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Semisymmetry of Generalized Folkman Graphs

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A regular edge- but not vertex-transitive graph is said to be *semisymmetric*. The study of semisymmetric graphs was initiated by Folkman, who, among others, gave constructions of several infinite families such graphs [8]. In this paper a generalization of his construction for orders a multiple of 4 is proposed, giving rise to some new families of semisymmetric graphs. In particular, one associated with the cyclic group of order n, $n \geq 5$, which belongs to the class of tetracirculants, that is, graphs admitting an automorphism with precisely four orbits, all of the same length. Semisymmetry properties of tetracirculants are investigated in greater detail, leading to a classification of all semisymmetric graphs of order $4p$, where p is a prime.

1. Introductory Remarks

Throughout this paper graphs are assumed to be finite, simple and, unless specified otherwise, undirected. For the group-theoretic concepts and notation not defined here we refer the reader to [5, 14]. Given a graph X we let $V(X)$, $E(X)$ and $\mathrm{Aut}\, X$ be the vertex set, the edge set and the automorphism group of X, respectively. We say that X is *vertex transitive* and *edge transitive* if $\mathrm{Aut}\, X$ acts transitively on $V(X)$ and $E(X)$, respectively. It is easily seen that an edge- but not vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of $\mathrm{Aut}\, X$. Moreover, if X is regular then these two parts have equal cardinality. A regular edge- but not vertex-transitive graph is called *semisymmetric*. The study of semisymmetric graphs was initiated by Folkman [8] who gave a construction of several infinite families of such graphs including, among others, a family of semisymmetric graphs of order $4p$, where p is an odd prime. The smallest graph in this construction has 20 vertices and happens to be the smallest semisymmetric graph. Inspired by Folkman's work the study of semisymmetric graphs has recently received a wide attention, resulting in a number of published articles (see [2, 3, 6, 9–11]). One of the main purposes of this article is to generalize Folkman's constructions.

We start by introducing the concept of orbital digraphs. Let H be a transitive permutation group acting on a set V and let $v \in V$. There is a 1–1 correspondence between the set of *suborbits* of H, that is, the set of orbits of the stabilizer H_v on V, and the set of *orbitals* of H, that is, the set of orbits in the natural action of H on $V \times V$, with the trivial suborbit $\{v\}$ corresponding to the diagonal $\{(v, v) : v \in V\}$. For an orbital Γ we let $S_{\Gamma,v} = \{w \mid (v, w) \in \Gamma\}$ denote the suborbit of H (relative to v) associated with Γ. Conversely, for a suborbit S of H relative to v we let $\Gamma_{S,v}$ be the associated orbital in the above 1–1 correspondence. The paired orbital Γ^{-1} of an orbital Γ is the orbital $\{(v, w) : (w, v) \in \Gamma\}$. If $\Gamma^{-1} = \Gamma$ we say that Γ is a *self-paired orbital*. Similarly, for a suborbit S of H relative to v we let $S^{-1} = S_{\Gamma^{-1},v}$ denote the *paired suborbit* of S. If $S^{-1} = S$ we say that S is self-paired. The *orbital digraph* $\vec{X}(H, V; \Gamma)$ *of* (H, V) *relative to* Γ, is the digraph with vertex set V and arc set Γ. The underlying undirected graph of $\vec{X}(H, V; \Gamma)$ will be called the *orbital graph of* (H, V) *relative to* Γ and will be denoted by $X(H, V; \Gamma)$. If $\Gamma = \Gamma^{-1}$ is a self-paired orbital then $X(H, V; \Gamma)$ admits a vertex- and arc-transitive action of H. On the other hand, if Γ is not self-paired then $X(H, V; \Gamma)$ admits a vertex- and edge- but not arc-transitive action of H, in short, a $\frac{1}{2}$-*arc-transitive action* of H.

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For a permutation τ of V contained in the normalizer of the permutation group H in the symmetric group $\mathrm{Sym}\, V$ we let Γ^τ denote the set $\{(x^\tau, y^\tau) \mid (x, y) \in \Gamma\}$. Since τ normalizes H, the set Γ^τ is also an orbital of H. If $v \in V$ is left fixed by τ and $S = S_{\Gamma,v}$, then the set $S^\tau = \{s^\tau \mid s \in S\}$ is the suborbit $S_{\Gamma^\tau,v}$ of H, which corresponds to the orbital Γ^τ relative to the vertex v.

The following construction, starting with a transitive permutation group H and its orbital Γ, is a generalization of Folkman's construction of semisymmetric graphs arising from abelian groups (see [8, Theorem 4]).

DEFINITION 1.1. Let H be a transitive permutation group on a set V, let Γ be its orbital, let $k \geq 2$ be an integer and let τ be a permutation of V contained in the normalizer of H in $\mathrm{Sym}\, V$ and such that $\tau^k \in H$. Let $\mathcal{B} = \{B_x \mid x \in V\}$ and $V_{0j} = \{x_{0j} \mid x \in V\}$, $j \in \mathbb{Z}_k$, be $k + 1$ copies of the set V. Let $Y(H, V, \Gamma, \tau, k)$ denote the graph with vertex set $\mathcal{B} \cup V_{00} \cup \cdots \cup V_{0k-1}$ and edge set $\{x_{0j} B_y \mid j \in \mathbb{Z}_k, (x, y) \in \Gamma^{\tau^j}\}$. Furthermore, let $V_{1j} = \{x_{1j} \mid x \in V\}$, $j \in \mathbb{Z}_k$, be k copies of the set V. The *generalized Folkman graph* $\mathcal{F}(H, V, \Gamma, \tau, k)$ has vertex set $\bigcup_{i,j\in\mathbb{Z}_k} V_{ij}$ and edge set $\{x_{0j} y_{1i} \mid i, j \in \mathbb{Z}_k, (x, y) \in \Gamma^{\tau^j}\}$.

