ (resp., $p^*$) is left adjoint to $p^!$ (resp., $p_*$), i.e.,*

$$R\,\mathrm{Hom}_{\mathcal{D}}(p^*\mathcal{M}, \mathcal{N}) = R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, p_*\mathcal{N})$$

$$R\,\mathrm{Hom}_{\mathcal{D}}(p_!\mathcal{M}, \mathcal{N}) = R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, p^!\mathcal{N}).$$

*Proof.* See the proof of Theorem 9.12 in [Bo].

LEMMA 3.8. *Let $p: X \to *$ be projection to the point and $\Delta: X \hookrightarrow X \times X$ be the diagonal. Then $Rp_\bullet R\,\mathrm{Hom}_{\mathcal{D}}(\bigstar\mathcal{M}, \mathcal{N}) = p_*\Delta^!(\bigstar\mathcal{M} \boxtimes \mathcal{N})$.*

## 4. THE GREEN CLASS OF $\mathcal{M}$

"... *we can say that there is only one formula (which we shall call "fundamental formula") in the whole theory of partial differential equations, no matter to which type they belong.*"

*J. Hadamard, Lectures on the Cauchy problem.*

## 0. *The Definition*

For any $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ the identity map in $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M})$ provides a canonical element

$$G_{\mathscr{M}} \in H^{\dim X}(\mathrm{DR}(\bigstar\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M})). \tag{22}$$

I will call it the *Green class of* $\mathscr{M}$.

The right hand side of (22) is a sheaf on $X$, and $G_{\mathscr{M}}$ is a canonical section of this sheaf. A more concrete way to think about it is this. Choose a locally projective resolution $\mathscr{M}^{\bullet}$ for $\mathscr{M}$. Take a Čech covering $\{\mathscr{U}_i\}$ of $X$ (in the classical or Zariski topology). Then there exists a section in the Čech complex $C(\mathscr{U}_{\bullet}, \mathrm{DR}(\bigstar\mathscr{M}^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}^{\bullet}))$ with coefficients in the complex of sheaves $\mathrm{DR}(\bigstar\mathscr{M}^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}^{\bullet})$ which represents the Green class.

## 1. *The Green Class and the Classical Green Formula*

Let $P$ be a differential operator. Set

$$\mathscr{P} = \frac{\mathscr{D}_X}{\mathscr{D}_X \cdot P} \qquad \text{and} \qquad \tilde{\bigstar}\mathscr{P} = \frac{\mathscr{D}_X^{\Omega}}{P \cdot \mathscr{D}_X^{\Omega}}.$$

Here $\mathscr{D}_X^{\Omega}$ is considered as a left $\mathscr{D}$-module with respect to the second structure. Note that $\mathrm{Hom}_{\mathscr{D}}(D_X^{\Omega}, C^{\infty}(X)) = \mathscr{A}^n(X)$. Let $v \in \mathscr{A}^n(X)$. According to the Green formula there exists an $(n-1)$-form $\omega_{n-1}(\varphi; P; v)$ on $X$ such that ($P^*$ is the adjoint operator on $\mathscr{A}^n(x)$)

$$P\varphi \cdot v - \varphi \cdot P^* v = d\omega_{n-1}(\varphi; P; v). \tag{23}$$

Of course neither the $(n-1)$-form $\omega_{n-1}(\varphi; P; v)$ nor its cohomology class $[\omega_{n-1}(\varphi; P; v)]$ are defined canonically by (23). However, there is a way to define a cohomology class in $H^{n-1}(X, \mathbb{R})$ starting form the Green formula. Namely, locally there exists an algebraic bidifferential operator

$$\omega_P \colon C^{\infty}(X) \otimes_{\mathscr{O}} \mathscr{A}^n(X) \to \mathscr{A}^{n-1}(X) \qquad \text{such that} \quad d\omega_P = P \otimes 1 - 1 \otimes P^*,$$

so $\omega_{n-1}(\varphi; P; v) := \omega_P(\varphi \otimes v)$. For two different algebraic bidifferential operators $\omega_P$ and $\omega'_P$ there exists an algebraic bidifferential operator

$$\omega''_P \colon C^{\infty}(X) \otimes_{\mathscr{O}} \mathscr{A}^n(X) \to \mathscr{A}^{n-2}(X) \qquad \text{such that} \quad d\omega''_P = \omega_P - \omega'_P$$

and so on. So choosing a covering and taking a partition of unity corresponding to it, we get a well defined cohomology class $[\omega_{n-1}(\varphi; P; v)]$. Below we explane how to get it without computations in local coordinates, using the $\mathscr{D}$-modules instead. (On the other hand, the approach we sketched leads to an equivariant cohomology class of the group of diffeomorphisms of $X$.)

LEMMA 4.1. *$\bigstar\mathscr{P}[-1]$ is isomorphic to $\tilde{\bigstar}\mathscr{P}$, so the Green class $G_{\bigstar\mathscr{P}}$ is an element of $H^{n-1}\mathrm{DR}(\tilde{\bigstar}\mathscr{P}\otimes_{\mathcal{O}}\mathscr{P})$.*

*Proof.* Let $\mathscr{D}_X \xrightarrow{P} \mathscr{D}_X$ be the obvious free resolution for $\mathscr{P}$. It is concentrated in degrees $[-1, 0]$. So

$$R\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{P}, \mathscr{D}_X^{\Omega}) = (\mathscr{D}_X^{\Omega} \xrightarrow{P^*} \mathscr{D}_X^{\Omega})[n]$$

(the complex is concentrated in degrees $[-n, -(n-1)]$), where $P^*: Q \to P\circ Q$. Recall that there is canonical involution on $\mathscr{D}_X^{\Omega}$ interchanging two left $\mathscr{D}_X$-structures. If we choose a volume form $\omega$ this involution sends $P\otimes\omega^{-1}$ just to $P^t\otimes\omega^{-1}$ where $P^t$ is the transposed to $P$ defined using $\omega$. The lemma follows immediately from these remarks.

Let

$$\mathscr{R}_{\mathscr{P}} := \mathscr{D}_X \xrightarrow{P} \mathscr{D}_X, \qquad \mathscr{R}_{\mathscr{P}^*} := \mathscr{D}_X^{\Omega} \xrightarrow{P^*} \mathscr{D}_X^{\Omega}$$

be the resolutions for $\mathscr{P}$ and $\tilde{\bigstar}\mathscr{P}$. Their tensor product over $\mathcal{O}$,

$$\begin{array}{ccc} \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega} & \xrightarrow{P\otimes 1} & \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega} \\ \uparrow {\scriptstyle \mathscr{R}_{\mathscr{P}}\otimes_{\mathcal{O}}\mathscr{R}_{\mathscr{P}^*} = 1\otimes P^*} & & \uparrow {\scriptstyle -P^*\otimes 1} \\ \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega} & \xrightarrow{P\otimes 1} & \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega} \end{array}$$

sits in degrees $[-(n-2), -n]$. Let

$$\cdots \xrightarrow{d} \mathscr{R}_{\mathscr{P}}\otimes_{\mathcal{O}}\mathscr{R}_{\mathscr{P}^*}\otimes_{\mathcal{O}}\Omega^{n-1} \xrightarrow{d} \mathscr{R}_{\mathscr{P}}\otimes_{\mathcal{O}}\mathscr{R}_{\mathscr{P}^*}\otimes_{\mathcal{O}}\Omega^{n}$$

be the de Rham complex $\mathrm{DR}(\mathscr{R}_{\mathscr{P}}\otimes_{\mathcal{O}}\mathscr{R}_{\mathscr{P}^*})$. Its degree $-(n-1)$ part is

$$\left(\mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^n \;\oplus\; \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^n\right) \;\oplus\; \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^{n-1}. \qquad (24)$$

Since $\mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^n = \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}$, there is a canonical element $(1\otimes 1, 1\otimes 1)$ in the left summand of (24).

Choose a covering $\{\mathscr{U}_i\}$ of $X$. Consider the Čech complex

$$C(\{\mathscr{U}_\bullet\}, \mathrm{DR}(\mathscr{R}_{\mathscr{P}}\otimes_{\mathcal{O}}\mathscr{R}_{\mathscr{P}^*})).$$

An $-(n-1)$-cocycle $\tilde{G}_{\mathscr{M}}$ in this complex such that the component in $C(\mathscr{U}_i, (\mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^n \oplus \mathscr{D}\otimes_{\mathcal{O}}\mathscr{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^n))$ is $(1\otimes 1, 1\otimes 1)$ represents the Green class. Its existence follows from general theory.

Let $\omega^{(i)}_{n-1}$ be the componenet of $\tilde{G}_{\mathcal{M}}$ in $C(\mathcal{U}_i, \mathcal{D}\otimes_{\mathcal{O}}\mathcal{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^{n-1})$, and $\omega^{(i,j)}_{n-2}$ be the component in $C(\mathcal{U}_{i,j}, \mathcal{D}\otimes_{\mathcal{O}}\mathcal{D}^{\Omega}\otimes_{\mathcal{O}}\Omega^{n-2})$. Then $d\omega^{(i,j)}_{n-2} = \omega^{(i)}_{n-1} - \omega^{(j)}_{n-1}$. To relate this cocycle with the discussion in Section 4.1 note that $\omega^{(i)}_{n-1}$ can be vied as an "algebraic bidifferential operator."

## 2. *The Green Formula and the Bar Construction*

Let $E^1$ and $E^2$ be vector bundles over an $n$-dimensional manifold $X$ and $E^1 \xrightarrow{P} E^2$ be a differential operator. Set $V_i := E^{i*}\otimes\mathcal{A}^n$. There are canonical pairings

$$\Gamma_0(X, E^i)\otimes\Gamma(X, V_i)\to\mathbb{R} \qquad (\varphi, g\otimes\omega)\to\int_X(\varphi, g)\omega.$$

So one has the adjoint operator $V_1 \xleftarrow{P^*} V_2$. It is a differential operator of the same order as $P$ uniquely defined by the property $(\varphi_1, P^*v_2) = (P\varphi_1, v_2)$.

Now suppose we have a sequence (not necessarily a complex) of differential operators

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} \cdots \xrightarrow{P_k} E_k.$$

Consider the sequence of adjoint differential operators

$$V_0 \xleftarrow{P_1^*} V_1 \xleftarrow{P_2^*} \cdots \xleftarrow{P_k^*} V_k.$$

THEOREM 4.2. *For any $k$ there exist forms $\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k)$ such that $\omega_n(\varphi; 1; v) := \varphi\cdot v$ and*

$$\begin{aligned} d\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k) &= \omega_{n-k+1}(P_1\varphi_0; P_2, ..., P_k; v_k) \\ &+ \sum_{i=1}^{k-1}(-1)^i\,\omega_{n-k+1}(\varphi_0; P_1, ..., P_i\circ P_{i+1}, ..., P_k; v_k) \\ &+ (-1)^k\,\omega_{n-k+1}(\varphi_0; P_1, ..., P_{k-1}; P_k^* v_k). \end{aligned}$$

## 3. *How to Compute the Green Class*

Let us call a $\mathcal{D}$-module $\mathcal{M}$ *excellent* if the object $\star\mathcal{M}$ is concentrated in just one degree, i.e., $H^i(\star\mathcal{M}) = 0$ for all $i$ but one. By the Roos theorem this degree is $-d_{\mathcal{M}}$. In this case set $\tilde{\star}\mathcal{M} := H^{-d_{\mathcal{M}}}(\star\mathcal{M})$. Consider a locally free resolution of a $\mathcal{D}$-module $\mathcal{M}$:

$$\mathcal{P}^{\bullet} = \{\mathcal{P}^{-k}\to\cdots\to\mathcal{P}^{-2}\to\mathcal{P}^{-1}\to\mathcal{P}^0\}.$$

Let

$$\star\mathcal{P}^\bullet = \{\tilde{\star}(\mathcal{P}^0) \to \tilde{\star}(\mathcal{P}^1) \to \tilde{\star}(\mathcal{P}^2) \to \cdots \to \tilde{\star}(\mathcal{P}^k)\}[d_X]$$

be the dual complex. Then $E^\bullet := \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}^\bullet, C^\infty(X))$ is a complex of differential operators between vector bundles:

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} \cdots \xrightarrow{P_k} E^k.$$

The adjoint complex

$$V_\bullet := \{V_k \xrightarrow{P_k^*} V_{k-1} \xrightarrow{P_{k-1}^*} \cdots \xrightarrow{P_1^*} V_0\}$$

is canonically isomorphic to $\mathrm{Hom}_{\mathcal{D}}(\star\mathcal{P}^\bullet, C^\infty(X))[-d_X-k]$.

Suppose that a $\mathcal{D}$-module $\mathcal{M}$ is excellent and admits a locally free resolution of the minimal possible length $k = d_{\mathcal{M}}$. (This usually happens in integral geometry.) Then

$$\mathrm{Sol}(\mathcal{M}, C^\infty(X)) = \mathrm{Ker}\, P_1 \qquad \text{and} \qquad \mathrm{Sol}(\tilde{\star}\mathcal{M}, D'(X)) = \mathrm{Ker}\, P_k^*.$$

Therefore for any $\varphi_0 \in \mathrm{Ker}\, P_1$ and $v_k \in \mathrm{Ker}\, P_k^*$ one has

$$d\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k) = 0.$$

THEOREM 4.3. *The cohomology class of the form* $\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k)$ *coincides with the Green class* $G_{\mathcal{M}}(\varphi_0; v_k)$.

*Proof.* It is similar to the proof of Lemma 4.1. Since $\mathcal{P}^{-i}$ is a locally free $\mathcal{D}$-module, there is a canonical element $1_i$ in

$$\mathcal{P}^{-i} \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}^{-i}, \mathcal{D}^{\Omega}) \otimes \Omega. \tag{25}$$

Namely, locally $\mathcal{P}^{-i} = \mathcal{D} \otimes_{\mathbb{C}} V$, so (25) is

$$V \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} \mathcal{D} \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{D}^{\Omega}) \otimes \Omega = \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$$

and we take $Id_V \otimes 1 \otimes 1$. A 0-cycle in $\mathrm{DR}(\mathcal{P} \otimes_{\mathcal{O}} \star\mathcal{P})$ whose component in $\mathcal{P} \otimes_{\mathcal{O}} \star\mathcal{P} \otimes_{\mathcal{O}} \Omega$ is $\sum 1_i$ represents the Green class.

## 5. GENERAL LINEAR MAPS AND NATURAL LINEAR MAPS BETWEEN SOLUTION SPACES

Denote by $R\,\mathrm{Hom}^{\mathrm{c}}(\cdot, \cdot)$ the $R\,\mathrm{Hom}$ with compact supports in the category of sheaves. For $\mathcal{M}, \mathcal{N} \in D^b(\mathcal{D}_X)$ set

$$R\,\mathrm{Hom}^{\mathrm{c}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) := R\Gamma_c(X; R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}))$$

We will define a canonical morphism

$$R\,\mathrm{Hom}^{\mathrm{c}}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2)) \otimes R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^{\infty}(X_1)) \\ \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, D'(X_2))[-n]. \tag{26}$$

Any linear map

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^{\infty}(X_1)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, D'(X_2))[-n] \tag{27}$$

continuous in an appropriate topology is given by a unique element in

$$R\,\mathrm{Hom}^{\mathrm{c}}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2)) \tag{28}$$

(this follows from Lemma 5.1), so the space (28) gives us the general linear maps (27). Our goal in this paper is to construct and study an interesting subspace in (28), the subspace of *natural* linear maps.

To make the relation with natural functionals clearer, we will spell out the construction of the map (26) using the Green class

$$G_{\mathscr{M}} \in H^{\dim X}(\mathrm{DR}(\bigstar\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M})) \tag{29}$$

and using the canonical morphism in $D^b(\mathscr{D}_X)$

$$i_{\mathscr{M}}\colon \mathscr{O}_X \to \bigstar\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}. \tag{30}$$

They are, of course, equivalent. To clarify the main point we will start from the case when $X_2$ is a point.