Observe that the generalized Folkman graph $\mathcal{F}(H, V, \Gamma, \tau, k)$ is obtained from $Y(H, V, \Gamma, \tau, k)$ by expanding each B_x to a k-tuple of vertices $x_{10}, x_{11}, \ldots, x_{1k-1}$ each retaining the neighbors' set of B_x. For the generalized Folkman graph $\mathcal{F}(H, V, \Gamma, \tau, k)$ it will be sometimes convenient to specify a suborbit S corresponding to Γ rather than Γ itself. The notation $\mathcal{F}(H, V, S, \tau, k)$ will then be used instead of $\mathcal{F}(H, V, \Gamma, \tau, k)$. The same applies to the graph $Y(H, V, \Gamma, \tau, k)$.

The generalized Folkman graphs are all regular and bipartite. Furthermore, letting $G = \langle H, \tau\rangle$ we can see that every element of G induces an automorphism of $\mathcal{F}(H, V, \Gamma, \tau, k)$ with H stabilizing all the sets V_{ij}, $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_k$, and τ stabilizing the sets V_{1j} and cyclically permuting the sets V_{0j}, $j \in \mathbb{Z}_k$. With abuse of notation, the symbols H, τ and G will also denote the corresponding induced actions on $\mathcal{F}(H, V, \Gamma, \tau, k)$ and $Y(H, V, \Gamma, \tau, k)$.

For every $x \in V$ the vertices x_{1j}, $j \in \mathbb{Z}_k$, have the same neighbors' sets in the graph $\mathcal{F} = \mathcal{F}(H, V, \Gamma, \tau, k)$. It follows that for each $x \in V$ the automorphism group $\mathrm{Aut}\,\mathcal{F}$ contains a copy of the symmetric group S_k fixing the sets V_{0j}, $j \in \mathbb{Z}_k$, pointwise, and acting on the set $\{x_{1j} \mid j \in \mathbb{Z}_k\}$ by permuting the indices $j \in \mathbb{Z}_k$. The group, generated by G and these automorphisms, acts transitively on the set of edges of $\mathcal{F}$ and has two orbits in its action on the set of vertices of $\mathcal{F}$, namely $\bigcup_{j\in\mathbb{Z}_k} V_{0j}$ and $\bigcup_{j\in\mathbb{Z}_k} V_{1j}$. The generalized Folkman graph $\mathcal{F}$ is therefore a regular bipartite edge-transitive graph with at most two vertex orbits. The fact that for each vertex in $\bigcup_{j\in\mathbb{Z}_k} V_{1j}$ there are at least $k - 1$ other vertices in $\bigcup_{j\in\mathbb{Z}_k} V_{1j}$ sharing the same set of neighbors in $\mathcal{F}$ gives rise to the following simple sufficient condition for the semisymmetry of generalized Folkman graphs. The proof is straightforward and is omitted.

PROPOSITION 1.2. *If no k distinct vertices in $\bigcup_{j\in\mathbb{Z}_k} V_{0j}$ have the same set of neighbors in the graph $Y(H, V, \Gamma, \tau, k)$, then the graph $\mathcal{F}(H, V, \Gamma, \tau, k)$ is semisymmetric.*

One of the main goals of this article is to give constructions of several infinite families of semisymmetric generalized Folkman graphs (see Section 2). Some of these constructions are immediate generalizations of the original Folkman's constructions of semisymmetric graphs of valency 4 corresponding to abelian groups (Examples 2.4 and 2.5). The others are new and arise in the context of alternating groups (Examples 2.6 and 2.7). We remark that the generalized Folkman graphs of the last two examples are associated with certain graphs admitting $\frac{1}{2}$-arc-transitive group actions. Namely, let X be a graph admitting a $\frac{1}{2}$-arc-transitive action of a subgroup H of $\mathrm{Aut}\, X$, and an arc-transitive action of a subgroup G of $\mathrm{Aut}\, X$, where H is of index 2 in G. Then there exists a non-self-paired orbital Γ of H such that $X =$

$X(H, V(X), \Gamma)$ and $\Gamma \cup \Gamma^{-1}$ is an orbital of G. Now let τ be an arbitrary element in $G \setminus H$. Then of course $\Gamma^\tau = \Gamma^{-1}$, $\tau^2 \in H$, and we can construct the generalized Folkman graph $\mathcal{F}(H, V(X), \Gamma, \tau, 2)$. Conversely, the generalized Folkman graph $\mathcal{F}(H, V, \Gamma, \tau, 2)$ gives rise to a graph admitting a $\frac{1}{2}$-arc-transitive action of H and an arc-transitive action of $G = \langle H, \tau\rangle$ provided Γ is not self-paired, and $\Gamma^\tau = \Gamma^{-1}$ is the paired orbital of Γ. The above described connection between these two families of graphs is the content of the sequel to this paper.

In Section 3 we give a number of preliminary results on n-polycirculants, that is, graphs admitting an automorphism having precisely n orbits, all of equal size. These results are used in Section 4, which is devoted to the study of semisymmetry properties of 4-polycirculants. Finally, building on the results from these two sections, we classify (and solve the isomorphism problem for) semisymmetric graphs of order $4p$, where p is a prime, in Section 5. In particular, we show that every such graph is a generalized Folkman graph.

2. Constructions

A sufficient condition for the semisymmetry of a generalized Folkman graph $\mathcal{F}(H, V, \Gamma, \tau, k)$ given in Proposition 1.2 is that no k vertices in $\bigcup_{j \in \mathbb{Z}_k} V_{0j}$ have the same neighbors set in the graph $Y(H, V, \Gamma, \tau, k)$. This gives rise to a particularly transparent condition when H contains a regular subgroup, as may be seen in the proposition and its corollary below.

PROPOSITION 2.1. *Let R be a group acting on itself by right multiplication, let A be a subgroup of* Aut R, *$H = \langle R, A\rangle$, and $r \in R$ such that $S = r^A$ have at least two elements. Furthermore, let τ be an automorphism of R normalizing H and such that $\tau^k \in H$. If $Sy \neq S$ for $y \neq 1$ and $Sy \neq S^\tau$ for all $y \in R$, then the generalized Folkman graph $\mathcal{F}(H, R, S, \tau, k)$ is semisymmetric.*

PROOF. The stabilizer of the identity element $1 \in R$ in the action of H on R equals A. By definition, S is an orbit of A and hence a suborbit of H. The orbital Γ arising from the suborbit S is the set $\{(y, sy) \mid y \in R, s \in S\}$. The orbital Γ^{τ^i}, $i \in \mathbb{Z}_k$, is then the set $\{(y, sy) \mid y \in R, s \in S^{\tau^i}\}$. Suppose that there are k different vertices in the union $\bigcup_{j \in \mathbb{Z}_k} V_{0j}$ with the same neighbors set in the graph $Y = Y(H, R, S, \tau, k)$. Let us assume first that two of them, say x_{0j} and y_{0j} belong to the same V_{0j}. By the definition of Y this implies that $S^{\tau^i} y = S^{\tau^i} x$ and so $S(yx^{-1})^{\tau^{-i}} = S$, contradicting one of the assumptions. We can thus assume that the neighbors' sets of $x_{00} \in V_{00}$ and $y_{01} \in V_{01}$, for some $x, y \in \mathbb{Z}_n$, are the same. But then $Sx = S^\tau y$ and so $S^\tau = Sxy^{-1}$, again a contradiction. □

COROLLARY 2.2. *Let R be a group acting on itself by right multiplication, let $r \in R$, let $\alpha \in$* Aut R *be such that $r^{\alpha^2} = r$, and let $t = r^\alpha$. Assume that*

(i) there exists $\tau \in$ Aut R *commuting with α and interchanging r with r^{-1};*
(ii) $r^2 t^{-2} \neq 1$;
(iii) $rtr^{-1}t^{-1} \neq 1$;
(iv) $(rt^{-1})^2 \neq 1$.

Then the graph $\mathcal{F}(\langle R, \alpha\rangle, R, \{r, t\}, \tau, 2)$ is semisymmetric.

PROOF. Since τ commutes with α it normalizes the group $\langle R, \alpha\rangle$. Moreover, since it interchanges r and r^{-1} it also interchanges t with t^{-1}. Hence $S^\tau = S^{-1}$. Suppose $Sy = S^\tau$ for some element $y \in R$. Then either $ry = r^{-1}$ and $ty = t^{-1}$, or $ry = t^{-1}$ and $ty = r^{-1}$. The first case contradicts (ii) whereas the second case contradicts (iii). On the other hand, if $Sy = S$ for some nonidentity element $y \in R$, we have $ry = t$ and $ty = r$, contradicting (iv). □

The semisymmetry of graphs constructed in Examples 2.4–2.7 may be deduced from the above two results (and the following simple lemma for graphs in Example 2.5).

For $x \in \mathbb{Z}_n$ and $A, B \subseteq \mathbb{Z}_n$ we define the sets $A + x = \{a + x \mid a \in A\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$.

LEMMA 2.3. *Let p be a prime and $A, B \subseteq \mathbb{Z}_p$ with $|B| \geq 2$. If $|A + B| = |A|$ then $A = \mathbb{Z}_p$.*

PROOF. Since $|A| = |A + B| = |\cup_{b \in B} (A + b)|$ we have that $A + b_1 = A + b_2$ for every pair $b_1, b_2 \in B$. Choose two distinct elements b_1 and b_2 in B and let $c = b_2 - b_1$. As $A + b_1 = A + b_2$ it follows that $A = A + kc$ for any $k \in \mathbb{Z}$. Since c is coprime with p we have that $A = A + x$ for all $x \in \mathbb{Z}_p$ and therefore $A = \mathbb{Z}_p$. □

EXAMPLE 2.4. This example of semisymmetric graphs of valency 4 arising from abelian groups is essentially due to Folkman (see [8, Theorem 4]), thus justifying the name generalized Folkman graphs. Let A be an abelian group and T an automorphism of A. Let $k > 1$ be an integer and $r \in A$ such that $2r \neq 0$. Suppose that $T^k(r) \in \{r, -r\}$ and $T^i(r) \notin \{r, -r\}$ for $1 \leq i \leq k - 1$. Furthermore, assume that there is $s \in A$ such that $2s = r$. Let H be a permutation group on the set A generated by the translations with elements of A and a permutation sending each $x \in A$ to $-x$. Observe that the set $S = \{-s, s\}$ is a suborbit of the action of H on A with respect to the identity element $0 \in A$. Moreover, T normalizes H. We can thus construct the generalized Folkman graph $\mathcal{F} = \mathcal{F}(H, A, T, S, k)$, which is semisymmetric. In fact, it is isomorphic to the graph, call it X, with vertex set $\mathbb{Z}_2 \times \mathbb{Z}_k \times A$, and edge set $\{(0, i, x)(1, j, x - b) \mid i, j \in \mathbb{Z}_2, x \in A, b \in \{0, T^i(r)\}\}$, whose semisymmetry was proved in [8, Theorem 4]. Namely, it is easy to check that $\Phi : \mathcal{F} \to X$, mapping according to the rule $x_{0i}^{\Phi} = (0, i, x - T^i(s))$ and $x_{1i}^{\Phi} = (1, i, x)$, is a graph isomorphism. Of course, the semisymmetry of $\mathcal{F}$ may also be proved using Proposition 2.1. We leave this to the reader.