1. *The Canonical Pairing via the Green Class*

Let $\varDelta_X$ be the orientation sheaf of $X$. Set $\tilde{D}'(X) := D'(X) \otimes_{\mathbb{Z}} \varDelta_X$. We will define the canonical pairing

$$R^i\,\mathrm{Hom}^{\mathrm{c}}_{\mathscr{D}_X}(\bigstar\mathscr{M}, \tilde{D}'(X)) \otimes R^{n-i}\,\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, C^{\infty}(X)) \to \mathbb{C}.$$

If $\mathscr{A}_i$, $\mathscr{B}_j$ are sheaves on $X$, $a \in R^i\,\mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2)$, and $b \in R^j\,\mathrm{Hom}(\mathscr{B}_1, \mathscr{B}_2)$, then the tensor product over $\mathbb{C}$ provides an element

$$a \otimes_{\mathbb{C}} b \in R^{i+j}\,\mathrm{Hom}(\mathscr{A}_1 \otimes_{\mathbb{C}} \mathscr{B}_1, \mathscr{A}_2 \otimes_{\mathbb{C}} \mathscr{B}_2).$$

If $\mathscr{A}_i$, $\mathscr{B}_j$ are sheaves of $\mathscr{O}$-modules on $X$ we can make a tensor product over $\mathscr{O}$:

$$a \otimes_{\mathscr{O}} b \in R^{i+j}\,\mathrm{Hom}(\mathscr{A}_1 \otimes_{\mathscr{O}} \mathscr{B}_1, \mathscr{A}_2 \otimes_{\mathscr{O}} \mathscr{B}_2).$$

Suppose $X$ is a smooth variety over $\mathbb{R}$ of dimension $n$ and

$$v \in R^i \operatorname{Hom}^{\mathrm{c}}_{\mathscr{D}_X}(\bigstar\mathscr{M}, D'(X)), \qquad f \in R^{n-i} \operatorname{Hom}_{\mathscr{D}_X}(\mathscr{M}, C^\infty(X)).$$

Their tensor product over $\mathscr{O}_X$ is an element

$$v \otimes_{\mathscr{O}} f \in R^n \operatorname{Hom}_{\mathscr{D}_X}(\bigstar\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M}, \tilde{D}'(X) \otimes_{\mathscr{O}} C^\infty(X)). \tag{31}$$

So the multiplication $m\colon \tilde{D}'(X) \otimes_{\mathscr{O}_X} C^\infty(X) \to \tilde{D}'(X)$ leads to an element

$$m(v \otimes_{\mathscr{O}} f) \in R^{i+j} \operatorname{Hom}_{\mathscr{D}_X}(\bigstar\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M}, \tilde{D}'(X)).$$

Applying this element to the Green class (29) we get a cohomology class

$$m(v \otimes_{\mathscr{O}} f)(G_{\mathscr{M}}) \in H^n_c(\mathrm{DR}(\tilde{D}'(X)[n]) = H^n_c(\tilde{X}, \mathbb{C}) = \mathbb{C}$$

where $\tilde{X} = X$ if $X$ is orientable, and it is a twofold covering given by the orientation class if it is not.

2. *The canonical Pairing via the Morphism* $i_{\mathscr{M}}$

Taking the Koszul resolution of the $\mathscr{D}$-module $\mathscr{O}_X$ we see that the complex of sheaves $R \operatorname{Hom}_{\mathscr{D}_X}(\mathscr{O}_X, D'(X))$ is canonically quasi-isomorphic to the De Rham complex of currents on $X$:

$$D'(X) \xrightarrow{d} D'(X) \otimes_{\mathscr{O}_X} \Omega^1_X \xrightarrow{d} \cdots$$
$$\xrightarrow{d} D'(X) \otimes_{\mathscr{O}_X} \Omega^{n-1}_X \xrightarrow{d} D'(X) \otimes_{\mathscr{O}_X} \Omega^n_X$$

(the last group sitting in degree $n$). If we take the $R \operatorname{Hom}$'s with compact support we get the De Rham complex of currents with compact support.

There is the trace map given by integration over the fundamental class of $X$:

$$\int_X : R^n \operatorname{Hom}^c_{\mathscr{D}_X}(\mathscr{O}_X, \tilde{D}'(X)) \to \mathbb{C}.$$

The composition of the morphism $i_{\mathscr{M}}$ (30) with the element $m(v \otimes_{\mathscr{O}} f)$ (31) gives

$$m \circ i_{\mathscr{M}}(v \otimes_{\mathscr{O}} f) \in R^n \operatorname{Hom}^c_{\mathscr{D}_X}(\mathscr{O}_X, \tilde{D}'(X)).$$

Applying $\int_X$ we get a pairing

$$v \otimes f \mapsto \int_X m \circ i_{\mathscr{M}}(v \otimes_{\mathscr{O}} f) \in \mathbb{C}.$$

Recall that there is the Grothendieck duality theory for topological vector spaces [Gr]. In particular, $C^\infty(X)$ has a natural topology of a Fréchet nuclear space, and $D_0'(X)$ has a natural topology of a dual to a Fréchet nuclear space, so they are topologically dual. An immediate consequence of this is the following simple duality lemma. For a more general result, see Theorem 6.1 in [KS].

LEMMA 5.1. *Suppose $X$ is a smooth variety over $\mathbb{R}$ of dimension $n$. Then*

$$R^i \operatorname{Hom}_{\mathscr{D}_X}(\mathscr{M}, C^\infty(X)) \tag{32}$$

*has a topology of a Fréchet nuclear space and*

$$R^{n-i} \operatorname{Hom}^c_{\mathscr{D}_X}(\star\mathscr{M}, D'(X)) \tag{33}$$

*has a topology of a dual to a Fréchet nuclear space. The spaces* (32) *and* (33) *are dual to each other.*

*Proof.* Consider first the classical case of a single differential operator. Let $P$ be a differential operator acting on smooth functions and $P^*$ the adjoint acting on the distributions with compact supports:

$$C^\infty(X) \xrightarrow{P} C^\infty(X)$$
$$D_0^n(X) \xleftarrow{P^*} D_0^n(X).$$

The canonical pairing boils down to the obvious duality between $\operatorname{Ker} P$ and the closure of $\operatorname{Coker} P^*$, and the closure of $\operatorname{Coker} P$ and $\operatorname{Ker} P^*$. The general statement for any $\mathscr{M} \in D^b(\mathscr{D}_X)$ we obtain similarly by taking a locally projective resolution.

3. *A Construction of the Map* (26)

Let $R \operatorname{Hom}^{c_1}$ be the $R \operatorname{Hom}$ with compact supports along the factor $X_1$. Choose

$$K \in R \operatorname{Hom}^{c_1}_{\mathscr{D}}(\star\mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2)) \qquad f \in R \operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^\infty(X_1)).$$

Their product $K \otimes_{\mathscr{O}_{X_1}} f$ over $X_1$ belongs to

$$R \operatorname{Hom}^{c_1}_{\mathscr{D}_{X_1 \times X_2}}(\star\mathscr{M}_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{M}_1 \boxtimes \mathscr{M}_2, \tilde{D}'(X_1) \otimes_{\mathscr{O}_{X_1}} C^\infty(X_1) \boxtimes D'(X_2)).$$

Using the multiplication map

$$m_{X_1}\colon \tilde{D}'(X_1) \otimes_{\mathscr{O}_{X_1}} C^\infty(X_1) \boxtimes D'(X_2) \to \tilde{D}'(X_1) \boxtimes D'(X_2)$$

we get a class

$$m_{X_1}(K \otimes_{\mathcal{O}_{X_1}} f) \in R\,\mathrm{Hom}^{c_1}_{\mathcal{D}_{X_1 \times X_2}}(\bigstar\mathcal{M}_1 \otimes \mathcal{M}_1 \mathcal{O}_{X\,1} \boxtimes \mathcal{M}_2, \tilde{D}'(X_1) \boxtimes D'(X_2)).$$

The canonical morphism $i_{\mathcal{M}_1}\colon \mathcal{O}_{X_1} \to \bigstar\mathcal{M}_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{M}_1$ provides an element

$$(m_{X_1} \circ i_{\mathcal{M}_1})(K \otimes_{\mathcal{O}_{X_1}} f) \in R\,\mathrm{Hom}^{c_1}_{\mathcal{D}_{X_1 \times X_2}}(\mathcal{O}_{X_1} \boxtimes \mathcal{M}_2, \tilde{D}'(X_1) \boxtimes D'(X_2)).$$

Applying $\int_{X_1}\colon R^n\,\mathrm{Hom}^c_{\mathcal{D}_{X_1}}(\mathcal{O}_{X_1}, \tilde{D}'(X_1)) \to \mathbb{C}$, we get

$$\bar{K}(f) \in R\,\mathrm{Hom}_{\mathcal{D}_{X_2}}(\mathcal{M}_2, D'(X_2)).$$

4. *Natural Functionals*

Recall that $R^{n-m}\,\mathrm{Hom}^c(\mathbb{C}, \Delta_X) = H_m(X, \mathbb{C})$ and the Poincare duality is given by

$$R^i\,\mathrm{Hom}(\mathbb{C}, \mathbb{C}) \otimes R^{n-i}\,\mathrm{Hom}^c(\mathbb{C}, \Delta_X) \to R^n\,\mathrm{Hom}^c(\mathbb{C}, \Delta_X) \xrightarrow{\int_X} \mathbb{C}.$$

The first arrow is the composition of Hom's. A tensor product over $\mathbb{C}$ provides a canonical map

$$R^j\,\mathrm{Hom}^c(\mathbb{C}, \Delta_X) \otimes R^i\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, D'(X)) \to R^{i+j}\,\mathrm{Hom}^c(\mathcal{M}, \tilde{D}'(X)).$$

Combining it with the canonical pairing we get a map

$$\langle \cdot, \cdot, \cdot \rangle_{\mathcal{M}}\colon$$
$$H_{i+j}(X, \mathbb{C}) \otimes R^i\,\mathrm{Hom}_{\mathcal{D}_X}(\bigstar\mathcal{M}, D'(X)) \otimes R^j\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, C^\infty(X)) \to \mathbb{C}. \quad (34)$$

By definition the natural functionals on the space $R^j\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, C^\infty(X))$ are the functionals $\langle \gamma, v, \cdot \rangle_{\mathcal{M}}$ provided by a homology class $\gamma \in H_{i+j}(X, \mathbb{C})$ and $v \in R^i\,\mathrm{Hom}_{\mathcal{D}_X}(\bigstar\mathcal{M}, \tilde{D}'(X))$.

5. *Natural Linear Maps*

There is a map

$$R\Gamma_c(X_1, \Delta_{X_1}) \otimes \mathrm{Hom}_{\mathcal{D}}(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2))$$
$$\to R\,\mathrm{Hom}^c_{\mathcal{D}}(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2)).$$

So we get a canonical morphism

$$R\Gamma_c(X_1, \Delta_{X_1}) \otimes R\,\mathrm{Hom}_{\mathcal{D}}(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2))$$
$$\otimes R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \to R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_2, D'(X_2))[-n]. \quad (35)$$

In particular, it induces a map

$$H_{i+j}(X_1, \mathbb{Z}) \otimes R^{i+k} \operatorname{Hom}_{\mathscr{D}_{X_1 \times X_2}}(\bigstar \mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2)) \\ \otimes R^j \operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^\infty(X_1)) \to R^k \operatorname{Hom}_{\mathscr{D}_{X_2}}(\mathscr{M}_2, D'(X_2)). \tag{36}$$

By definition, a natural linear map

$$\bar{K}_\gamma \colon R^j \operatorname{Hom}_{\mathscr{D}_{X_1}}(\mathscr{M}_1, C^\infty(X_1)) \to R^k \operatorname{Hom}_{\mathscr{D}_{X_2}}(\mathscr{M}_2, D'(X_2))$$

is given by a "kernel"

$$K = K(x_1, x_2) \in R^{i+k} \operatorname{Hom}_{\mathscr{D}_{X_1 \times X_2}}(\bigstar \mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2))$$

and $\gamma \in H_{i+j}(X_1, \mathbb{Z})$.

## 6. *Examples*

Suppose $\mathscr{M}$ is an excellent $\mathscr{D}_X$-module, $m := d_{\mathscr{M}}$. Recall that $\tilde{\bigstar} \mathscr{M} := H^{-m}(\bigstar \mathscr{M})$ is the dual system to $\mathscr{M}$. Taking $k = m$, $i = 0$, $l = m$ we get

$$\operatorname{Hom}_{\mathscr{D}}(\mathscr{M}, C^\infty(X)) \otimes \operatorname{Hom}_{\mathscr{D}}(\tilde{\bigstar} \mathscr{M}, D'(X)) \otimes H_m(X, \mathbb{R}) \to \mathbb{R}.$$

Let $\tilde{G}_{\mathscr{M}}(\cdot, \cdot)$ be a cocycle in $\mathrm{DR}(\tilde{\bigstar} \mathscr{M} \otimes_{\mathscr{O}} \mathscr{M})$ representing the Green class. Let $f(x)$ be a smooth solution of the system $\mathscr{M}$. Choose a distributional solution $v(x)$ of $\tilde{\bigstar} \mathscr{M}$. Then we get a closed differential form $\tilde{G}_{\mathscr{M}}(v(x), f(x))$ of degree $d_{\mathscr{M}}$ on $X$. Choose a cycle $\gamma$ of dimension $d_{\mathscr{M}}$ in $X$. Then

$$\langle \gamma, v, f \rangle = \int_\gamma \tilde{G}_{\mathscr{M}}(v(x), f(x))$$

is a functional on smooth solutions of $\mathscr{M}$.

EXAMPLE 0. Suppose $\mathscr{M} = \mathscr{D}_X$. Then $\bigstar \mathscr{M} = \mathscr{D}_X^\Omega[n]$ and $\tilde{\bigstar} \mathscr{M} = \mathscr{D}_X^\Omega$. Recall that $\operatorname{Hom}_{\mathscr{D}}^c(\mathscr{D}_X^\Omega, D'(X)) = D_0^n(X)$. We get the usual pairing $C^\infty(X) \otimes D_0^n(X) \to \mathbb{C}$.

The following examples show a wider class of functionals on solution spaces than the natural functionals we just defined above. The point is that sometimes we can integrate the differential form $\tilde{G}_{\mathscr{M}}(v(x), f(x))$ not only over cycles, but also over some chains (which do not represent a homology class in the general sense), still getting a functional on smooth solutions of a system $\mathscr{M}$.

EXAMPLE 1. Let $\mathscr{M}$ be a $\mathscr{D}$-module on $\mathbb{R}^n$ corresponding to the system $x_1 \cdot f = x_2 \cdot f = \cdots = x_m \cdot f = 0$, $0 < m < n$. Then

$$R\,\mathrm{Hom}^i_{\mathscr{D}}(\mathscr{M}, C^\infty(X)) = 0 \qquad \text{for} \quad i \neq m,$$

$$R\,\mathrm{Hom}^m_{\mathscr{D}}(\mathscr{M}, C^\infty(X)) = \frac{C^\infty(X)}{(C^\infty(X) \cdot x_i)},$$

and

$$R\,\mathrm{Hom}^i_{\mathscr{D}}(\mathscr{M}, D'(X)) = 0 \qquad \text{for} \quad i \neq 0$$

$$\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}, D'(X)) = \delta(x_1)\,\delta(x_2)\,\cdots\delta(x_m) \cdot D'(X).$$

So there is a natural pairing

$$R\,\mathrm{Hom}^m_{\mathscr{D}}(\mathscr{M}, C^\infty(X)) \otimes \mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, D'(X)) \to \mathbb{R}.$$

It should correspond to the case $i = 0, j = m, k = m$. However, $H_m(\mathbb{R}^n, \mathbb{R}) = 0$ in any topological sense.

*Comparing the General and Natural Functionals.* Let $P$ be a differential operator on $X$. Recall that we get the general functionals on Ker $P$ from the closure of Coker $P^*$; see the proof of Proposition 5.1. The natural functionals we get in a different way. Take $f \in \mathrm{Ker}\, P$ and $v \in D'(X)$, $v \in \mathrm{Ker}\, P^*$. Note that if, for example, $P$ is an operator with constant coefficients, then the restriction of $\mathrm{Ker}\, P \cap D'_0(X) = 0$, so it is essential that $v$ is not necessarily compactly supported. Then make the Green class $[\omega_{n-1}(v; P; f)]$ and integrate it over a homology class $[\gamma]$. A simple example is given in Example 2 below.

One advantage of natural functionals on Ker $P$ is that they correspond to "functions," i.e., elements of the subspace Ker $P^*$, rather than to elements of the quotient Coker $P^*$.

EXAMPLE 2. Let $\mathscr{L}_a$ be the system $(\sum_{i=1}^n x_i \partial_{x_i} - a)\, f(x) = 0$ on $\mathbb{R}^n \backslash 0$. Then $\tilde{\bigstar} \mathscr{L}_a = \mathscr{L}_{-a-n}$. Consider the following differential $(n-1)$-form:

$$\sigma_n(x, dx) := \sum_{i=1}^{n} (-1)^{i-1}\, x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \tag{37}$$

Then

$$\tilde{G}_{\mathscr{L}_a}(v(x), f(x)) = v(x)\, f(x)\, \sigma_n(x, dx).$$

Let $\gamma$ be an $(n-1)$-cycle generating $H_{n-1}(\mathbb{R}^n \backslash 0)$. Then

$$\int_{\gamma} v(x) \cdot f(x)\, \sigma_n(x, dx) \tag{38}$$

provides a nondegenerate pairing between the smooth solutions of $\mathscr{L}_a$ and $\mathscr{L}_{-a-n}$.

EXAMPLE 3. Consider $\mathbb{C}^n$ as a real manifold. Let $\mathscr{L}_{a,b}$ be the following system in $\mathbb{C}^n \backslash 0$:

$$\left(\sum_{i=1}^{n} z_i \partial_{z_i} - a\right) f(z, \bar{z}) = 0, \qquad \left(\sum_{i=1}^{n} \bar{z}_i \partial_{\bar{z}_i} - b\right) f(z, \bar{z}) = 0$$

Then $\tilde{\bigstar} \mathscr{L}_{a,b} = \mathscr{L}_{-a-n,\, -b-n}$. Then

$$\tilde{G}_{\mathscr{L}_{a,b}}(v(z, \bar{z}), f(z, \bar{z})) = v(z, \bar{z}) \cdot f(z, \bar{z}) \cdot \sigma_n(z, dz) \wedge \sigma_n(\bar{z}, d\bar{z}).$$

Let $\Gamma$ be a chain intersecting any one-dimensional subspace in $\mathbb{C}^n$ with multiplicity one. Then

$$\int_{\Gamma} v(z, \bar{z})\, f(z, \bar{z})\, \sigma_n(z, dz) \wedge \sigma_n(\bar{z}, d\bar{z}) \tag{39}$$

provides a pairing between the solutions of $\mathscr{L}_{a,b}$ and $\mathscr{L}_{-a-n,\, -b-n}$. However, $H_{2n-2}(\mathbb{C}^n \backslash 0) = H_{2n-2}(S^{2n-1}) = 0$!

A chain $\Gamma$ can be considered as a discontinuous "section" of the Hopf bundle $\mathbb{C}^n \backslash 0 \to \mathbb{C}P^{n-1}$. A better way to think about this integral is the following. The form we integrate can be pushed down to $\mathbb{C}P^{n-1}$, so we integrate over the fundamental cycle.

## 7. *Composition of Natural Maps between Smooth Solution Spaces*

We cannot define in general a morphism

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, D'(X_1)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, D'(X_2))$$

using distributional kernels because of the lack of multiplication of distributions, and *a priori* there is no way to compose operators

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^{\infty}(X_1)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, D'(X_2))$$

and

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^{\infty}(X_2)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_3, D'(X_3)).$$

However, the natural linear maps constructed using smooth kernels can be composed. Namely, suppose

$$K_{12} \in R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2, C^{\infty}(X_1 \times X_2)), \qquad \gamma_1 \in H_{l_1}(X_1, \mathbb{R})$$
$$K_{23} \in R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_2 \boxtimes \mathscr{M}_3, C^{\infty}(X_2 \times X_3)), \qquad \gamma_2 \in H_{l_2}(X_2, \mathbb{R}).$$

They define the corresponding natural maps

$$\bar{K}_{12}^{\gamma_1}\colon R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^{\infty}(X_1)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^{\infty}(X_2))$$
$$\bar{K}_{23}^{\gamma_2}\colon R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^{\infty}(X_2)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_3, C^{\infty}(X_3)).$$

Their composition is given by the data

$$K_{23} \circ K_{12} \in R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_3, C^{\infty}(X_1 \times X_3)), \qquad \gamma_1 \in H_{l_1}(X_1, \mathbb{R}),$$

where the kernel $K_{23} \circ K_{12}$ is constructed as follows. Let

$$\Delta_2\colon X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$$

be the diagonal imbedding and $\pi_2\colon X_1 \times X_2 \times X_3 \to X_1 \times X_3$ be the projection. Then

$$\Delta_2^! C^{\infty}(X_1 \times X_2 \times X_2 \times X_3) = C^{\infty}(X_1 \times X_2 \times X_3)[-d_{X_2}]$$

and

$$H_l(X_2, \mathbb{Z}) \to R^{d_{X_2}-l}\,\mathrm{Hom}_{\mathscr{D}}(\pi_{2_*} C^{\infty}(X_1 \times X_2 \times X_3), C^{\infty}(X_1 \times X_3)).$$

Therefore one has the canonical morphism

$$H_l(X_2, \mathbb{Z}) \to R^l\,\mathrm{Hom}_{\mathscr{D}}(\pi_{2_*} \Delta_2^! C^{\infty}(X_1 \times X_2 \times X_2 \times X_3), C^{\infty}(X_1 \times X_3)).$$

According to Lemma 3.8 one has $G_{\mathscr{M}} \in p_* \Delta^!(\mathscr{M} \boxtimes \bigstar\mathscr{M})$. Therefore

$$\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_3 \xrightarrow{id \boxtimes G_{\mathscr{M}} \boxtimes id} \pi_{2_*} \Delta_2^!(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2 \boxtimes \bigstar\mathscr{M}_2 \boxtimes \mathscr{M}_3). \tag{40}$$

There is a canonical map

$$R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2, C^{\infty}(X_1 \times X_2)) \otimes R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_2 \boxtimes \mathscr{M}_3, C^{\infty}(X_2 \times X_3))$$
$$= R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2 \boxtimes \bigstar\mathscr{M}_2 \boxtimes \mathscr{M}_3, C^{\infty}(X_1 \times X_2 \times X_2 \times X_3))$$
$$\xrightarrow{G_{\mathscr{M}} \times \gamma_2} R\,\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_3, C^{\infty}(X_1 \times X_3))$$

provided by the morphism (40) and the morphism

$$\gamma_2 \in R^{-l_2}\,\mathrm{Hom}_{\mathscr{D}}(\pi_{2_*}\Delta_2^! C^\infty(X_1 \times X_2 \times X_2 \times X_3), C^\infty(X_1 \times X_3)).$$

THEOREM 5.2. *The kernel* $K_{23} \circ K_{12}$ *coincides with* $(G_{\mathscr{M}} \times \gamma_2)(K_{23} \otimes K_{12})$.

*Proof.* Follows immediately from the definitions.

EXAMPLE. Suppose $\mathscr{M}_i$ are excellent $\mathscr{D}_{X_i}$-modules. In this case usually the natural smooth kernels are just functions

$$K_{12}(x_1, x_2) \in \mathrm{Hom}_{\mathscr{D}}(\tilde{\bigstar}\mathscr{M}_1 \boxtimes \mathscr{M}_2, C^\infty(X_1 \times X_2))$$

and

$$K_{23}(x_2, x_3) \in \mathrm{Hom}_{\mathscr{D}}(\tilde{\bigstar}\mathscr{M}_2 \boxtimes \mathscr{M}_3, C^\infty(X_1 \times X_2)),$$

and the composition is defined by the natural kernel

$$K_{13}(x_1, x_3) = \int_{\gamma_2} G_{\mathscr{M}_2}(K_{12}(x_1, x_2), K_{23}(x_2, x_3)).$$

## 6. NATURAL LINEAR MAPS FOR EXCELLENT $\mathscr{D}$-MODULES

### 1. *The General Scheme*

Let $\mathscr{M}$ and $\mathscr{N}$ be excellent $\mathscr{D}$-modules on manifolds $X$ and $Y$, i.e., $\tilde{\bigstar}\mathscr{M} := (\bigstar\mathscr{M})[-d_{\mathscr{M}}]$ is a $\mathscr{D}$-module. Let $c_{\mathscr{M}} := \mathrm{codim}\, SS\mathscr{M} = \dim X - d_{\mathscr{M}}$. Then solutions

$$f \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{M}, C^\infty(X)) \qquad \text{and} \qquad g \in \mathrm{Hom}_{\mathscr{D}}(\tilde{\bigstar}\mathscr{M}, D'(X))$$

provide a homomorphism

$$H^{c_{\mathscr{M}}}\mathrm{DR}(\tilde{\bigstar}\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M}) \xrightarrow{\bar{f} \otimes \bar{g}} H^{c_{\mathscr{M}}}\mathrm{DR}(C^\infty(X) \otimes_{\mathscr{O}} D'(X))$$
$$\xrightarrow{m} H^{d_{\mathscr{M}}} D'^{\bullet}(X).$$

The Green class of $\mathscr{M}$ goes under this map to a cohomology class of degree $d_{\mathscr{M}}$ on $X$. Recall that we put $\mathrm{DR}(\mathscr{M})$ in degrees $[-\dim X, 0]$, while the smooth de Rham complex $\mathscr{A}^{\bullet}(X)$ is sitting in degrees $[0, \dim X]$.

Let us define a natural linear map

$$I\colon \mathrm{Sol}(\mathscr{M}, C^\infty(X)) \to \mathrm{Sol}(\mathscr{N}, D'(Y))$$

by a kernel

$$K_I(x, y) \in \mathrm{Sol}(\tilde{\bigstar}\mathscr{M} \boxtimes \mathscr{N}, D'(X \times Y)) \tag{41}$$

and a cycle $\gamma_X$ of dimension $d_{\mathscr{M}}$ in $X$ as follows. Let $\tilde{G}_{\mathscr{M}}(\cdot, \cdot)$ be a cocycle in the Čech complex of a covering of $X$ with coefficients in $\mathrm{DR}(\tilde{\bigstar}\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M})$ representing the Green class. (In integral geometry one may usually take a cocycle in the complex $\mathrm{DR}(\tilde{\bigstar}\mathscr{M} \otimes_{\mathscr{O}} \mathscr{M})$). Using solutions $K_I(x, y)$ of $\tilde{\bigstar}\mathscr{M}$ (where $y$ is considered as a parameter) and $f(x)$ of $\mathscr{M}$ we get a closed differential form $\tilde{G}_{\mathscr{M}}(K_I(x, y), f(x))$ of degree $d_{\mathscr{M}}$ on $X$. Set

$$f(x) \mapsto \int_\gamma \tilde{G}_{\mathscr{M}}(K_I(x, y), f(x)) \in \mathrm{Sol}(\mathscr{N}, D'(Y)). \tag{42}$$

Under certain assumptions on the wave front of the kernel $K_I(x, y)$, which we will assume below, the integral over cycle $\gamma$ makes sense and the image of (41) lies in $C^\infty(Y)$. Then a (natural) inverse for $I$ is an integral transformation

$$J\colon \mathrm{Sol}(\mathscr{N}, C^\infty(Y)) \to \mathrm{Sol}(\mathscr{M}, C^\infty(X))$$

$$J\colon \varphi(x) \mapsto \int_{\gamma_Y} \tilde{G}_{\mathscr{N}}(K_J(x, y), \varphi(y)) \tag{43}$$

defined via a certain $d_{\mathscr{N}}$-cycle $\gamma_Y$ in $Y$ and a kernel

$$K_J(x, y) \in \mathrm{Sol}(\mathscr{M} \boxtimes \tilde{\bigstar}\mathscr{N}, D'(X \times Y)). \tag{44}$$

This data defines also a transformation

$$J^t\colon \mathrm{Sol}(\tilde{\bigstar}\mathscr{M}, C^\infty(X)) \to \mathrm{Sol}(\tilde{\bigstar}\mathscr{N}, C^\infty(Y))$$

$$g(x) \mapsto \int_{\gamma_X} \tilde{G}_{\mathscr{M}}(g(x), K_J(x, y)).$$

There is a canonical map

$$\langle \cdot, \cdot, \cdot \rangle_{\mathscr{M}}\colon \mathrm{Sol}(\tilde{\bigstar}\mathscr{M}, C^\infty(X)) \otimes \mathrm{Sol}(\mathscr{M}, C^\infty(X)) \otimes H_{d_{\mathscr{M}}}(X, \mathbb{R}) \to \mathbb{R}$$

$$\langle g, f, \gamma_X \rangle_{\mathscr{M}} := \int_{\gamma_X} \tilde{G}_{\mathscr{M}}(g(x), f(x)). \tag{45}$$

So if we choose a homology class $\gamma_X$ we get a pairing

$$\langle g, f\rangle_{\mathscr{M}} := \langle g, f, \gamma_X\rangle_{\mathscr{M}} \tag{46}$$

and a similar one for $\mathscr{N}$.

THEOREM 6.1 (The Plancherel Formula). *Let $J$ be a natural inverse for $I$: $J \circ I = id_X$. Then for $f \in \mathrm{Sol}(\mathscr{M}, C^\infty(X))$, $g \in \mathrm{Sol}(\tilde{\bigstar}\mathscr{M}, C^\infty(X))$ one has*

$$\langle g, f, \gamma_X\rangle_{\mathscr{M}} = \langle J^t g, If, \gamma_Y\rangle_{\mathscr{N}}.$$

*Proof.* $\langle g, f, \gamma_X\rangle_{\mathscr{M}} = \langle g, J \circ If, \gamma_Y\rangle_{\mathscr{N}}$. So the theorem follows from

LEMMA 6.2. *Let $\varphi \in \mathrm{Sol}(\mathscr{N}, C^\infty(Y))$ and $g \in \mathrm{Sol}(\tilde{\bigstar}\mathscr{M}, C^\infty(X))$. Then*

$$\langle g, J\varphi, \gamma_X\rangle_{\mathscr{M}} = \langle J^t g, \varphi, \gamma_Y\rangle_{\mathscr{N}}. \tag{47}$$

*Proof.* The Green class is multiplicative with respect to the $\boxtimes$-product. So we can set $\tilde{G}_{\mathscr{M} \boxtimes \mathscr{N}} := \tilde{G}_{\mathscr{M}} \boxtimes \tilde{G}_{\mathscr{N}}$. Consider the following solutions

$$g(x) \boxtimes \varphi(y) \in \mathrm{Sol}(\tilde{\bigstar}\mathscr{M} \boxtimes \tilde{\bigstar}\mathscr{N}, C^\infty(X \times Y))$$
$$K_J(x, y) \in \mathrm{Sol}(\mathscr{M} \boxtimes \mathscr{N}, D'(X \times Y)).$$

They are solutions to the dual systems. So there is a pairing

$$\langle g(x) \boxtimes \varphi(y), K_J(x, y), \gamma_X \times \gamma_Y\rangle_{\mathscr{M} \boxtimes \mathscr{N}}.$$

We can evaluate it computing first the pairing along $X$ and then along $Y$. In this case we get the right-hand side of (47). Computing first the pairing along $Y$ and then along $X$ we get the left-hand side of (47).

The kernel $K_J$ is a much simpler (and fundamental) object than the actual integral transformation $J$. The reasons are the following:

(1) The kernel $K_J$ is a canonically defined distribution, while the formula for $J\varphi(x)$ depends on a cocycle $\tilde{G}_{\mathscr{N}}$ representing the Green class.

(2) Explicit calculation of cocycle $\tilde{G}_{\mathscr{N}}$ can be a nontrivial problem and so the final formula for the right-hand side of (43) could be quite complicated even for a very simple kernel $K_J$.

So the problem of inversion of the transformation $I$ splits into three steps:

*Step* 1. Find a distribution (44).

*Step* 2. Compute a cocycle $\tilde{G}_{\mathscr{N}}$ for the Green class.

*Step* 3. Find a cycle $\gamma_Y$.

The distribution (44) should be uniquely defined if it exists. However, it may not exist. The Green class always exists. Different cocycles representing it together with different choices of cycles $\gamma_Y$ provide the diversity of concrete inversion formulas. I will demonstrate below how this general scheme works in the simplest concrete problems.

## 2. *The Fourier Transform of Homogeneous Functions and the Radon Transform*

As everybody knows the Fourier transform in an $n$-dimensional real vector space $V_n$ is defined by the formula

$$S(V_n) \to S(V_n^*); \qquad f(x) \to \tilde{f}(\xi) := \int f(x)\, e^{2\pi i \langle x, \xi \rangle}\, d^n x.$$

The inverse operator is $f(\xi) \to \int f(\xi)\, e^{-2\pi i \langle x, \xi \rangle}\, d^n \xi$. Using the Plancherel formula one can define the Fourier transform of generalized functions.

Let $\Phi_\lambda^+(\mathbb{R}P^{n-1})$ (resp., $\Phi_\lambda^-(\mathbb{R}P^{n-1})$) be the space of even (resp., odd) smooth homogeneous functions $f(x)$ on $\mathbb{R}^n \backslash 0$ of degree $\lambda$: $f(ax) = |a|^\lambda f(x)$, $a > 0$. Also let $\Psi_\lambda(V_n)$ be the space of homogeneous distribution of degree $\lambda$ in $V_n$, and $\Psi_\lambda(V_n) = \Psi_\lambda^+(V_n) \oplus \Psi_\lambda^-(V_n)$ be the decomposition on even and odd parts.

Then $\Phi_\lambda(\mathbb{R}P^{n-1}) \subset \Psi_\lambda(V_n)$. This inclusion is not an isomorphism for integral $\lambda = -k$, $k \geqslant n$. One has

$$\Psi_{-k}(V_n)/\Phi_{-k}(\mathbb{R}P^{n-1}) = S^{k-n}(V_n)$$
$$= \{\delta - \text{functions of degree } -k \text{ at } 0\}.$$

The Fourier transform of generalized functions provides an isomorphism

$$\tilde{\mathscr{F}}_\lambda \colon \Psi_{-\lambda-n}^{\pm}(V_n) \to \Psi_\lambda^{\pm}(V_n').$$

Restricting to $\Phi_{-\lambda-n}^{\pm}(\mathbb{R}P^{n-1})$ we get a map

$$\mathscr{F}_\lambda \colon \Phi_{-\lambda-n}^{\pm}(\mathbb{R}P^{n-1}) \to \Phi_\lambda^{\pm}((\mathbb{R}P^{n-1})').$$

*It is remarkable that there is another way to define the operator* $\mathscr{F}_\lambda$. Let me recall that the space of the homogeneous degree $\lambda$ generalized function on $\mathbb{R}$ is two-dimensional and splits on the even and odd components (with respect to the involution $x \to -x$) generated by

$$\frac{|x|^\lambda}{\Gamma\left(\frac{\lambda+1}{2}\right)} \quad \text{and} \quad \frac{|x|^\lambda \operatorname{sgn} x}{\Gamma\left(\frac{\lambda+2}{2}\right)}.$$

They are both analytic on $\lambda$ on the whole complex plane. One has

$$\left.\frac{|x|^{\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)}\right|_{\lambda=-2k-1} = \frac{(-1)^k\, k!}{(2k)!} \cdot \delta^{(2k)}(x); \tag{48}$$

$$\left.\frac{|x|^{\lambda}\operatorname{sgn} x}{\Gamma\left(\frac{\lambda+2}{2}\right)}\right|_{\lambda=-2k} = \frac{(-1)^k\,(k-1)!}{(2k-1)!} \cdot \delta^{(2k-1)}(x). \tag{49}$$

Let $\gamma_{n-1}$ be a cycle generating $H_n(\mathbb{R}^n\backslash 0;\mathbb{Z})$. The kernel

$$K_\lambda^+(\xi, x) := \frac{|\langle \xi, x\rangle|^{\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)}$$

and the cycle $\gamma_{n-1}$ defines the operator

$$I_\lambda^+ : \Phi_{-\lambda-n}(\mathbb{R}P^{n-1}) \to \Phi_\lambda((\mathbb{R}P^{n-1})')$$

$$I_\lambda^+ : f(x) \to \frac{1}{2}\int_{\gamma_{n-1}} f(x)\,\frac{|\langle \xi, x\rangle|^{\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)}\,\sigma_n(x, dx).$$

The odd kernel

$$K_\lambda^-(\xi, x) := \frac{|\langle \xi, x\rangle|^{\lambda}\cdot \operatorname{sgn}(\langle \xi, x\rangle)}{\Gamma\left(\frac{\lambda+2}{2}\right)}$$

defines an integral transformation

$$(I_\lambda^- f)(\xi) = \int_{\gamma_{n-1}} f(x)\, K_\lambda^-(\xi, x)\,\sigma_n(x, dx).$$

PROPOSITION 6.3. $\mathscr{F}_\lambda^+ = \pi^{1/2+\lambda}\Gamma(-\lambda/2)\cdot I_\lambda^+$, $\mathscr{F}_\lambda^- = i\cdot \pi^{1/2+\lambda}\Gamma(-\lambda/2)\cdot I_\lambda^-$.