We would like to mention that the existence of an element s such that $2s = r$ is not needed neither in the definition of the graph X nor in the proof of its semisymmetry (but is essential in the definition of the corresponding graph $\mathcal{F}$ and the graph isomorphism Φ). Namely, letting $A = \mathbb{Z}_8$, $T(x) = 3x$ and $r = 1$, the above construction gives us a semisymmetric graph X (on 32 vertices), which does not belong to the family of generalized Folkman graphs, as defined in this article. (Observe that the subgroup of Aut X fixing the four copies of $\mathbb{Z}_8$ does not act equivalently on these copies.) □

EXAMPLE 2.5. Let $V = \mathbb{Z}_n$, let S be a nontrivial subgroup of the multiplicative group $\mathbb{Z}_n^*$, let $k \geq 2$ be an integer, and $a \in \mathbb{Z}_n^*$ be such that $a^k \in S$, while $a^i \notin S$ for all $1 \leq i < k$. Furthermore let τ be an automorphism of $\mathbb{Z}_n$ mapping each $x \in \mathbb{Z}_n$ to ax and $H = \{x \mapsto sx + b \mid s \in S, b \in \mathbb{Z}_n\}$ a subgroup of the group of all affine transformations of $\mathbb{Z}_n$. Clearly, τ normalizes H and S is a suborbit of H relative to the identity element $0 \in \mathbb{Z}_n$. We can thus construct the generalized Folkman graph $\mathcal{F} = \mathcal{F}(H, \mathbb{Z}_n, \tau, S, k)$. If $S \neq x + aS$ for all $x \in \mathbb{Z}_n$, and $S \neq x + S$ for all $x \in \mathbb{Z}_n \setminus \{0\}$, then $\mathcal{F}$ is semisymmetric by Proposition 2.1. The last condition on a and S is always fulfilled if $n = p \geq 5$ is a prime. Namely, $S = x + a^{\epsilon} S$, $\epsilon \in \{0, 1\}$, implies $S = xS + a^{\epsilon} S$, and by Lemma 2.3 we have that $x = 0$ and $a^{\epsilon} \in S$, a contradiction. Moreover, it can be easily seen that $\mathcal{F}$ is a connected graph in this case.

In the last section we will prove that, with the sole exception of one of the two semisymmetric graphs of order 20, every semisymmetric graph of order $4p$, $p \geq 5$ a prime, is isomorphic to the generalized Folkman graph $\mathcal{F}(H, \mathbb{Z}_p, \tau, S, 2)$. □

For convenience we take S_n and A_n to be the groups of all permutations and all even permutations, respectively, on the set $\mathbb{Z}_n$.

EXAMPLE 2.6. Let $R = A_4$, $r = (0, 1, 2) \in A_4$, $\alpha \in \mathrm{Aut}\, A_4$ a conjugation by the transposition $(2, 3)$ and $\tau \in \mathrm{Aut}\, A_4$ a conjugation by the transposition $(0, 1)$. Let H be the group generated by α and the right regular action of R. For $t = r^{\alpha}$, the conditions (i)–(iv) of Corollary 2.2 are easily checked. It follows that for $S = \{r, t\}$ the generalized Folkman graph $\mathcal{F} = \mathcal{F}(\langle R, \alpha\rangle, R, S, \tau, 2)$ is semisymmetric. Moreover, since $rt^{-1} = (1, 2, 3)$ and $r^2t^{-2} = (0, 2, 3)$ generate A_4, it follows that $\mathcal{F}$ is a connected graph of order 48 and valency 4. $\square$

EXAMPLE 2.7. Let $n \geq 5$ be odd, and $R = A_n$. Let $r = (0, 1, \ldots, n-1)$, $a = (1, 2, \ldots, \frac{n-1}{2})(n-1, \ldots, \frac{n+3}{2}, \frac{n+1}{2})$ and let b be the permutation of $\mathbb{Z}_n$ mapping i to $-i$ for each i. Furthermore let α and τ be conjugations by a and b, respectively, let $A = \langle\alpha\rangle$ and let H be the group generated by α and the right regular action of R. Then the set $S = r^A$ equals $\{r_0, r_1, \ldots, r_{\frac{n-3}{2}}\}$, where $r_i = r^{\alpha^i}$ for $i = 0, 1, \ldots, n-1$.

The generalized Folkman graph $\mathcal{F}(H, R, \tau, S)$ is connected, semisymmetric, of order $2n!$, and has valency $\frac{n-1}{2}$. The proof of its semisymmetry is based on Proposition 2.1. Observe first that $r^{\tau} = r^{-1}$. Since the permutations a and b commute, the same holds for the automorphisms τ and α, and so τ normalizes H and maps r_i to r_i^{-1} for every $i \in \{0, \ldots, \frac{n-3}{2}\}$. It remains to check that $S \neq Sy$ for all $y \in R \setminus \{1\}$, and that $S^{-1} \neq Sy$ for all $y \in R$. Suppose first that $S = Sy$ for some $y \neq 1$. Let r_i and r_j be two elements of S such that $r_0 = r_i y$ and $r_1 = r_j y$. It follows that $r_i^{-1}r_0 = r_j^{-1}r_1$ and so $a^i r^{-1} a^{-i} r = a^j r^{-1} a^{-j-1} r a$. If $i < j$ then we can rewrite the equation as $a^{-i} = r a^{j-i} r^{-1} a^{-j-1} r a r^{-1}$. The left-hand side of this equation fixes 0, which is not the case for its right-hand side. A similar contradiction is obtained when $j < i$. The case $S^{-1} = Sy$ for some $y \in R$ is dealt with in a similar fashion.

To prove that $\mathcal{F}$ is connected it suffices to show that the permutations of the form $s_1s_2^{-1}$ and $s_1^2s_2^{-2}$, where $s_1, s_2 \in S$, generate the alternating group $R = A_n$. It may be easily checked that for every $i \in \{1, \ldots, \frac{n-3}{2}\}$, the permutation $r_0r_i^{-1}$ is the 5-cycle $(0, \frac{n-1}{2}, n-1, n-i-1, i)$. It is easily seen that these 5-cycles generate a primitive subgroup of R. For $n \geq 11$ we may thus apply the classical result of Marggraf (see [14, Theorem 13.5]) to deduce that this subgroup is in fact the whole alternating group R. As for $n \in \{5, 7, 9\}$, the proof of the above fact may be done using as one of the generators also the permutation $r_0^2r_1^{-2}$. We leave out the details of the computations. (Note that $r_0^2r_1^{-2} = (0, n-2, n-3, \frac{n-1}{2}, n-1, \frac{n-3}{2}, 1)$ for $n \geq 7$ and $r_0^2r_1^{-2} = (0, 3, 4)$ for $n = 5$.) $\square$

We note that in the previous examples the corresponding suborbits S and S^{τ} are paired to each other. The corresponding generalized Folkman graphs therefore arise from graphs admitting a $\frac{1}{2}$-arc-transitive action of H and an arc-transitive action of $\langle H, \tau\rangle$.

3. PRELIMINARY LEMMAS ON POLYCIRCULANTS

Let $k \geq 1$ and $n \geq 2$ be integers. A graph of order kn admitting an automorphism π with k orbits of length n is called a (k, n, π)-polycirculant. In particular, a $(2, n, \pi)$-polycirculant, a $(3, n, \pi)$-polycirculant, and a $(4, n, \pi)$-polycirculant will be called an (n, π)-bicirculant, an (n, π)-tricirculant, and an (n, π)-tetracirculant, respectively. (The automorphism π will sometimes be omitted from the above notations.)

The generalized Folkman graph $\mathcal{F}(V, H, \Gamma, \tau, k)$ is a $(2k, |V|, \pi)$-polycirculant whenever the group H contains a cyclic subgroup $\langle\pi\rangle$ of order $|V|$. For example, the smallest semisymmetric graph (on 20 vertices) is a tetracirculant. In Section 4 we will investigate semisymmetry properties of tetracirculants. (Note that there are no semisymmetric circulants, bicirculants or tricirculants.) It is the purpose of this section to prove a number of lemmas that will be needed in the next two sections in the analysis of semisymmetry properties of tetracirculants. We start with the following simple observation.

LEMMA 3.1. *Let A and B be subsets of $\mathbb{Z}_n$. Then the following hold.*

$$\sum_{x \in \mathbb{Z}_n} |A \cap (B + x)| = |A|\,|B|, \tag{1}$$

$$\sum_{x \in \mathbb{Z}_n \setminus \{0\}} |A \cap (B + x)| = |A|\,|B| - |A \cap B|. \tag{2}$$

The next two lemmas deal with the structure of bicirculants. We remark that, using the classification of 2-transitive permutation groups, the first one could be proved in a more general setting, that is, for (regular) bipartite graphs with the setwise stabilizer of one of the two parts of the bipartition being a 2-transitive permutation group. However, assuming that the graph is a bicirculant the result can be proved without the classification theorem.

LEMMA 3.2. *Let X be a bipartite (n, π)-bicirculant, with the bipartition consisting of the two orbits of π, which is neither totally disconnected nor a complete bipartite graph. If H is a subgroup of* $\mathrm{Aut}\, X$ *containing π, fixing both orbits of π and acting* 2*-transitively on one of them, then H acts faithfully and* 2*-transitively on both orbits of π.*

PROOF. Let U and W be the two orbits of π, U being the one on which H acts 2-transitively, and let $u \in U$, $w \in W$ be two vertices of X. Define τ to be a permutation of $V(X)$ which interchanges u^{π^i} with $w^{\pi^{-i}}$ for each $i \in \mathbb{Z}_n$. It is easy to see that τ is an automorphism of X. Let G denote the setwise stabilizer of U and W in $\mathrm{Aut}\, X$. We will first show that the actions of G on U and W are isomorphic. For any $g \in G$ the permutation $g^\tau = \tau^{-1} g \tau$ is clearly an element of G which shows that the mapping $g \mapsto g^\tau$ is an automorphism of G. Also, for any $x \in U$ and $g \in G$ we have $(x^g)^\tau = (x^\tau)^{g^\tau}$, implying that the pair $(g \mapsto g^\tau, x \mapsto x^\tau)$ is an isomorphism of the actions of G on U and W. Since G acts 2-transitively on U it acts 2-transitively on both U and W.

Assume now that the action of H is not faithful on one of the sets U and W, say U (W respectively), and let K denote the kernel of this action. If K was transitive on W (U respectively) the graph X would be either totally disconnected or complete, contradicting our assumption. On the other hand, if K was not transitive on W (U respectively) the orbits of K on W (U respectively) would be blocks of imprimitivity for the action of G on W (U respectively) which contradicts the fact that G acts 2-transitively on both U and W.