Set

$$f_\lambda(x) := \pi^{\lambda/2}\frac{|x|^{\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)}, \qquad g_\lambda(x) := \pi^{\lambda/2}\frac{|x|^{\lambda}\operatorname{sgn}(x)}{\Gamma\left(\frac{\lambda+2}{2}\right)}.$$

LEMMA 6.4.

$$\mathscr{F}(f_\lambda(x)) = f_{-1-\lambda}(\xi), \qquad \mathscr{F}(g_\lambda(x)) = i \cdot g_{-1-\lambda}(\xi). \tag{50}$$

*Proof.* See p. 173 in [GS] for an equivalent formula.

*Proof of the Proposition.* Using the polar coordinates $x = r \cdot s$ where $s \in S^{n-1}$, $|s| = 1$, we have $d^n x = r^{n-1}\, dr\, ds_{n-1}$ where $ds_{n-1}$ is the standard volume form on the unit sphere in $\mathbb{R}^n$. Then for $f(x) \in \Phi_{-\lambda-n}(\mathbb{R}P^{n-1})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x)\, e^{2\pi i \langle \xi, x \rangle} d^n x \\
&\quad = \frac{1}{2} \int_{-\infty}^{\infty} |r|^{-\lambda-n}\, e^{2\pi i r \cdot \langle \xi, s \rangle} r^{n-1}\, dr \int_{S^{n-1}} f(e)\, ds_{n-1} \\
&\quad = \frac{1}{2} \pi^{\lambda+1/2} \Gamma\left(\frac{-\lambda}{2}\right) \int_{S^{n-1}} f(e) \frac{|\langle \xi, s \rangle|^{\lambda}}{\Gamma\left(\dfrac{\lambda+1}{2}\right)} ds_{n-1} \\
&\quad = \pi^{-\lambda-1/2} \Gamma\left(\frac{-\lambda}{2}\right) I_\lambda^+ f.
\end{aligned}$$

The proof in the case of odd functions is completely similar.

COROLLARY 6.5.

$$I^+_{-\lambda-n} \circ I^+_\lambda = \frac{\pi^n}{\Gamma\left(\dfrac{-\lambda}{2}\right)\Gamma\left(\dfrac{\lambda+n}{2}\right)} \cdot Id$$

$$I^-_{-\lambda-n} \circ I^-_\lambda = \frac{\pi^n}{\Gamma\left(\dfrac{-\lambda}{2}\right)\Gamma\left(\dfrac{\lambda+n}{2}\right)} \cdot Id.$$

In particular, using (48) we see that $I_{-1}$ is just a projectively invariant version of the Radon transform

$$(I_{-1} f)(\xi) = \int_{\gamma_m} f(x)\, \delta(\langle \xi, x \rangle)\, \sigma_{m+1}(x, dx),$$

and the inversion formula looks as follows. When $n$ is even

$$f(y)=\frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-2}}\int_{\gamma_{n-1}}\hat{f}(\xi)\,\delta^{(n-2)}(\langle\xi,y\rangle)\,\sigma_n(\xi,d\xi).$$

When $n$ is odd

$$f(y)=\frac{(-1)^{n/2-1}\,(n-2)!}{2(2\pi)^{n-1}}\int_{\gamma_{n-1}}\hat{f}(\xi)(\langle\xi,y\rangle)^{-n+1}\,\sigma_n(\xi,d\xi).$$

The operator $I^+_{-\lambda-n}$ is defined on the space of all homogeneous degree $\lambda$ functions. However, it is zero on the subspace of odd functions. The reason is this. A sphere in $\mathbb{R}^n\backslash 0$ representing the generator in $H_{n-1}(\mathbb{R}^n\backslash 0)$ has canonical coorientation "out of the origin." The involution $x\to -x$ preserves it. So it acts on the class $\gamma_{n-1}$ in the same way as it acts on the orientation class of $\mathbb{R}^n$ and hence on the form $\sigma_{n-1}(x,dx)$, by multiplication by $(-1)^n$. So if $f(x)$ is an odd function the integral $\int_{\gamma_{n-1}} f(x)\,\sigma_n(\xi,d\xi)$ vanishes because the contributions of the opposite parts of the sphere cancel each other.

From our point of view these results look as follows: Let

$$L_\lambda:=\sum_{i=1}^{n} x_i\partial_{x_i}-\lambda$$

be the Euler operator. Denote the corresponding $\mathscr{D}$-module by $\mathscr{L}_\lambda$. Then $\Phi_\lambda(\mathbb{R}P^{n-1})$ is the space of smooth even solutions of $\mathscr{L}_\lambda$.

It follows from Lemma 4.1 that $\star\mathscr{L}_\lambda=\mathscr{L}_{-\lambda-n}[1]$ and the Green class of $\mathscr{L}_\lambda$ is

$$G_{\mathscr{L}_\lambda}(\varphi;v)=\varphi\cdot v\cdot\sigma_n(x,dx).$$

So pairing (46) looks in this case as follows:

$$\Phi_\lambda(\mathbb{R}P^{n-1})\otimes\Phi_{n-\lambda}(\mathbb{R}P^{n-1})\to\mathbb{R}$$

$$f(x)\otimes g(x)\mapsto\int_{\gamma_{n-1}} f(x)\,g(x)\,\sigma_n(x,dx).$$

Note that

$$K^{\pm}_\lambda(x,\xi)\in \operatorname{Sol}(\mathscr{L}_\lambda\boxtimes\mathscr{L}_\lambda,\mathscr{D}'(\mathbb{R}^n\backslash 0\times\mathbb{R}^n\backslash 0))^{\pm}. \tag{51}$$

One has $\mathcal{L}_\lambda = \tilde{\bigstar}\mathcal{L}_{-\lambda-n}$, so the integral transformation $I_\lambda^\pm$ is just the natural linear map provided by the kernel (51).

3. *The Complex Space*

Let $\lambda$ and $\mu$ be complex numbers such that $n := \lambda - \mu$ is an integer. Let

$$\Phi_{\lambda,\mu}(\mathbb{C}P^m) := \{f \mid f(az, \bar{a}\bar{z}) = a^\lambda \bar{a}^\mu f(z, \bar{z})\}$$

be the space of smooth homogeneous function in $\mathbb{C}^{m+1}\backslash 0$ of the bidegree $\lambda, \mu$. Consider the kernel

$$K_\lambda^{\mathbb{C}}(x, \xi) = \frac{\langle \xi, x \rangle^\lambda \cdot \langle \bar{\xi}, \bar{x} \rangle^\mu}{\Gamma\left(\dfrac{s + |n| + 2}{2}\right)}$$

where $s = \lambda + \mu$. It is a homogeneous generalized function. It defines an integral transformation

$$I_{\lambda,\mu}^{\mathbb{C}} \colon \Phi_{-\lambda-m-1,\,-\mu-m-1}(\mathbb{C}P^m) \to \Phi_{\lambda,\mu}((\mathbb{C}P^m)')$$

$$f(z, \bar{z}) \to \int_{\mathbb{C}P^m} f(z, \bar{z})\, K_\lambda^{\mathbb{C}}(z, \xi)\, \sigma_m(z, dz) \wedge \sigma_m(\bar{z}, d\bar{z}).$$

Here the integral has the following meaning. The form we integrate can be pushed down to $\mathbb{C}P^m$, so we integrate over the fundamental cycle. One has (see [GGV])

$$K_\lambda^{\mathbb{C}}(z, \xi)|_{\lambda=-k-1,\,\mu=-l-1} = \frac{\pi(-1)^{k+l+1}\, j!}{k!\, l!}\, \delta^{k,l}(z, \bar{z})$$

where $j = \min(k, l)$. In particular, applying the above results to the case $k=0$, $l=0$ we come to the Radon transform of smooth homogeneous functions of degree $(-m, -m)$ in $\mathbb{C}P^m$:

$$I_m^{\mathbb{C}} \colon \Phi_{-m,\,-m}(\mathbb{C}P^m) \to \Phi_{-1,\,-1}((\mathbb{C}P^m)')$$

$$(I_m^{\mathbb{C}} f)(\xi) = \hat{f}(\xi) = (i/2)^m \int_{\mathbb{C}P^m} f(x)\, \delta(\langle \xi, x \rangle)\, \sigma_m(x, dx) \wedge \sigma_m(\bar{x}, d\bar{x}).$$

The projectively invariant inversion formula is

$$(J_m^{\mathbb{C}} f)(y) = c_m^{\mathbb{C}} \int_{\mathbb{C}P^m} \hat{f}(\xi)\, \delta^{(m-1,\, m-1)}(\langle \xi, y \rangle)\, \sigma_m(\xi, d\xi) \wedge \sigma_m(\bar{\xi}, d\bar{\xi})$$

where $c_m^{\mathbb{C}} = (-1)^{m-1}\,(m-1)!\,(\pi)^{-2m+2}\,(i/2)^m$.

## 7. INTEGRAL GEOMETRY ON THE FAMILY OF SPHERES

### 1. *The Integral Transformation*

Let

$$S^m = \{x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2 = 0\}/\mathbb{R}^*$$

be a sphere in $\mathbb{R}P^{m+1}$. The stereographic projection identifies the family of its hyperplane sections with the family of all spheres in $\mathbb{R}^m$.

Let $Q_{m+1} := \{x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2 = 0\}$ be a cone in $\mathbb{R}^{m+2} \backslash 0$. It has two connected components, $Q_{m+1}^+$ in the half space $x_{m+2} > 0$ and its opposite $Q_{m+1}^-$.

Denote by $\Phi_\lambda^+(S^m)$ the space of all smooth homogeneous functions of degree $\lambda$ on the cone $Q_{m+1}^+$. Let $SO(m+1, 1)_0$ be the connected component of the unity of the group $O(m+1, 1)$. It acts on $Q_{m+1}^+$.

Let $\beta_m$ be a hyperplane section of $Q_{m+1}^+$ which is isomorphic to a sphere. The orientation of $\mathbb{R}^{m+2}$ provides an orientation of $\beta_m$: the cycle $\beta_m$ is cooriented out of the origin in the cone, and the cone itself is coorientated outside of the convex component in $\mathbb{R}^{m+2}$. Let $\beta_m^+$ be a cycle oriented in this way. Its homology class is a generator of $H_m(Q_{m+1}^+, \mathbb{Z})$.

LEMMA 7.1. *There is a nondegenerate $O(m+1, 1)$-invariant pairing*

$$\langle \cdot, \cdot \rangle_{S^m} : \Phi_{-\lambda-m}(S^m) \otimes \Phi_\lambda(S^m) \to \mathbb{R}$$

*defined by the formula*

$$\langle f, g \rangle_{S^m} := \int_{\beta_m^+} \delta(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2)\, f(x)\, g(x)\, \sigma_{m+2}(x, dx).$$

Here we integrate the closed $m$-form on $Q_{m+1}^+$. By definition it is the restriction to $Q_{m+1}^+$ of any form $\alpha_m$ satisfying the condition

$$d(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2) \wedge \alpha_m = f(x)\, g(x)\, \sigma_{m+2}(x, dx).$$

The restriction is well defined on $Q_{m+1}^+$.

*Proof.* The $SO(m+1, 1)_0$-invariance is obvious.

Let $\xi_1, ..., \xi_{m+2}$ be coordinates in $(\mathbb{R}^{m+2})'$ dual to $x_i$ and $\langle \xi, x \rangle = \sum \xi_i x_i$. Consider the kernel

$$K_\lambda^+(\xi, x) := \frac{|\langle \xi, x \rangle|^\lambda}{\Gamma\left(\frac{\lambda+1}{2}\right)}. \tag{52}$$

Set

$$\Delta := \partial_{\xi_1}^2 + \cdots + \partial_{\xi_{m+1}}^2 - \partial_{\xi_{m+2}}^2; \qquad L_\lambda := \sum_{i=1}^{m+2} \xi_i \partial_{\xi_i} - \lambda.$$

Let us denote by $\mathscr{M}_\lambda$ the $\mathscr{D}$-module on $\mathbb{R}^{m+2}$ corresponding to the system

$$\mathscr{M}_\lambda\colon \quad L_\lambda f = 0, \qquad (x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2)\, f = 0,$$

and by $\mathscr{N}_\lambda$ the $\mathscr{D}$-module on $(\mathbb{R}^{m+2})'$ corresponding to the system of differential equations

$$\mathscr{N}_\lambda\colon \quad L_\lambda \varphi = 0, \quad \Delta\varphi = 0.$$

Then

$$K_\lambda^+(\xi, x) \in \mathrm{Sol}(\mathscr{M}_\lambda \boxtimes \mathscr{N}_\lambda, D'(\mathbb{R}^{m+2} \times (\mathbb{R}^{m+2})'))^+$$

is an even solution of this system. Note that $\mathscr{M}_\lambda = \tilde{\bigstar} \mathscr{M}_{-\lambda-m}$. So the kernel $K_\lambda^+(\xi, x)$ defines an operator

$$I_\lambda^+ : \Phi_{-\lambda-m}(S^m) \to \mathrm{Sol}(\mathscr{N}_\lambda)^+;$$

$$(I_\lambda^+ f)(\xi) = \int_{\beta_m^+} \delta(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2)\, f(x)\, K_\lambda^+(\xi, x)\, \sigma_{m+2}(x, dx). \tag{53}$$

Consider the following domain:

$$\tilde{\Gamma}_0 := \{\xi \mid \xi_1^2 + \cdots + \xi_{m+1}^2 - \xi_{m+2}^2 = 0\}$$

$$\tilde{\Gamma}_1 := \{\xi \mid \xi_1^2 + \cdots + \xi_{m+1}^2 > \xi_{m+2}^2\}.$$

*Remark.* The functions $I_\lambda^\pm f(\xi)$ are *a priori* smooth only in the complement to the cone $\tilde{\Gamma}_0$. Indeed, the integral transform $I_\lambda^+$, for instance, is written in affine coordinates as

$$I_\lambda^+(f)(\xi) = \int f(x_1, ..., x_{m+1})\, \delta(x_1^2 + \cdots + x_{m+1}^2 - 1)\, \frac{|\langle \xi', x\rangle + s|^\lambda}{\Gamma\left(\frac{\lambda+1}{2}\right)}\, d^{m+1}x$$

where $\xi = (\xi', s)$ and $\langle \xi', x\rangle = \sum \xi_i x_i$. Set $\xi_1 = 1$, $\xi_i = 0$ for $i > 1$. Then

$$I_\lambda^+(f)(1, 0, ..., 0; s) = \int \tilde{f}(x_1)\, \frac{|x_1 + s|^\lambda}{\Gamma\left(\frac{\lambda+1}{2}\right)}\, dx_1$$

where $\tilde{f}(x_1) := \int f(x)\, \delta(x_1^2 + \cdots + x_{m+1}^2 - 1)\, dx_2 \cdots dx_{m+1}$. The function $\tilde{f}(x_1)$ vanishes outside of the segment $[-1, 1]$, is smooth inside of it, but not smooth near $x_1 = \pm 1$. The integral $\int |x|^\lambda f(x)\, dx$ is regularized near $x = 0$ in the assumption that the function $f(x)$ is smooth near zero.

Similarly, the kernel

$$K_\lambda^-(\xi, x) := \frac{|\langle \xi, x\rangle|^\lambda \cdot \operatorname{sgn}(\langle \xi, x\rangle)}{\Gamma\left(\frac{\lambda+2}{2}\right)}$$

is an odd solution of the system $\mathscr{M}_\lambda \boxtimes \mathscr{N}_\lambda$. It defines an intertwiner operator for the group $O(m+1, 1)$:

$$I_\lambda^- : \Phi_{-\lambda-m}(S^m) \to \operatorname{Sol}(\mathscr{N}_\lambda)^-;$$

Note that

$$\Phi_\lambda(S^m) = \operatorname{Sol}(\mathscr{M}_\lambda|_{\mathbb{R}^{m+2}\backslash 0}, C^\infty(\mathbb{R}^{m+2}\backslash 0))^+.$$

So the operators $I_\lambda^\pm$ are natural linear operators between smooth solution spaces.