We can thus assume that H acts faithfully on both U and W. If the action of H was primitive on W it would follow by [14, Theorem 25.3] that $n = p$ is a prime (by assumption H contains a regular cyclic group of order n). By the well known Burnside theorem on groups of prime degree [13, Theorem 7.3], the order of H would be less than $p(p-1)$, and such a group could not be 2-transitive on U, a contradiction. The action of H on W is therefore imprimitive. Since H contains the cyclic element π the blocks of imprimitivity $\mathcal{B} = \{B_0, \ldots, B_{k-1}\}$ are orbits of π^k for some divisor k of n. We can thus assume that $B_i = \{w^{\pi^{i+jk}} \mid j \in \mathbb{Z}_r \setminus \{0\}\}$, where $r = n/k$. Let us denote the corresponding orbits of π^k on U by $C_i = \{u^{\pi^{i+jk}} \mid j = \mathbb{Z}_r \setminus \{0\}\}$, and let S_{ij}, $i, j \in \mathbb{Z}_k$, denote the set of those $s \in \mathbb{Z}_r$ for which $u^{\pi^i} w^{\pi^{j+sk}}$ is an edge of X. Note that S_{ij} is a symbol of the induced bipartite (r, π^k)-bicirculant $X[C_i, B_j]$. Also, $S_{(i+t)(j+t)} = S_{ij}$ for any $i, j, t \in \mathbb{Z}_k$ (just apply π^t). Let K denote the kernel of the action of H on $\mathcal{B}$. Then K is not trivial since it contains π^k. Since H is 2-transitive on U, the kernel K is transitive on U which shows that the cardinalities of the set S_{ij} are independent on i. Combining this fact by the previous argument we obtain that there exists d such that $|S_{ij}| = d$ for all $i, j \in \mathbb{Z}_k$.

Since the action of H on U is 2-transitive it follows that $|N(u, u^{\pi^{tk}})| = |N(u, u^{\pi^{i+tk}})|$ for any $i \in \mathbb{Z}_k$ and $t \in \mathbb{Z}_r$. This is equivalent to

$$\sum_{j=0}^{k-1} |S_{0j} \cap (S_{0j} + t)| = \sum_{j=0}^{k-1} |S_{0j} \cap (S_{ij} + t)|.$$

By taking the sum over all $t \in \mathbb{Z}_r$ we get, by Lemma 3.1,

$$k(d^2 - d) = kd^2 - \sum_{j=0}^{k-1} |S_{0j} \cap S_{ij}|,$$

which implies $S_{0j} = S_{ij}$ for any $i, j \in \mathbb{Z}_k$. But then the vertices u^{π^i}, $i \in \mathbb{Z}_k$, share the same neighbors. The set of all vertices of U together with this set of neighbors forms a block of the action of H on U. But then, since H is 2-transitive on U, all vertices of U have the same neighbors and X is a complete bipartite graph, contradicting our assumption on X. □

LEMMA 3.3. *Let X be an (n, π)-bicirculant, $U, W \subset V(X)$ the orbits of π, and H a subgroup of* Aut X *which fixes U and W setwise. For $u \in U$ and $w \in W$ let $S = \{s \in \mathbb{Z}_n \mid u \sim w^{\pi^s}\}$. If $S = -S + c$ for some $c \in \mathbb{Z}_n$ and H acts 2-transitively on U, then $|S| \in \{0, 1, n-1, n\}$.*

PROOF. Let $u_i = u^{\pi^i}$ and $w_i = w^{\pi^i}$. Since H acts 2-transitively on U, it follows that $\lambda = |N(u_0, u_k) \cap W| = |S \cap (S + k)|$ is constant for all $k \in \mathbb{Z}_n \setminus \{0\}$. Replacing S by $-S + c$, we have $\lambda = |S \cap (-S + c + k)|$ for all $k \in \mathbb{Z}_n \setminus \{0\}$. Therefore

$$|S \cap (-S + c + k)| = \begin{cases} \lambda & \text{if } k \neq 0, \\ |S| & \text{if } k = 0. \end{cases} \tag{3}$$

But letting R be the set $S \cap (S + \frac{n}{2})$ if n is even and the empty set if n is odd, it follows by [1, Lemma 2.2] that

$$|S \cap (-S + c + k)| = \begin{cases} \text{even} & \text{if } \ c + k \in (2S \setminus 2R)^{\mathrm{C}}, \\ \text{odd} & \text{if } \ c + k \in 2S \setminus 2R. \end{cases} \tag{4}$$

Combining (3) and (4) we see that either $\mathbb{Z}_n \setminus \{c\} \subseteq 2S \setminus 2R$ or $\mathbb{Z}_n \setminus \{c\} \subseteq 2S \setminus 2R^{\mathrm{C}}$. Now if n is odd then $2S \setminus 2R = S$ and so $\mathbb{Z}_n \setminus \{c\}$ is either a subset of S or a subset of S^{C}. Hence $|S| \in \{0, 1, n-1, n\}$. The case n even is somewhat trickier. If $\mathbb{Z}_n \setminus \{c\} \subseteq 2S \setminus 2R$, then $n - 1 \leq |2S \setminus 2R| \leq \frac{n}{2}$. Hence $n \leq 2$ and the result follows. If $\mathbb{Z}_n \setminus \{c\} \subseteq 2S \setminus 2R^{\mathrm{C}}$ then either $2S = 2R$ or $2S = 2R \cup \{2s\}$ for $s \in S \setminus R$. In the first case $S = R = S + \frac{n}{2}$. Therefore $\lambda = |S|$ and S is clearly either $\mathbb{Z}_n$ or the empty set. In the second case S is either $R \cup \{s\}$ or $R \cup \{s + \frac{n}{2}\}$. In both cases $\lambda |S \cap (S + \frac{n}{2})| = |S| - 1$. Then by Lemma 3.1 we have $|S|^2 - |S| = (n-1)(|S| - 1)$, giving us the quadratic equation $|S|^2 - n|S| + n - 1 = 0$. It follows that $|S|$ is either 1 or $n - 1$. □

We wrap up this section with a lemma which deals with imprimitive groups of degree $2p$, where p is a prime, and in which bicirculants are present only implicitly. For a permutation group G acting on a set V and a subset $B \subseteq V$ we let $G(B)$ denote the pointwise stabilizer of B in G.

LEMMA 3.4. *Let G be a transitive permutation group acting on a set V of cardinality* $2p$, *and let $\mathcal{B}$ be a complete system of imprimitivity of G consisting of blocks of length* 2. *Then either:*

(i) G has also blocks of length p; or
(ii) for any pair $B, B' \in \mathcal{B}$ there exists $g \in G$ such that g fixes B pointwise and B' setwise but not pointwise.

PROOF. Since G is a transitive group of degree $2p$, there is an element π of G of order p. Let K denote the kernel of the action of G on $\mathcal{B}$, and let $\bar{G} = G/K$. We are going to distinguish two different cases.

CASE 1. $K = 1$.
Then $G = \bar{G}$. If G is solvable, then $\bar{G}$ is a solvable group of degree p, and consequently $\langle \pi \rangle = \langle \bar{\pi} \rangle$ is normal in G. Its two orbits are blocks of length p of G.
Suppose that $G = \bar{G}$ is nonsolvable. Then by [14, Theorem 11.7] it is 2-transitive on $\mathcal{B}$. Let X be an orbital graph of G on V having at least one edge between two different blocks in $\mathcal{B}$. Clearly, there are edges in X between any two blocks in $\mathcal{B}$, in view of 2-transitivity of G. Moreover, because of arc-transitivity of X, any pair of blocks $B, B' \in \mathcal{B}$ induces isomorphic bipartite graphs $X[B, B']$. We now follow step by step the second paragraph of Case 2 in the proof of [12, Theorem 6.2, p.79] and deduce that G has also two blocks of length p (which are in fact orbits of π), with the only exception occurring when $X[B, B'] \cong K_{2,2}$. But in this case $X \cong K_p[\overline{K_2}]$ and (ii) follows since X is an orbital graph of G. (In this case the subdegrees of G are $1, 1, 2p-2$.)

CASE 2. $K \neq 1$.
Suppose that (ii) is not true. Then there must exist $B \in \mathcal{B}$ and $s \in \mathbb{Z}_p^*$ such that $K_{(B)} \leq K_{(B^{\pi^s})}$. But $K_{(B)}$ and $K_{(B^{\pi^s})}$ are conjugate in G (via π^s). Hence $K_{(B)} = K_{(B^{\pi^s})}$. With a repeated conjugation by π^s we see that $K_{(B')} = K_{(B)}$ for any $B' \in \mathcal{B}$. It follows that $K_{(B)} = 1$ and $K \cong \mathbb{Z}_2$.
If $\bar{G}$ is solvable, then $\langle \bar{\pi} \rangle$ is normal in $\bar{G}$. Consequently its preimage $\langle \pi, K \rangle \cong \mathbb{Z}_{2p}$ is normal in G and therefore $\langle \pi \rangle$ is normal in G, giving rise to two blocks of length p.
Suppose that $\bar{G}$ is nonsolvable. Again by [14, Theorem 11.7] it is 2-transitive on $\mathcal{B}$. As in Case 1, let X be an orbital graph of G on V having at least one edge between two different blocks in $\mathcal{B}$. Clearly, there are edges in X between any two blocks in $\mathcal{B}$, in view of 2-transitivity of G. Furthermore, the bipartite graphs $X[B, B']$ are either all isomorphic to $K_{2,2}$ or all isomorphic to $2K_2$. Now the first case implies that G has subdegrees $1, 1, 2p-2$, contradicting the assumption that (ii) is not satisfied. Hence $X[B, B'] \cong 2K_2$ and we can then use [12, Lemma 6.1] to deduce that G has two blocks of length p. □

4. SEMISYMMETRY OF TETRACIRCULANTS

The purpose of this section is to investigate semisymmetry properties of tetracirculants. Since all semisymmetric graphs are necessarily bipartite, restricting ourselves to those bipartite (n, π)-tetracirculants for which each of the two bipartition sets is a union of two orbits of π seems natural.

Let n be a positive integer and let S_{00}, S_{01}, S_{10} and S_{11} be nonempty subsets of $\mathbb{Z}_n$. Define the graph $X = \mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$ to have vertex set $\mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and edge set $\{(x, 0, i)(y, 1, j) \mid i, j \in \mathbb{Z}_2,\ y - x \in S_{ij}\}$. We will use shorthand notations x_{ij} for the vertex $(x, i, j) \in V(X)$, and $V_{ij} = V_{ij}(X)$ for the set $\{x_{ij} \mid x \in \mathbb{Z}_n\}$, $i, j \in \mathbb{Z}_2$. Furthermore, we let $U = U(X) = V_{00} \cup V_{01}$ and $W = W(X) = V_{10} \cup V_{11}$. The permutation π defined by the rule ${x_{ij}}^{\pi} = (x+1)_{ij}$ is an automorphism of X having four orbits of length n. The graph X is therefore an (n, π)-tetracirculant. Conversely, if X is a bipartite (n, π)-tetracirculant for which each of the two bipartition sets is a union of two orbits of π, then

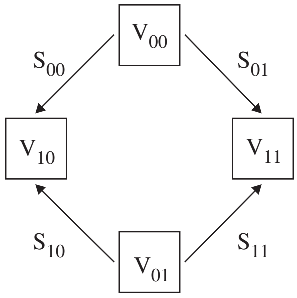

FIGURE 1. Bipartite tetracirculant $\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$.