*In this chapter we will work with the restriction of the functions $I_\lambda^\pm$ to the domain $\tilde{\Gamma}_1$.* Our first goal is to invert the operator

$$I_\lambda^\pm : \Phi_{-\lambda-m}(S^m) \to \operatorname{Sol}(\mathscr{N}_\lambda|_{\tilde{\Gamma}_1})^\pm.$$

## 2. *The Green Class*

Now we take the crucial step. Consider the following $m$-form:

$$\omega_m(\varphi; v) := \sum_{i \leqslant i < j \leqslant m+2} (-1)^{i+j-1} \times (\xi_i \cdot \varepsilon_j (v \cdot \varphi'_{\xi_j} - v'_{\xi_j} \cdot \varphi) - \xi_j \cdot \varepsilon_i (v \cdot \varphi'_{\xi_i} - v'_{\xi_i} \cdot \varphi)) \times d\xi_1 \wedge \cdots \hat{d\xi_i} \cdots \hat{d\xi_j} \cdots \wedge d\xi_{m+2} \tag{54}$$

Here $\varepsilon_{m+2} = -1$ and $\varepsilon_j = 1$ if $j \neq m+2$. Let $\omega_{m+1}(\varphi; \Delta; v)$ be the Green form for the Laplacian $\Delta$:

$$\omega_{m+1}(\varphi; \Delta; v) = \sum_{1 \leqslant j \leqslant m+2} (-1)^{j-1} \varepsilon_j (\varphi_{\xi_j} \cdot v - \varphi \cdot v_{\xi_j}) \times d\xi_1 \wedge \cdots \hat{d\xi_j} \cdots \wedge d\xi_{m+2}. \tag{55}$$

Then (54) is the contraction of the Green form (55) with the Euler vector field $L$:

$$\omega_m(\varphi; v) = -\tfrac{1}{2} i_L \omega_{m+1}(\varphi; \Delta; v).$$

*Remark.* More generally, for any homogeneous differential operator $P$ with constant coefficients in $\mathbb{R}^n$ the Green form for the system $Pf = 0$, $L_a f = 0$ is equal to $-\frac{1}{2} i_L \omega_{n-1}(\varphi; \Delta; v)$.

It can be also written as follows:

$$\omega_m(\varphi; v) = \left[ \xi v, \varepsilon \cdot \frac{\partial}{\partial \xi} \varphi, d\xi, ..., d\xi \right] - \left[ \xi \varphi, \varepsilon \cdot \frac{\partial}{\partial \xi} v, d\xi, ..., d\xi \right].$$

Here $[\xi v, \varepsilon \cdot (\partial / \partial \xi) \varphi, d\xi, ..., d\xi]$ means the determinant of the following matrix:

$$\begin{pmatrix} \xi_1 v & \varepsilon_1 \cdot \varphi'_{\xi_1} & d\xi_1 & \cdots & d\xi_1 \\ \xi_2 v & \varepsilon_2 \cdot \varphi'_{\xi_2} & d\xi_2 & \cdots & d\xi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_{m+2} v & \varepsilon_{m+2} \cdot \varphi'_{\xi_{m+2}} & d\xi_{m+2} & \cdots & d\xi_{m+2} \end{pmatrix}.$$

LEMMA 7.2. *The form $\omega_m(\varphi; v)$ can be pushed down to $\Gamma$.*

*Proof.* This is an easy calculation.

THEOREM 7.3. (a) $\mathcal{N}_a = \tilde{\bigstar} \mathcal{N}_b$ *where* $a+b+m=0$.

(b) *The form* $\omega_m(\varphi; v)$ *represents the Green class* $G_{\mathcal{N}_a}(\varphi; v)$ *of the system* $\mathcal{N}_a$.

COROLLARY 7.4. *The form* $\omega_m(\varphi; v)$ *is closed if the functions* $\varphi$ *and* $v$ *satisfy the following systems of differential equations*:

$$L_a \varphi = 0, \quad \Delta\varphi = 0 \quad \textit{and} \quad L_b v = 0, \quad \Delta v = 0 \qquad \textit{where} \quad a+b+m=0.$$

In the rest of this Section we will use this corollary extensively (but not the fact that $\omega_m(\varphi; v)$ represents the Green class). So let me first give a straightforward proof independent of the proof of Theorem 7.3.

*Proof.* One has

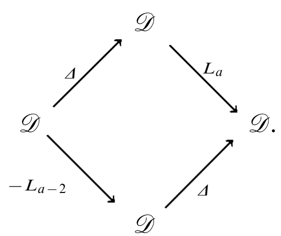
$$d\omega_m(v; \varphi) = \sum_{1 \leqslant j \leqslant m+2} (-1)^{j-1} \left(\xi_j (v \cdot \Delta\varphi - \Delta v \cdot \varphi)\right.$$
$$\left. - \varepsilon_j \cdot L_{-m-1}(v \cdot \varphi'_{\xi_j} - v'_{\xi_j} \cdot \varphi)\right) d\xi_1 \wedge \cdots \wedge \hat{d\xi_j} \wedge \cdots \wedge d\xi_{m+2}.$$

Indeed, by applying $(\partial/\partial\xi_i)\, d\xi_j$ to $\omega_{n-2}(v; \varphi)$ we get

$$\sum_{1 \leqslant i < j \leqslant m+2} (-1)^{j-1} \left( \xi_j (v \cdot \varphi''_{\xi_i \xi_i} - v''_{\xi_i \xi_i} \cdot \varphi) - \xi_i \frac{\partial}{\partial \xi_i} \varepsilon_j \cdot (v \cdot \varphi'_{\xi_j} - \varphi \cdot \varphi'_{\xi_j}) \right.$$
$$\left. - (j-1)\, \varepsilon_j \cdot (v \cdot \varphi'_{\xi_j} - \varphi \cdot v'_{\xi_j}) \right) d\xi_1 \wedge \cdots \wedge \hat{d\xi_j} \wedge \cdots \wedge d\xi_{m+2}.$$

Similarly, we compute the contribution of $(\partial/\partial\xi_j)\, d\xi_j$ and take the sum.

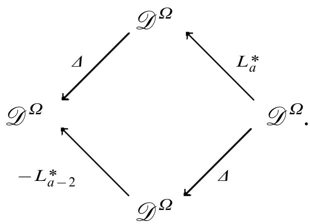
*Proof of Theorem* 7.3. Consider a complex of $\mathcal{D}$-modules $\mathcal{D} \xrightarrow{d} \mathcal{D}^2 \xrightarrow{d} \mathcal{D}$ sitting in degrees $[-2, 0]$ ($d$ has degree $+1$) which we visualize as:



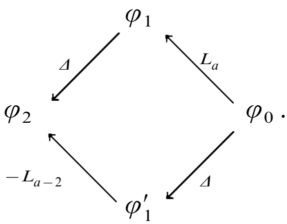

One has $[L_a, \Delta] = -2\Delta$, so $\Delta L_{-1} + (-L_{-3})\, \Delta = 0$; i.e., we get a complex.

This is a resolution of the $\mathscr{D}$-module $\mathscr{N}_a$. Indeed, consider a filtration on $\mathscr{D}$ such that the degree of $x$ and $\partial/\partial x$ is $+1$. Then both $L_a$ and $\Delta$ have degree $+2$. By shifting the filtration in the second term of the resolution down by 2 and in the third down by 4 we get a filtered complex. The associated graded quotient complex is a Koszul resolution. So our complex is also a resolution. The part a) follows easily from this.

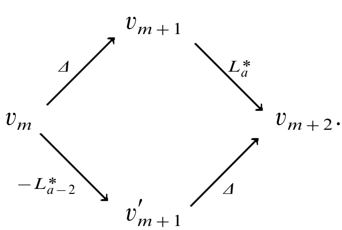
To calculate the Green class we use theorem (4.3) for this resolution. The complex $\bigstar\mathscr{P}^\bullet := \mathrm{Hom}_{\mathscr{D}}(\mathscr{P}^\bullet, \mathscr{D}^\Omega)[m+2]$ is concentrated in degrees $[-(m+2), -m]$ and looks as follows:




A homomorphism of $\mathscr{D}$-modules $\mathscr{D}_X \to C^\infty(X)$ is determined by its value at $1 \in \mathscr{D}_X$. So one can represent the complex $\mathrm{Hom}_{\mathscr{D}}(\mathscr{P}^\bullet, C^\infty(\mathbb{R}^{m+2}))$ by the following picture where $\varphi_0, \varphi_1, \varphi_1', \varphi_2$ are the values of the corresponding homomorphisms at 1:




Similarly, one can draw a picture for $\mathrm{Hom}_{\mathscr{D}}(\bigstar\mathscr{P}^\bullet, C^\infty(\mathbb{R}^{m+2}))$:



Recall that $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{\Omega}, C^{\infty}(X)) = \mathscr{A}^m(X)$. So $v$'s are the forms of top degree. Then

$$(L_a \varphi_0 \cdot v_{m+1} - \varphi_0 \cdot L_a^t v_{m+1}) + (\Delta\varphi_0 \cdot v'_{m+1} - \varphi_0 \cdot \Delta v'_{m+1}) \\ = d\omega_{m+1}(\varphi_0 ; v_{m+1}, v'_{m+1})$$

and

$$(\Delta\varphi_1 \cdot v_m - \varphi_1 \cdot \Delta v_m) + ((-L_{a-2})\, \varphi'_1 \cdot v_m - \varphi'_1 \cdot (-L_{a-2})^t\, v_m) \\ = d\omega_{m+1}(\varphi_1, \varphi'_1 ; v_m)$$

where

$$\omega_{m+1}(\varphi_0 ; v_{m+1}, v'_{m+1}) := \varphi_0 \cdot v_{m+1} \sigma_{m+2}(\xi, d\xi) \\ + \sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial \varphi_0}{\partial \xi_i} \cdot v'_{m+1} - \varphi_0 \cdot \frac{\partial v'_{m+1}}{\partial \xi_i} \right) (-1)^{i-1} \\ \times d\xi_1 \wedge \cdots \wedge \hat{d\xi_i} \wedge \cdots \wedge d\xi_{m+2}$$

and

$$\omega_{m+1}(\varphi_1, \varphi'_1 ; v_m) := -\varphi'_1 \cdot v_m \sigma_{m+2}(\xi, d\xi) \\ + \sum_{i=1}^{m+2} \varepsilon_i \left( \frac{\partial \varphi_1}{\partial \xi_i} v_m - \varphi_1 \frac{\partial v_m}{\partial \xi_i} \right) (-1)^{i-1} \\ \times d\xi_1 \wedge \cdots \wedge \hat{d\xi_i} \wedge \cdots \wedge d\xi_{m+2}.$$

Further,

$$\omega_{m+1}(\varphi_0 ; \Delta v_m, -L_{a-2}^t v_m) + \omega_{m+1}(L_a \varphi_0, \Delta\varphi_0 ; v_m) = d\omega_m(\varphi_0 ; v_m)$$

where $\omega_m(\varphi; v)$ is the Green form (54).

3. *Construction of the Inverse Operator*

We have defined in Section 7.1 the domain $\tilde{\Gamma}_1 = \{\xi_1^2 + \cdots + \xi_{m+1}^2 > \xi_{m+2}^2\}$. Let $\Gamma_1 = \tilde{\Gamma}_1/\mathbb{R}_+^*$ be the manifold of all oriented rays inside $\tilde{\Gamma}_1$. Its closure $\Gamma = \Gamma_0 \cup \Gamma_1$ parametrizes *oriented* hyperplane sections of the sphere $S^m$ (here $\Gamma_0 = \tilde{\Gamma}_0/\mathbb{R}_+^*$).

$\Gamma = S^{m+1} \backslash (\mathscr{D}_+ \cup \mathscr{D}_-)$ where $\mathscr{D}_+$ is a ball $\{\xi_1^2 + \cdots + \xi_{m+1}^2 < \xi_{m+2}^2\}/(\mathbb{R}^*)^+$ and $\mathscr{D}_- = -\mathscr{D}_+$. Therefore $H_m(\Gamma, \mathbb{Z}) = \mathbb{Z}$. Consider the cycle $\gamma_m$ of rays in the hyperplane $\xi_{m+2} = 0$. It is cooriented by the function $\xi_{m+2}$ (or, more invariantly, by the choice of one of the balls $\mathscr{D}_+$). So orientation of

$\mathbb{R}^{m+2}$ provides an orientation of the cycle. Denote by $\gamma_m^+$ the cycle oriented in this way. Its homology class is a generator of $H_m(\Gamma, \mathbb{Z})$.

There is a nondegenerate pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{N}_\lambda}: \quad \mathrm{Sol}(\mathcal{N}_\lambda)^+ \otimes \mathrm{Sol}(\mathcal{N}_{-\lambda-m})^- \to \mathbb{R};$$

$$\langle \varphi, v \rangle_{\mathcal{N}_\lambda} := \int_{\gamma_m^+} \omega_m(\varphi, m).$$

*Remark.* This pairing would have been zero if $\varphi$ and $v$ have the same parity. Indeed, in this case the involution $\xi \mapsto -\xi$ multiplies the form $\omega_m(\varphi, v)$ by $(-1)^{m+2}$ and the cycle $\gamma_m$ by $(-1)^{m+1}$, so the contributions to the integral coming from the antipodal parts of the cycle are cancelled.

Let $K$ be a compact hypersurface in $\Gamma$. Its homology class $[K] \in H_n(\Gamma)$ is equal to $d(K) \cdot [\gamma_M^+]$. The integer $d(K)$ is the intersection number of the class $[K]$ with the cycle consisting of all oriented spheres passing through a given point $x \in S^m$ and tangent to a given hyperplane in $T_x S^m$.

According to part (a) of Theorem 7.3 $\tilde{\star}\mathcal{N}_\lambda = \mathcal{N}_{-m-\lambda}$. So by our general philosophy the kernel $K_\lambda^-(\xi, x)$ defines integral operators

$$\begin{gathered} J_\lambda^+ : \mathrm{Sol}_{C^\infty}(\mathcal{N}_{-\lambda-m})^+ \to \Phi_\lambda(S^m) \\ (J_\lambda^+ \varphi)(\xi) = \tfrac{1}{2} \int_K \omega_m(\varphi; K_\lambda^-(\xi, x)), \end{gathered} \tag{56}$$

and similarly the even kernel $K_\lambda^+(\xi, x)$ provides the operators

$$J_\lambda^- : \mathrm{Sol}_{C^\infty}(\mathcal{N}_{-\lambda-m})^- \to \Phi_\lambda(S^m).$$

Note that $J_\lambda^+$ is defined by an odd kernel and $J_\lambda^-$ by an even kernel.

Further, there are operators

$$(J_{-\lambda-m}^\pm)^t : \Phi_\lambda(S^m) \to \mathrm{Sol}_{C^\infty}(\mathcal{N}_{-\lambda-m})^\mp$$

$$(J_{-\lambda-m}^\pm)^t\,(g) = \int_{\beta_m^+} \delta(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2)\, g(x)\, K_\lambda^\mp(x, \xi)\, \sigma_{m+2}(x, dx).$$

THEOREM 7.5. (a) *These operators are intertwiners for the group* $SO(m+1, 1)_0$.

(b) *For any m-cycle* $K \in \Gamma$ *one has*

$$d(K) \cdot \langle f, g \rangle_{S^m} = c(\lambda) \cdot \int_K \omega_m(I_\lambda^\pm f; (J_{-\lambda-m}^\pm)^t\, g)$$

*where*

$$c(\lambda)=\frac{\pi^{m+1}}{\Gamma\left(\frac{-\lambda}{2}\right)\Gamma\left(\frac{\lambda+m+2}{2}\right)}.$$

(c) *In particular,* $d(K)\cdot J^{\pm}_{-\lambda-m}\circ I^{\pm}_{\lambda}=c(\lambda)\cdot Id.$

The part (b) can be viewed as the universal form of the Plancherel theorem for the integral transformation $I^{\pm}_{\lambda}$.

*Proof.* (a) The operator $J^{\pm}_{\lambda}$ is an intertwiner for the following three reasons.

1. A group element $g\in \mathrm{SO}(m+1,1)_0$ sends form $\omega_m(\varphi, v)$ to the form $\omega_m(g\cdot\varphi, g\cdot v)$. Indeed, the form $\omega_m$ is a cocycle representing the Green class for the system $\mathcal{N}_\lambda$. This system as well as the volume form in $\mathbb{R}^{m+2}$ is invariant under the action of the group $\mathrm{SO}(m+1,1)_0$.

2. A *connected* Lie group acts trivially on the homology.

In the definition of the inverse operator $J^{-}_{\lambda}$ we can integrate over an $m$-cycle $\tilde{K}\subset(\mathbb{R}^{m+2})'$ projecting to $K$. So $J^{\pm}_{\lambda}$ *a priori* is defined for any smooth function $\varphi(\xi)$. However, it commutes with the group action only on the subspace $\mathrm{Sol}(\mathcal{N}_\lambda, C^\infty(\mathbb{R}^{m+2}))$. Indeed, $g$ moves the cycle $\tilde{K}$ to a different cycle $g\tilde{K}$ *homologous to the initial one*. To compare the integrals we use the Stokes formula for the form $\omega_m(\varphi; K_\lambda(\xi, x))$. The integrals will be the same only if the form is closed. This happens only if $\varphi(\xi)\in \mathrm{Sol}_{C^\infty}(\mathcal{N}_\lambda)$.

(b) Let $n=(0:\cdots:0:1:1)$ be the "North pole" in $S^m$. The variety $\Gamma_n$ parametrizing the hyperplane sections of the sphere $S^m$ passing through the point $n$ is a hyperplane given by the equation $\xi_{m+1}+\xi_{m+2}=0$.

It is sufficient to prove these formulas for one cycle $K$. Let $\pi_n\colon(x_1, ..., x_{m+2})\to(x_1, ..., x_m, x_{m+1}-x_{m+2})$ be the projection along the line $n$. Set $\tilde{x}:=(x_1, ..., x_m)$, $v:=x_{m+1}-x_{m+2}$. Assuming that $f\in\Phi_\lambda(S^m)$ vanishes near the line $n$. Then $\pi_n$ identifies $f|_{Q^+_{m+2}}$ with a function $\varphi(\tilde{x}, v):=f(\tilde{x}, v, -(x_1^2+\cdots+x_m^2)/v)$ on the hyperplane $x_{m+2}=0$, which vanish at $v\leqslant\varepsilon$, $\varepsilon>0$. Let $\alpha^+_m:=\pi_n(\gamma^+_m)$. The restriction of $I^+_\lambda f(\xi)$ to $\xi_{m+1}+\xi_{m+2}=0$ can be written as

$$(-1)^m\int_{\alpha^+_m}\frac{\varphi(\tilde{x}, v)}{v}\,\frac{|\tilde{\xi}\tilde{x}+v\cdot(\xi_{m+1}-\xi_{m+2})/2|^{\lambda}}{\Gamma((\lambda+1)/2)}\,\sigma_{m+1}(\tilde{x}, v).$$

This and the following lemma show that for $K=\Gamma_n$ part (b) reduces to the Plancherel theorem and the inversion formula for the generalized Radon transform in the projective space (see Sections 6.2–6.3). Set $\xi'=(\xi_1, ..., \xi_{m+1})$.

LEMMA 7.6. *The restriction of the form $\omega_m(\varphi; v)$ to $\Gamma_n$ is equal to*

$$\omega_m(\varphi; v)|_{\Gamma_n} = (-1)^{m+1} \left(v \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})\, \varphi - \varphi \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})\, v\right) \times \sigma_{m+1}(\xi', d\xi').$$

Integrating by parts we get $2 \cdot (-1)^m\, \varphi \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})\, v\, \sigma_{m+1}(\xi', d\xi')$.

4. *An Example: The Radon Transform Along the Hyperplane Sections of a Sphere*

The generalized functions (52) have no poles on $\lambda$. One has

$$K^{+}_{-(2k+1)}(\xi, x) = \frac{(-1)^k\, k!}{(2k)!} \cdot \delta^{(2k)}(\langle \xi, x \rangle),$$

$$K^{-}_{-2k}(\xi, x) = \frac{(-1)^k\, (k-1)!}{(2k-1)!} \cdot \delta^{(2k-1)}(\langle \xi, x \rangle),$$

$$K^{+}_{-2k}(\xi, x) = \frac{(-1)^k\, 2^k \sqrt{\pi}}{(2k-1)!!} \langle \xi, x \rangle^{-2k},$$

$$K^{-}_{-(2k+1)}(\xi, x) = \frac{(-1)^k\, 2^k \sqrt{\pi}}{(2k-1)!!} \langle \xi, x \rangle^{-2k-1}.$$

So we get the following integral transformation: For $f \in \Phi_{1-m}(S^m)$ set

$$If(\xi) = \int_{\beta_m^+} \delta(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2)\, f(x)\, \delta(\langle \xi, x \rangle)\, \sigma_{m+2}(x, dx).$$

The function $If(x)$ is zero outside $\Gamma$. Consider the following kernel:

$$K_{-(m-1)}(\xi, x) := \delta^{(m-2)}(\langle \xi, x \rangle) \quad \text{for odd } m$$

and

$$\langle \xi, x \rangle^{-(m-1)} \quad \text{for even } m.$$

This defines an integral transformation acting on $g \in \Phi_{-1}(S^m)$:

$$(J^t)\, g(\xi) = \int_{\beta_m^+} \delta(x_1^2 + \cdots + x_{m+1}^2 - x_{m+2}^2) \times g(x)\, K_{-(m-1)}(\langle \xi, x \rangle)\, \sigma_{m+2}(x, dx).$$

THEOREM 7.7. (a) *For any $m$-cycle $K \in \Gamma$ one has*

$$d(K) \cdot \langle f, g \rangle_{S^m} = c_m \cdot \int_K \omega_m(If; J^t g)$$

*where* $-c_m = (-1)^{(m-1)/2}/(2\pi)^{m-1}$ *for odd* $m$ *and* $= (-1)^{m/2}\,(m-1)!/(2\pi)^m$ *for even* $m$.

(b) *In particular,*

$$d(K)\cdot f(x) = c_m \cdot \int_K \omega_m(If;\, K_{-(m-1)}(\xi, x)). \tag{57}$$

So the inversion formula is local for odd $m$ and nonlocal for even $m$.

Theorem 7.7 is a special case of Theorem 7.5.

The inverse operator $J$ provided by the kernel $K_{-(m-1)}(\xi, x)$ looks as follows:

$$\begin{aligned}(J\varphi)(x) := \int_K \sum_{1\leqslant i<j\leqslant n} &(-1)^{i+j-1}\,(\varphi(\xi)(\xi_i x_j - \xi_j x_i)\,\delta^{(m-1)}(\langle \xi, x\rangle)\\ &-(\xi_i \varphi'_{\xi_j} - \xi_j \varphi'_{\xi_i})\,\delta^{(m-2)}(\langle \xi, x\rangle))\\ &\times d\xi_1 \wedge \cdots \wedge \hat{d}\xi_i \wedge \cdots \wedge \hat{d}\xi_j \wedge \cdots \wedge d\xi_n\end{aligned}$$

for odd $m$ and

$$\begin{aligned}(J\varphi)(x) := \int_K \sum_{1\leqslant i<j\leqslant n} &(-1)^{i+j-1}\,(\varphi(\xi)(\xi_i x_j - \xi_j x_i)\langle \xi, x\rangle^{-m}\\ &-(\xi_i \varphi_{\xi_j} - \xi_j \varphi_{\xi_i})\langle \xi, x\rangle^{-(m-1)})\\ &\times d\xi_1 \wedge \cdots \wedge \hat{d}\xi_i \wedge \cdots \wedge \hat{d}\xi_j \wedge \cdots \wedge d\xi_n\end{aligned}$$

for even $m$.

## 5. *Admissible Families of Spheres*

Restricting the integral operator $I^{\pm}_{\lambda}$ to a family $K$ of oriented spheres we get an integral transformation

$$I^{\pm}_{\lambda, K}\colon \Phi^{\pm}_{\lambda}(S^m) \to \Psi^{\pm}_{-\lambda-m}(K).$$

*A priori* the restriction of the form $\omega_m(\varphi; v)$ to a hypersurface $K$ depends not only on the restriction of the functions $\varphi$ and $v$ on $K$, but also on their first derivatives in the direction normal to $K$. Therefore for general $K$ the right-hand side of (57) cannot be computed if we know only $I^{\pm}_{\lambda, K}(f)$. So it does not give an inversion formula for the integral transformation $I^{\pm}_{\lambda, K}$.

DEFINITION 7.8. A hypersurface $K \subset \Gamma$ is called admissible if the restriction of the form $\omega_m(\varphi; v)$ to $K$ depends only on the restrictions of smooth solutions $\varphi \in \mathrm{Sol}_{C^\infty}(\mathcal{N}_\lambda)$, $v \in \mathrm{Sol}_{C^\infty}(\mathcal{N}_\lambda)$ to $K$.

This means that there exists a bidifferential operator $\nu\colon C^\infty(K)^{\otimes 2} \to \mathcal{A}^m(K)$ such that for any $\varphi$, $v$ as above $\omega_m(\varphi; v)|_K = \nu(\varphi|_K, v|_K)$.

It is worth comparing this definition of admissibility with the one usually used in integral geometry; see [G3].

Let $C$ be a submanifold in $S^m$. Consider the family $\Gamma_C$ of oriented hyperplane sections of the sphere $S^m$ tangent to $C$. For example, if $C$ is a point then $\Gamma_C$ is the set of all sections passing through it and $d(\Gamma_C)=1$.

LEMMA 7.9. *For any $C \subset S^m$ the hypersurface $\Gamma_C$ is admissible.*

*Proof.* For $C=n$ this follows from Lemma 7.6. Indeed, the vector field $(\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})$ is tangent to the hyperplane $\Gamma_n$.

In general, we proceed as follows. The form $\omega_m(\varphi; v)$ is given by a bidifferential operator of first order (see (54)), so its restriction to $K$ is determined by the restriction of the functions $\varphi$ and $v$ to the first infinitesimal neighborhood of $K$. Let $\eta \in \Gamma_C$ and $t(\eta)$ be the tangency point of the hyperplane $\langle \eta, x \rangle = 0$ with $C$. Then the tangent space to $\Gamma_K$ at a point $t(\eta)$ coincides with $\Gamma_{t(\eta)}$.

## 6. *Inversion of the Integral Transform Related to an Admissible Family*

The restriction of the form $\omega_m(I^{\pm}_{\lambda}(f); K^{\mp}_{\lambda}(\xi, x))$ to $\Gamma_C$ depends only $I^{\pm}_{\lambda, \Gamma_C} f$. So one can expect the inversion formula

$$d(\Gamma_C)\, f(x) = c(\lambda) \cdot \int_{\Gamma_C} \omega_m(I^{\pm}_{\lambda}(f); K^{\mp}_{\lambda}(\xi, x))), \tag{58}$$

similar to (57). However, the cycle $\Gamma_C$ lies in the closure $\Gamma$ of $\Gamma_0$, while the function $I^{\pm}_{\lambda}(f)$ was well defined only inside of $\Gamma_0$. For the same reason the form $\omega_m(I^{\pm}_{\lambda}(f); K^{\mp}_{\lambda}(\xi, x))$ is closed only inside of $\Gamma_0$ (and outside of $\Gamma$). So it is *a priori* unclear whether the formula makes sense and if it is possible to use the Stokes theorem.

To avoid this trouble we consider the integral transformation $I^{\pm}_{\lambda, \Gamma_C}$ only on the subspace $C^{\infty}(S^m, C)$ of the functions vanishing in a very small neighborhood of the subvariety $C$ in $S^m$.

Let $\hat{C} \in \Gamma$ be the subvariety of spheres of radius zero with centers at points of $C$. Let $\Psi^{\pm}_{\lambda}(\Gamma_C; \hat{C})$ be the subspace of $\Psi^{\pm}_{\lambda}(\Gamma_C)$ consisting of functions smooth near $\hat{C}$. Then $I^{\pm}_{\lambda} f$ is smooth in a neighborhood of $\hat{C}$. So we get an integral transformation

$$I^{\pm}_{\lambda, \Gamma_C}\colon \Phi_{-\lambda-m}(S^m, C) \to \Psi^{\pm}_{\lambda}(\Gamma_C; \hat{C}).$$

Now we may apply the Stokes formula near $\hat{C}$. Assuming this, let us perturb the cycle $\Gamma_C$ near the boundary of $\Gamma$ by moving it a little bit inside of $\Gamma$. Geometrically, this means that we replace small spheres tangent to $C$ by small spheres close to them which are not tangent to $C$.

*Remarks.* (1) The cycle $K$ becomes homologous to 0 in the sphere $S^{m+1}$ parametrizing *all* oriented hyperplanes.

(2) One can deform smoothly the cycle $K$ out of the domain $\Gamma$. However, doing this we must cross *all* the points of the boundary $\Gamma_1$ of $\Gamma$. Therefore we *cannot* use the Stokes formula to compare

$$\int_K \omega_m(I_\lambda^\pm(f); K_\lambda^\mp(\xi, x))) \qquad \text{and} \qquad \int_{K'} \omega_m(I_\lambda^\pm(f); K_\lambda^\mp(\xi, x))),$$

where $K$ is inside $\Gamma_0$ and $K'$ is outside $\Gamma$. This is very natural: otherwise we would prove that they are equal, and so are equal to zero since the cycle $K'$ is homologous to zero in the complement to $\Gamma$.

So we can reduce the investigation of the integral to the study of a similar integral over a cycle $K$ inside $\Gamma_0$, which was done above. Therefore we come to the following conclusion:

THEOREM 7.10. *For an admissible family $\Gamma_C$ the operator $J_\lambda^\pm$ provides an operator*

$$J_{\lambda, \Gamma_C}^\pm : \Psi_\lambda^\pm(\Gamma_C, \hat{C}) \to \Phi_{-\lambda-m}(S^m, C)$$

*such that*

$$c(\lambda) \cdot I_{\lambda, \Gamma_C}^\pm \circ J_{\lambda, \Gamma_C}^\pm = d(K) \cdot Id.$$

7. *Geometry of the Family of Spheres*

The group $SO(m+1, 1)$ acts on the family of all spheres in $S^m$. A remarkable fact is that a bigger symmetry group, $SO(m+1, 2)$, acts as a group of contact transformations on the family of all spheres (including the points, which are spheres of zero radius!).

Namely, let

$$X_{m+1} := \{\eta_1^2 + \cdots + \eta_{m+1}^2 - \eta_{m+2}^2 - \eta_{m+3}^2 = 0\}/\mathbb{R}_+^*$$

be the $(m+1)$-dimensional quadric of signature $(m+1, 2)$. Its affine part $\eta_{m+3} \neq 0$ is isomorphic to the hyperboloid $\Gamma_0 = \{\xi_1^2 + \cdots + \xi_{m+1}^2 - \xi_{m+2}^2 = 1\}$. The complement to the affine part is the projectivization of the cone $\{\xi_1^2 + \cdots + \xi_{m+1}^2 - \xi_{m+2}^2 = 0\}$, i.e., it is a sphere $\{\xi_1^2 + \cdots + \xi_{m+1}^2 = 1\}$. The quadric $X_{m+1}$ parametrizes all oriented hyperplane sections of the sphere $S^m$. The hyperboloid $\Gamma_0$ parametrizes all oriented spheres of non zero radius.

Let $A \subset S^m \times X_{m+1}$ be the incidence subvariety. Consider the double bundle corresponding to this family and its symplectization:

$$\begin{array}{ccc} & A & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ S^m & & X_{m+1} \end{array} \qquad \begin{array}{ccc} & N_A^*(S^m \times X_{m+1}) & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ T^*S^m & & T^*X_{m+1} \end{array}$$

Let

$$\Sigma := \pi_2(N_A^*(S^m \times X_{m+1})) \subset T^*X_{m+1}.$$

Then $\Sigma_\xi := T_\xi^* X_{m+1} \cap \Sigma$ is a nondegenerate quadratic cone in the bundle cotangent to $\xi$. This cone is dual to the cone in the space tangent to the quadric at the point $\xi$ given by the intersection of the quadric with the hyperplane in the projective space tangent to the quadric at $\xi$.

The hypersurface $\Sigma$ is foliated on bicharacteristics curves. This foliation is invariant under the action of the multiplicative group $\mathbb{R}^*$ on $T^*X_{m+1}$.

LEMMA 7.11. (a) *Projection along the bicharacteristics gives the* $\mathbb{R}^*$*-equivariant fibration*

$$\pi_\Sigma\colon (\Sigma \backslash \{\textit{zero section}\}) \to (T^*S^m \backslash \{\textit{zero section}\}).$$

(b) *The projection of a bicharacteristic to* $X_{m+1}$ *consists of all spheres tangent to a given hyperplane at a given point.*

So the manifold of all bicharacteristics is identified with the projectivization of the bundle cotangent to $S^m$.

Geometrically, $P(\Sigma \backslash \{\text{zero section}\})$ is the set of all pairs

$$\{\text{a contact element } h \text{ at a point } x \in S^m, \text{ a sphere tangent to } h \text{ at } x\}.$$

The group $SO(m+1, 2)$ acts on $X_{m+1}$ and hence on $\Sigma$. Thanks to the lemma the group $SO(m+1, 2)$ acts as a group of homogeneous symplectomorphisms on $T^*S^m$. It preserves the family of homogeneous Lagrangian subvarieties given by the bundles conormal to spheres (including the spheres of zero radius).

## 8. *The Hamilton–Jacobi Method for Description of Admissible Families of Spheres*

A hypersurface $K' \subset X_{m+1}$ is characteristic if its conormal bundle in $X_{m+1}$ is contained in $\Sigma$; i.e., for any nonsingular $\xi \in K'$ the tangent plane $T_\xi K'$ is tangent to the "light cone" $\Sigma_\xi^* \subset T_\xi X_{m+1}$.

PROPOSITION 7.12. *An irreducible hypersurface $K' \subset X_{m+1}$ is admissible if and only if it is characteristic.*

*Proof.* We already proved in Lemma 7.9 that if $K'$ is characteristic then it is admissible. Let us prove the converse statement. Since $\omega_m(\varphi; v)$ is given by a bidifferential operator of order $(1, 1)$ it is enough to check that the restriction of the differential form $\omega_m(\varphi; v)$ to any noncharacteristic hyperplane does depend on the derivatives of $\varphi$ and $v$ in the direction transversal to this hyperplane. The group $SO(m+1, 1)$ acts transitively on the variety of noncharacteristic hyperplanes in the tangent spaces $T_\xi X_{m+1}$. So it is sufficient to check the statement above for the hyperplane $\xi_{m+2}=0$. One has

$$\omega_m(\varphi; v)|_{\xi_{m+2}=0} = \sum_{1 \leqslant i \leqslant m+1} (-1)^{i+m-1} \xi_i \times (v \cdot \varphi'_{\xi_{m+2}} - v'_{\xi_{m+2}} \cdot \varphi)\, d\xi_1 \wedge \cdots \widehat{d\xi_i} \cdots \wedge d\xi_{m+1}.$$

The proposition follows.

The following lemma is well known:

LEMMA 7.13. *Any algebraic irreducible homogeneous Lagrangian subvariety in $T^*X$ is isomorphic to the conormal bundle to an irreducible algebraic subvariety $Y \subset X$.*

THEOREM 7.14. *Any admissible hypersurface in $\Gamma$ is a piece of a hypersurface $\Gamma_C$ for a certain $C \subset S^m$.*

*Proof.* We may assume that $K'$ is irreducible. According to Proposition 7.12 $N^*_{K'} X_{m+1}$ is a Lagrangian subvariety in $\Sigma$, so $\pi_\Sigma$ projects it down to a Lagrangian subvariety in $T^*S^m$, which by the above lemma must have the form $N^*_C S^m$.

## 8. HOLONOMIC KERNELS AND THEIR COMPOSITION: THE BICATEGORY OF $\mathscr{D}$-MODULES

### 1. *Motivations*

As we emphasized before the composition of natural linear maps defined by distributional kernels does not always exist. However, when it is defined we come to the problem of the computation of the composition. Many important problems of analysis can be considered as special cases of this one. For instance, in integral geometry both the integral transformation and its inverse should be treated as natural linear maps between solution spaces of $\mathscr{D}$-modules, so to invert an integral transformation we should be able to compute the composition of natural linear maps.

Let us assume for a moment that $\mathscr{M}_i$ are excellent $\mathscr{D}$-modules. Then usually the natural kernels are *distributions satisfying the holonomic system of differential equations*. This means that the image of the homomorphism

$$\tilde{\bigstar}\mathscr{M}_1 \boxtimes \mathscr{M}_2 \to D'(X_1 \times X_2) \tag{59}$$

provided by the kernel

$$K_{12}(x_1, x_2) \in \mathrm{Hom}_{\mathscr{D}}(\tilde{\bigstar}\mathscr{M}_1 \boxtimes \mathscr{M}_2, D'(X_1 \times X_2))$$

is a holonomic $\mathscr{D}$-module. Let us denote it by $\mathscr{K}_{12}$ and by $\tilde{\bigstar}\mathscr{M}_1 \boxtimes \mathscr{M}_2 \xrightarrow{\alpha_{12}} \mathscr{K}_{12}$ denote the corresponding morphism of $\mathscr{D}$-modules. So (59) is a composition

$$\tilde{\bigstar}\mathscr{M}_1 \boxtimes \mathscr{M}_2 \xrightarrow{\alpha_{12}} \mathscr{K}_{12} \hookrightarrow D'(X_1 \times X_2).$$

The idea of keeping only the first arrow suggests the following definition:

DEFINITION 8.1. A holonomic kernel on $X_1 \times X_2$ is a collection $(\mathscr{M}_1, \mathscr{M}_2, \mathscr{K}_{12}; \alpha)$ where

$$\mathscr{M}_1 \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_1}), \qquad \mathscr{M}_2 \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_2}), \qquad \mathscr{K}_{12} \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_1 \times X_2}),$$

and

$$\alpha \in R\,\mathrm{Hom}_{\mathscr{D}_{X_1 \times X_2}}(\bigstar\mathscr{M}_1 \boxtimes \mathscr{M}_2, \mathscr{K}_{12}).$$

A holonomic kernel is a finer algebraic version of a holonomic distribution on $X_1 \times X_2$ than the $\mathscr{D}$-module which this distribution satisfies.

EXAMPLE. Suppose that $\mathscr{M}_i = \mathscr{D}_{X_i}$ for $i = 1, 2$. Then $\tilde{\bigstar}\mathscr{D}_{X_1} = \mathscr{D}_{X_1}$ and $\mathscr{D}_{X_1} \times \mathscr{D}_{X_2} = \mathscr{D}_{X_1 \times X_2}$. Morphisms of $\mathscr{D}$-modules $\mathscr{D}_{X_1 \times X_2} \to \mathscr{K}$ are defined by their value on the generating section 1 and correspond just to the *sections* of $\mathscr{K}_{12}$.

For instance, if $X_1 = X_2 = \mathbb{A}^1$ and $\mathscr{K}_{12}$ is the $\mathscr{D}$-module of the delta functions on the diagonal, the morphism above correspond to sections $f(x)\,\delta^{(k)}(x - y)$.

It seems that the notion of a bicategory is the appropriate language for discussing the holonomic kernels and their composition.

## 2. *Bicategories*

A complete definition of a (lax) bicategory is given in [Be] or in [KV, p. 200]. In particular, a notion of bicategory $\mathscr{C}$ includes the following data:

a set $\mathrm{Ob}\mathscr{C}$ of objects;

for any two objects a set of 1-morphisms from $A$ to $B$; and

for any two 1-morphisms $\alpha_1, \alpha_2$ between $A$ and $B$ a set of 2-morphisms between $\alpha_1$ and $\alpha_2$.

For any two objects $A_1$ and $A_2$ of a bicategory there is a category $\mathrm{Mor}_1(A_1, A_2)$ of 1-morphisms from $A_1$ to $A_2$. The objects in this category are 1-morphisms from $A_1$ to $A_2$; the morphisms between any two given 1-morphisms from $A_1$ to $A_2$ are given by the 2-morphisms between these 1-morphisms.

The composition of 1-morphisms provides a bifunctor

$$\mathrm{Mor}_1(A_1, A_2) \times \mathrm{Mor}_1(A_2, A_3) \to \mathrm{Mor}_1(A_1, A_3).$$

The archetypal example is the bicategory of all categories. Its objects are categories and for any two categories $\mathscr{A}$ and $\mathscr{B}$ the category $\mathrm{Mor}_1(\mathscr{A}, \mathscr{B})$ is the category of functors from $\mathscr{A}$ to $\mathscr{B}$.

## 3. *A Bicategory of $\mathscr{D}$-Modules*

Below we work in the derived category. In particular, all morphisms are morphisms in the derived category.

The objects of the bicategory are pairs $(X, \mathscr{M})$ where $X$ is an algebraic variety over a field $k$ (char $k = 0$) and $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$.

By definition, 1-morphisms between the two objects $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are holonomic kernels

$$\star\mathscr{M} \boxtimes \mathscr{N} \xrightarrow{\alpha} K.$$

It is the composition of 1-morphisms which makes the whole story relevant to integral geometry. Roughly speaking, it answers the question, "What system of differential equations satisfies the kernel of the composition of two natural maps?" motivated by Section 5.7 above.

Let $\Delta_2\colon X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$ be the diagonal embedding of $X_2$ and $\pi_2\colon X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_3$ be the projection. Consider the objects $(X_i, \mathscr{M}_i)$ where $i = 1, 2, 3$.

Definition 8.2. The composition of 1 morphisms

$$\star\mathscr{M}_1 \boxtimes \mathscr{M}_2 \xrightarrow{\alpha_{12}} \mathscr{K}_{12} \qquad \text{and} \quad \star\mathscr{M}_2 \boxtimes \mathscr{M}_3 \xrightarrow{\alpha_{23}} \mathscr{K}_{23}$$

is the 1-morphism

$$\star\mathscr{M}_1 \boxtimes \mathscr{M}_3 \xrightarrow{\alpha_{13}} \mathscr{K}_{13}$$

where

$$\mathcal{K}_{13} = \mathcal{K}_{12} \circ \mathcal{K}_{23} := \pi_{2_*} \Delta_2^!(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23})$$

and the morphism $\alpha_{13}$ is the composition of the morphisms $id \boxtimes G \boxtimes id$ and $\alpha_{12} \boxtimes \alpha_{23}$:

$$\begin{aligned}\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_3 &\xrightarrow{id \boxtimes G \boxtimes id} \pi_{2_*} \Delta_2^!(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \bigstar\mathcal{M}_2 \boxtimes \mathcal{M}_3) \\ &\xrightarrow{\alpha_{12} \boxtimes \alpha_{23}} \pi_{2_*} \Delta_2^!(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}).\end{aligned}$$

A 2-morphism between 1-morphisms

$$\bigstar\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_1} \mathcal{K}_1 \quad \text{and} \quad \bigstar\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_2} \mathcal{K}_2$$

is a morphism $\varphi_{12}: \mathcal{K}_1 \to \mathcal{K}_2$ making the following diagram commutative:

$$\begin{array}{ccccc} & & \bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_3 & & \\ & \swarrow^{\alpha_1} & & \searrow^{\alpha_2} & \\ \mathcal{K}_1 & & \xrightarrow{\quad\varphi_{12}\quad} & & \mathcal{K}_2\,. \end{array}$$

A 2-morphism between a holonomic kernel $\alpha'_{13}: \bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_3 \to \mathcal{K}'_{13}$ and the composition $\mathcal{K}_{12} \circ \mathcal{K}_{23}$ of holonomic kernels $\alpha_{12}: \bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_3 \to \mathcal{K}_{12}$ and $\alpha_{23}: \bigstar\mathcal{M}_2 \boxtimes \mathcal{M}_3 \to \mathcal{K}_{23}$ is provided by the following commutative diagram:

$$\begin{array}{ccc} \bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_3 & \xrightarrow{id \boxtimes G \boxtimes id} & R\pi_{2_*} \Delta_2^!(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \bigstar\mathcal{M}_2 \boxtimes \mathcal{M}_3) \\ \downarrow{\scriptstyle \alpha'_{13}} & & \downarrow{\scriptstyle R\pi_{2_*} \Delta_2^!(\alpha_{12} \boxtimes \alpha_{23})} \\ \mathcal{K}'_{13} & \longrightarrow & R\pi_{2_*} \Delta_2^!(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}) \end{array}$$

The composition of 2-morphisms is defined in an obvious way.

*The Identity* 1-*Morphism* $Id_{\mathcal{M}}$. For any $\mathcal{M} \in D^b_{\mathrm{coh}}(\mathcal{D}_X)$ there is a canonical morphism

$$I_{\mathcal{M}}: \bigstar\mathcal{M} \boxtimes \mathcal{M} \to \delta_\Delta[d_X]$$

corresponding via (15) to the identity map $Id \in \mathrm{Hom}_{\mathcal{D}_X}(\bigstar\mathcal{M}, \bigstar\mathcal{M})$.

We will say that the 1-morphism $\alpha_{23}$ is weakly inverse to the 1-morphism $\alpha_{12}$ if there is a 2-morphism from the identity 1-morphism $Id_{\mathcal{M}}$ to the composition of 1-morphisms $\alpha_{23} \circ \alpha_{12}$. This means that the following diagram is commutative:

$$\begin{array}{ccc} \bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_1 & \xrightarrow{id \boxtimes G \boxtimes id} & R\pi_{2_*} \Delta_2^!(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \bigstar\mathcal{M}_2 \boxtimes \mathcal{M}_1) \\ \downarrow{\scriptstyle I_{\mathcal{M}_1}} & & \downarrow{\scriptstyle R\pi_{2_*}\Delta_2^!(\alpha_{12} \boxtimes \alpha_{23})} \\ \delta_\Delta & \xrightarrow{\varphi} & R\pi_{2_*} \Delta_2^!(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}) \end{array}$$

*Remark.* These definitions make sense for any (not necessarily holonomic) $\mathcal{K}_{ij} \in D^b_{\text{coh}}(\mathcal{D}_X)$.

### 3. *On Composition of Holonomic Kernels*

Recall that for $\mathcal{M}, \mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_X)$ one has

$$\mathcal{M} \overset{!}{\otimes} \mathcal{N} := \Delta^!(\mathcal{M} \boxtimes \mathcal{N}) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}[-d_X]$$

where $\Delta: X \hookrightarrow X \times X$ is the diagonal embedding.

DEFINITION 8.3. Let $\mathcal{K} \in D^b_{\text{coh}}(\mathcal{D}_{X_1 \times X_2})$. Then it defines a functor

$$\bar{\mathcal{K}}: D^b_{\text{coh}}(\mathcal{D}_{X_1}) \to D^b_{\text{coh}}(\mathcal{D}_{X_2}) \qquad \bar{\mathcal{K}}(\mathcal{M}) := p_{2_*}(\mathcal{K} \overset{!}{\otimes} p_1^! \mathcal{M}).$$

This is motivated by the following proposition (compare with Section 8.2):

PROPOSITION 8.4.

$$R\,\mathrm{Hom}_{\mathcal{D}_{X_1 \times X_2}}(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{K}_{12}) = R\,\mathrm{Hom}_{\mathcal{D}_{X_2}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1)).$$

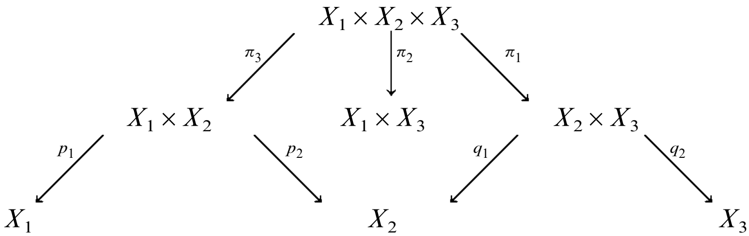
*Proof.* Let $p_i: X_1 \times X_2 \to X_i$ be natural projections. We have

$$\begin{aligned} &R\,\mathrm{Hom}_{\mathcal{D}}(\bigstar\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{K}_{12}) \\ &\quad = R\,\mathrm{Hom}_{\mathcal{D}}(p_1^!(\bigstar\mathcal{M}_1)[-d_{X_2}] \otimes_{\mathcal{O}} p_2^!(\mathcal{M}_2)[-d_{X_1}], \mathcal{K}_{12}) \\ &\quad = R\,\mathrm{Hom}_{\mathcal{D}}(p_2^!(\mathcal{M}_2)[-d_{X_1}], \bigstar p_1^! \bigstar(\mathcal{M}_1)[d_{X_2}] \otimes_{\mathcal{O}} \mathcal{K}_{12}) \\ &\quad = R\,\mathrm{Hom}_{\mathcal{D}}(p_2^*(\mathcal{M}_2)[d_{X_1}], \mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^!(\mathcal{M}_1)[-d_{X_2}]) \\ &\quad = R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_2, p_{2_*}(\mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^!(\mathcal{M}_1)[-d_{X_1} - d_{X_2}]) \\ &\quad = R\,\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1)). \end{aligned}$$

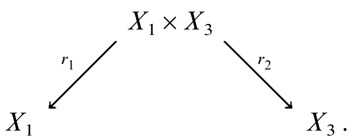
PROPOSITION 8.5. *There is a natural isomorphism of functors*:

$$\mathscr{K}_{23} \mathbin{\bar{\circ}} \mathscr{K}_{12} = \bar{\mathscr{K}}_{23} \circ \bar{\mathscr{K}}_{13}.$$

*Proof.* Consider the following diagrams:




and




Let $\Delta_2: X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$ be the diagonal imbedding of $X_2$.

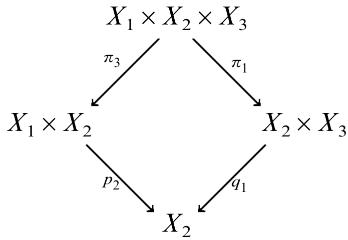
LEMMA 8.6. *Let* $\mathscr{K}_{12} \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_1 \times X_2})$ *and* $\mathscr{K}_{23} \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_2 \times X_3})$. *Then*

$$\mathscr{K}_{12} \circ \mathscr{K}_{23} = \pi_{2_*}(\pi_3^! \mathscr{K}_{12} \otimes_{\mathscr{O}} \pi_1^! \mathscr{K}_{23}).$$

*Proof.* It follows immediately from $\pi_3 = \tau_{12} \circ \Delta_2$ and $\pi_1 = \tau_{23} \circ \Delta_2$. One has

$$\begin{aligned}
K_{23}(K_{12}(\mathscr{M})) &= q_{3_*}(q_2^! p_{2_*}(p_1^! \mathscr{M}_1 \overset{!}{\otimes} K_{12}) \overset{!}{\otimes} K_{23}) \\
&= q_{3_*}(\pi_{1_*} \pi_3^! (p_1^! \mathscr{M}_1 \overset{!}{\otimes} K_{12}) \overset{!}{\otimes} K_{23}) \\
&= q_{3_*}(\pi_{1_*}(\pi_1^! \mathscr{M}_1 \overset{!}{\otimes} \pi_{12}^! K_{12}) \overset{!}{\otimes} K_{23}) \\
&\overset{1}{=} q_{3_*} \pi_{1_*}(\pi_1^! \mathscr{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) \\
&= \pi_{3_*}(\pi_1^! \mathscr{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) \\
&= r_{3_*} \pi_{2_*}(\pi_1^! \mathscr{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) \\
&= r_{3_*} \pi_{13_*}(\pi_{2!} r_1^! \mathscr{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) \\
&\overset{2}{=} r_{3_*}(r_1^! \mathscr{M}_1 \overset{!}{\otimes} K_{12} \circ K_{23}).
\end{aligned}$$

Here (1) is by the base change for the diagram




and (2) is by projection formula $f_*(A \overset{!}{\otimes} f^! B) = f_* A \otimes {}^! B$.

Lemma 8.7. *The composition of* 1-*morphisms*

$$(\mathscr{M}_1, \mathscr{M}_2, \mathscr{K}_{12}; \alpha_{12}) \quad \textit{on} \;\; X_1 \times X_2$$

*and*

$$(\mathscr{M}_2, \mathscr{M}_3, \mathscr{K}_{23}; \alpha_{23}) \quad \textit{on} \;\; X_2 \times X_3$$

*is a* 1-*morphism*

$$(\mathscr{M}_1, \mathscr{M}_3, \mathscr{K}_{13}; \alpha_{13}) \qquad \textit{on} \quad X_1 \times X_3$$

*where* $\mathscr{K}_{13} := \mathscr{K}_{23} \circ \mathscr{K}_{12}$ *and* $\alpha_{23}$ *is the composition*

$$\mathscr{M}_3 \to \bar{\mathscr{K}}_{23}(\mathscr{M}_2) \xrightarrow{\bar{\mathscr{K}}_{23}(\tau_{12})} \bar{\mathscr{K}}_{23}(\bar{\mathscr{K}}_{12}(\mathscr{M}_1)) = \bar{\mathscr{K}}_{13}(\mathscr{M}_1).$$

## 4. *Natural Linear Maps Provided by Algebraic Kernels*

Let $\mathscr{M}_i \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_i})$, $i = 1, 2$, and $\mathscr{K}_{12} \in D^b_{\mathrm{coh}}(\mathscr{D}_{X_1 \times X_2})$. Suppose we are given the following data:

(1) an algebraic kernel

$$\alpha_{12} \in R\,\mathrm{Hom}_{\mathscr{D}_{X_2}}(\mathscr{M}_2, \mathscr{K}_{12}(\mathscr{M}_1));$$

$$K_{12} \in R\,\mathrm{Hom}_{\mathscr{D}_{X_1 \times X_2}}(\mathscr{K}_{12}, \mathscr{O}_{X_1 \times X_2})$$

(2) an element

$$\gamma \in R\,\mathrm{Hom}_{\mathscr{D}}(p_{2_*} \mathscr{O}(X_1 \times X_2), \mathscr{O}(X_2)).$$

We will construct a linear map

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^\infty(X_1)) \to R\,\mathrm{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^\infty(X_2))$$

related to this data. Namely, by the functoriality

$$\begin{aligned} &R\operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^\infty(X_1)) \\ &\longrightarrow R\operatorname{Hom}_{\mathscr{D}}(p_1^! \mathscr{M}_1, p_1^! C^\infty(X_1)) \\ &\longrightarrow R\operatorname{Hom}_{\mathscr{D}}(p_1^! \mathscr{M}_1, C^\infty(X_1 \times X_2)[d_{X_2}]) \\ &\xrightarrow{K_{12}\otimes} R\operatorname{Hom}_{\mathscr{D}}(\mathscr{K}_{12} \otimes_{\mathscr{O}} p_1^! \mathscr{M}_1, C^\infty(X_1 \times X_2)[d_{X_2}]) \\ &\longrightarrow R\operatorname{Hom}_{\mathscr{D}}(p_{2_*}(\mathscr{K}_{12} \otimes_{\mathscr{O}} p_1^! \mathscr{M}_1), p_{2_*} C^\infty(X_1 \times X_2)[d_{X_1}]) \\ &\xrightarrow{\gamma} R\operatorname{Hom}_{\mathscr{D}}(\mathscr{K}_{12}(\mathscr{M}_1), C^\infty(X_2)[-d_{X_2}]). \end{aligned}$$

The morphism $\alpha_{12}$ provides the last arrow,

$$R\operatorname{Hom}_{\mathscr{D}}(\mathscr{K}_{12}(\mathscr{M}_1), C^\infty(X_2)[-d_{X_2}]) \to R\operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^\infty(X_2)[-d_{X_2}]).$$

If

$$\alpha_{12} \in R^f \operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_2, \mathscr{K}_{12}(\mathscr{M}_1)), \qquad K_{12} \in R^k \operatorname{Hom}_{\mathscr{D}}(\mathscr{K}_{12}, \mathscr{O})$$

and

$$\gamma \in R^{d_{X_1} - l} \operatorname{Hom}_{\mathscr{D}}(p_{2_*} C^\infty(X_1 \times X_2), C^\infty(X_2)),$$

then we get a linear map

$$R^j \operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_1, C^\infty(X_1)) \to R^{j+k+l-d_{X_1}} \operatorname{Hom}_{\mathscr{D}}(\mathscr{M}_2, C^\infty(X_2)).$$

## 5. *Algebraic Version of the Radon Transform of (Holonomic) Functions*

Any 1-morphism $\gamma\colon A_2 \to A_3$ provides a functor

$$F_\beta\colon \operatorname{Mor}_1(A_1, A_2) \to \operatorname{Mor}_1(A_1, A_3) \qquad \alpha \mapsto \beta \circ \alpha.$$

There is an object ★ corresponding to the one-dimensional vector space considered as a $\mathscr{D}$-module over a point.

The category $\operatorname{Mor}_1(\bigstar, (X, \mathscr{M}))$ looks as follows: Its objects are pairs, a holonomic complex of $\mathscr{D}$-modules $\mathscr{L}$ on $X$ and a morphism $\alpha\colon \mathscr{M} \to \mathscr{L}$. The morphisms are provided by $\varphi\colon \mathscr{L}_1 \to \mathscr{L}_2$, making the corresponding diagram commutative. We will call it the category of $\mathscr{D}$-modules under $\mathscr{M}$ on $X$.

Therefore the 1-morphisms $(\alpha, \mathscr{K})\colon (X, \mathscr{M}) \to (X, \mathscr{N})$ provide functors from the category of $\mathscr{D}$-modules under $\mathscr{M}$ on $X$ to the category of $\mathscr{D}$-modules under $\mathscr{N}$ on $Y$.

## 6. *Examples*

Let me first discuss the analytic properties of the Radon transform in $\mathbb{R}^2$:

$$\varphi(x, y) \mapsto \hat{\varphi}(\xi_1, \xi_2, s) := \int \varphi(x, y)\, \delta(\xi_1 x + \xi_2 y - s)\, dx\, dy.$$

The 1-form

$$\kappa\hat{\varphi}(\xi_1, \xi_2) := \hat{\varphi}'_s(\xi_1\, d\xi_2 - \xi_2\, d\xi_1)$$

is closed on the subvariety $\xi_1 x + \xi_2 y - s = 0$. Here $(x, y)$ is a given point. The integral of this 1-form over any cycle in the $(\xi_1, \xi_2)$ plane is zero.

Consider the line through the point $(x, y)$ corresponding to $\xi = (\xi_1, \xi_2)$. On a line minus a point ($(x, y)$ in our case), there is canonical multiplicatively invariant measure $(dt/t)$. Let $L(\xi) := \int \varphi(x - \xi_2 t, y + \xi_1 t)(dt/t)$ be the integral over this measure. Then

$$\int_{\xi}^{\eta} (\kappa\hat{\varphi})(\xi_1, \xi_2) = L(\eta) - L(\xi) \tag{60}$$

where we integrate over any path connecting points $\xi$ and $\eta$.

In particular, in the affine picture

$$\begin{gathered} \varphi(x, y) \mapsto \int_{\infty}^{\infty} \varphi(x, ax + b)\, dx, \quad \kappa\hat{\varphi} = \hat{\varphi}'_b\, da \\ \int \hat{\varphi}'_b(a, y - ax)\, da = \int_{-\infty}^{\infty} \varphi(x, y) \frac{dy}{y}, \end{gathered} \tag{61}$$

the 1-form $\kappa\hat{\varphi}$ is exact on the image of functions vanishing at the point $(x, y)$. For example,

$$\kappa(I(x\varphi)) = (I(x\varphi))'_b\, da = (I\varphi)'_a\, da = d(I\varphi).$$

Formula (60) follows immediately from this.

Now let us turn to the $\mathscr{D}$-module picture. Set

$$X_1 = \{(x, y)\} = \mathbb{R}^2, \quad X_2 = \{(a, b)\} = \mathbb{R}^2, \quad X_3 = \{(x', y')\} = \mathbb{R}^2,$$

and $\mathscr{M}_{X_i} = \mathscr{D}_{X_i}$. Note that $\bigstar\mathscr{D}_{X_i} \boxtimes \mathscr{D}_{X_{i+1}} = \mathscr{D}_{X_i \times X_{i+1}}[d_{X_i}]$. Set

$$\delta(A) = \delta(y - ax - b) \qquad \text{and} \quad \delta(A') = \delta(y' - ax' - b)$$

$$\mathscr{K}_{12} := \mathscr{D}_{X_1 \times X_2} \cdot \delta(A), \qquad \mathscr{K}_{23} := \mathscr{D}_{X_2 \times X_3} \cdot \delta(A')$$

$$\alpha_{12}[-2]\colon 1_{X_1 \times X_2} \mapsto \delta(A), \qquad \alpha_{12}[-2]\colon 1_{X_2 \times X_3} \mapsto \delta^{(1)}(A').$$

The $\mathscr{D}$-module $\mathscr{K}_{13}$ has a more complicated structure which can be described as follows:

$$0 \to \mathcal{O}_{X_1 \times X_3} \oplus \delta_{\Delta_{13}} \to \mathscr{K}_{13} \to \delta_V \to 0. \tag{62}$$

Here $\Delta_{13} \subset X_1 \times X_2$ is the diagonal and $V$ is the divisor of pairs of points $(p, p')$ with $x = x'$ (i.e., the vertical line through $p$ contains the point $p'$). Let

$$j: X_1 \times X_3 \backslash V \hookrightarrow X_1 \times X_3 \qquad i: V \hookrightarrow X_1 \times X_3 \qquad f: V \backslash \Delta_{13} \hookrightarrow V.$$

Then (62) is the Bauer sum of the following two standard extensions:

$$0 \to \mathcal{O}_{X_1 \times X_3} \to j_* \, j^* \mathcal{O}_{X_1 \times X_3} \to \delta_V \to 0$$

and

$$i_*(0 \to \delta_{\Delta_{13}} \to f_! f^* \mathcal{O}_V \to \mathcal{O}_V \to 0).$$

To see this consider the variety $\mathscr{A} := \{p, l, p'\} \subset X_1 \times X_2 \times X_3$ such that $p, p' \in l$ and its closure $\bar{\mathscr{A}}$ is in $X_1 \times \bar{X}_2 \times X_3$. Note that $\bar{\mathscr{A}}$ is the blow up of the diagonal $\Delta_{13}$ in $X_1 \times X_3$.

Then $\mathscr{K}_{13} = \pi_{2_*} \mathcal{O}_{\mathscr{A}}$. One has $\bar{\pi}_{2_*} \mathcal{O}_{\bar{\mathscr{A}}} = \mathcal{O}_{X_1 \times X_3} \oplus \delta_{\Delta_{13}}$. Further, note that $\bar{\mathscr{A}} \backslash \mathscr{A}$ projects isomorphically to $V$. So one has

$$0 \to \mathcal{O}_{\bar{\mathscr{A}}} \to g_* \mathcal{O}_{\mathscr{A}} \to \delta_V \to 0.$$

Taking a direct image of this extension to $X_1 \times X_3$ we get (62).

PROPOSITION 8.8. *The formula*

$$\delta(x - x')\, \delta(y - y') \mapsto \delta(y - ax - b) \otimes \delta^{(1)}(y' - ax' - b)\, da\, db \tag{63}$$

*defines a homomorphism of $\mathscr{D}$-modules $\delta_\Delta \to \mathscr{K}_{13}$ and hence a 2-morphism $Id_{\mathscr{D}_{X_1}} \Rightarrow (\alpha_{13}, \mathscr{K}_{13})$.*

*Proof.* We have to show that by applying to the right-hand side of (63) any differential equations which the left-hand side satisfies, we will get an exact 2-form in the de Rham complex with respect to $(a, b)$ variables. This follows from the formulas

$$(x - x') \cdot \delta(A) \otimes \delta^{(1)}(A')\, da\, db = d(\delta(A) \otimes \delta(A')(x\, da + db))$$

$$(y - y') \cdot \delta(A) \otimes \delta^{(1)}(A')\, da\, db = d(\delta(A) \otimes \delta(A')\, a(x\, da + db))$$

$$(\partial_x + \partial_{x'})\, \delta(A) \otimes \delta^{(1)}(A')\, da\, db = d(\delta(A) \otimes \delta^{(1)}(A')\, a\, da)$$

$$(\partial_y + \partial_{y'})\, \delta(A) \otimes \delta^{(1)}(A')\, da\, db = d(\delta(A) \otimes \delta^{(1)}(A)\, da).$$

One can see the structure of the extension (62) from this point of view. Namely, $\delta(A)\otimes\delta(A')\,da\,db$ is a generator of $\mathcal{K}_{13}$, so

$$(x-x')\ \delta(A)\otimes\delta(A')\,da\,db$$

is the generator of the submodule $\mathcal{O}_{X_1\times X_3}$ and $\delta(A)\otimes\delta^{(1)}(A')\,da\,db$ generates the submodule $\delta_{\Delta_{13}}$.

*The Radon Transform over the Lines in the Space.*

$$X_1=\{(x,\,y,\,z)\}=\mathbb{A}^3,$$
$$X_2=\{(a_1,a_2,b_1,b_2)\}=\mathbb{A}^4,$$
$$X_3=\{(x',\,y',\,z')\}=\mathbb{A}^3,$$

and

$$\mathcal{M}_{X_i}=\mathcal{D}_{X_i}\quad\text{for } i=1,3;\qquad \mathcal{M}_{X_2}=\mathcal{D}_{X_2}\cdot\left(\frac{\partial^2}{\partial a_1\,\partial b_2}-\frac{\partial^2}{\partial a_2\,\partial b_1}\right).$$

Note that $\bigstar\mathcal{M}_{X_2}=\mathcal{M}_{X_2}$ [3].

Let $A\subset X_1\times X_2$ be the correspondence $\{y-a_1x-b_1=0,\ z-a_2x-b_2=0\}$ defining our family of lines. Set

$$\delta(A)=\delta(y-a_1x-b_1)\cdot\delta(z-a_2x-b_2),$$
$$\delta(A')=\delta(y'-a_1x'-b_1)\cdot\delta(z'-a_2x'-b_2)$$

Then

$$\mathcal{K}_{12}:=\mathcal{D}_{X_1\times X_2}\cdot\delta(A)\qquad \mathcal{K}_{23}:=\mathcal{D}_{X_2\times X_3}\cdot\delta(A')$$
$$\alpha_{12}\colon 1_{X_1\times X_2}\mapsto\delta(A)[-3]\qquad \alpha_{23}\colon 1_{X_2\times X_3}\mapsto\delta(A')[-3].$$

PROPOSITION 8.9. *The formula*

$$\delta(x-x')\,\delta(y-y')\,\delta(z-z')\mapsto G_{\mathcal{M}_2}(\delta(A'),\delta(A))$$

*defines a homomorphism of $\mathcal{D}$-modules $\delta_\Delta\to\mathcal{K}_{23}\circ\mathcal{K}_{12}$ providing a 2-morphism $Id_{\mathcal{D}X_1}\Rightarrow(\alpha_{23},\mathcal{K}_{23})\circ(\alpha_{12},\mathcal{K}_{12})$.*

Here

$$\begin{aligned}G_{\mathcal{M}_2}(g,f)=\tfrac{1}{2}\big(&(g'_{a_1}\cdot f-g\cdot f'_{a_1})\,da_1\wedge da_2\wedge db_1\\&-(g'_{a_2}\cdot f-g\cdot f'_{a_2})\,da_1\wedge da_2\wedge db_2\\&+(g'_{b_1}\cdot f-g\cdot f'_{b_1})\,da_1\wedge db_1\wedge db_2\\&-(g'_{b_2}\cdot f-g\cdot f'_{b_2})\,da_2\wedge db_1\wedge db_2\big).\end{aligned}$$

*Relation with the "form $\kappa$" of* [*GGrS*]. The Green class can represented by another cocycle

$$\kappa_{\mathscr{M}} := \delta(A) \otimes \left( \frac{\partial}{\partial b_1} \delta(A')\, da_1 + \frac{\partial}{\partial b_2} \delta(A')\, da_2 \right)$$
$$\wedge\, (x\, da_1 + db_1) \wedge (x\, da_2 + db_2).$$

The great advantage of this is "locality:" it is a cocycle in the de Rham complex with support in the incidence subvariety $A$.

The expression $(\partial f/\partial b_1)\, da_1 + (\partial f/\partial b_2)\, da_2$ is the "1-form $\kappa$". It is a 1-form on the incidence subvariety $A$.

## ACKNOWLEDGMENTS

I am very grateful to Joseph Bernstein for many stimulating discussions and suggestions.

This work was partially supported by NSF Grant DMS-9500010.

The paper was finished during my stay at the Max-Planck-Institute fur Mathematik in Bonn, whose financial support and hospitality are gratefully acknowledged.

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