X is isomorphic to the graph $\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$ for some $S_{ij} \subseteq \mathbb{Z}_n$. Namely, choose four vertices u_{00}, u_{01}, u_{10}, u_{11}, one from each orbit of π, and let x_{ij} denote the vertex ${u_{ij}}^{\pi^x}$ for any $x \in \mathbb{Z}_n$. For a pair $i, j \in \mathbb{Z}_2$ let S_{ij} denote the set $\{s \in \mathbb{Z}_n \mid u_{0i} \sim s_{1j}\}$. A pair $x_{0i}y_{1j}$ is an edge of X if and only if $y - x \in S_{ij}(X)$. The graph X is clearly isomorphic to the graph $\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$. The quadruple $(S_{00}, S_{01}, S_{10}, S_{11})$ will be called *the symbol of X with respect to the quintuple* $(u_{00}, u_{01}, u_{10}, u_{11}; \pi)$ (or with respect to π, in short). It will be useful to have a similar notation for certain bipartite bicirculants. Let X be a bipartite (n, π)-bicirculant where the bipartition is given by the two orbits U and V of π. For $u \in U$ and $v \in V$, we let $S = \{s \in \mathbb{Z}_n \mid u \sim v^{\pi^s}\}$ be the *symbol of X with respect to the triple* $(u, v; \pi)$ (or with respect to π, in short).

We remark that an n-tetracirculant is a generalized Folkman graph if and only if it is isomorphic to the graph $\mathcal{T}(R, R, T, T)$ for some subsets $R, T \subseteq \mathbb{Z}_n$. This suggests that such graphs might be possible candidates for semisymmetric tetracirculants. The proposition below, summarizing a part of Example 2.5, deals with the special case where T is a coset of the multiplicative subgroup R in $\mathbb{Z}_n^*$.

PROPOSITION 4.1. *Let S be a nontrivial subgroup of the $\mathbb{Z}_n^*$ and $a \in \mathbb{Z}_n^*$ such that $a^2 \in S$ but $a \notin S$. If $S \neq x + aS$ for all $x \in \mathbb{Z}_n$, and $S \neq x + S$ for all $x \in \mathbb{Z}_n \setminus \{0\}$, then the tetracirculant $\mathcal{T}(S, S, aS, aS)$ is semisymmetric. If n is prime the last two conditions are always satisfied and, moreover, the corresponding semisymmetric graphs are connected.*

It would be interesting to see if there are semisymmetric tetracirculants which are generalized Folkman graphs, but not of the form given by the above proposition, and furthermore, if there are semisymmetric tetracirculants which are not generalized Folkman graphs? We would like to pose these two questions as open problems.

PROBLEM 4.2. Does there exist a semisymmetric n-tetracirculant with symbol (R, R, T, T), where R and T are subsets of $\mathbb{Z}_n$ which are not cosets of some subgroup of $\mathbb{Z}_n^*$?

PROBLEM 4.3. Is there a semisymmetric tetracirculant which is not a generalized Folkman graph?

In this section we will give some partial answers to the above two problems. A complete solution for the special case of p-tetracirculants is the main ingredient of Section 5. Proposition 4.6 below indicates that under certain extra conditions the answer to Problem 4.2 might be negative. We start by giving two simple lemmas. The proof of the first one is left to the reader.

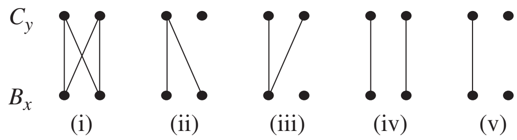

FIGURE 2. Bipartite graphs induced by pairs of blocks.

LEMMA 4.4. *Let y, z, w be arbitrary elements of $\mathbb{Z}_n$, let $a, b, c \in \mathbb{Z}_n^*$, and let σ belong to the dihedral group $\langle(1243), (23)\rangle$. Then:*

(i) $\mathcal{T}(S_1, S_2, S_3, S_4) \cong \mathcal{T}(S_1 + y, S_2 + w, S_3 + y + z, S_4 + z + w)$;
(ii) $\mathcal{T}(S_1, S_2, S_3, S_4) \cong \mathcal{T}(aS_1, cS_2, abS_3, bcS_4)$;
(iii) $\mathcal{T}(S_1, S_2, S_3, S_4) \cong \mathcal{T}(S_{1^\sigma}, S_{2^\sigma}, S_{3^\sigma}, S_{4^\sigma})$.

LEMMA 4.5. *Let n be a positive integer, and S_1, S_2 and S_3 be arbitrary nonempty subsets of $\mathbb{Z}_n$. Then the graph $\mathcal{T}(S_1, S_2, S_3, S_1)$ is not semisymmetric.*

PROOF. The permutation σ, defined by the rule $x_{ij}^\sigma = (-x)_{1-i,1-j}$, is an automorphism of $\mathcal{T}(S_1, S_2, S_3, S_1)$ interchanging the two sets of bipartition, and so $\mathcal{T}(S_1, S_2, S_3, S_1)$ is not semisymmetric. □

For a transitive permutation group G with an imprimitivity block system $\mathcal{B}$ and a permutation π we say that π and $\mathcal{B}$ are *orthogonal* if each orbit of π intersects each block in $\mathcal{B}$ in exactly one point.

PROPOSITION 4.6. *Let X be a semisymmetric (n, π)-tetracirculant with bipartition (U, W) and assume that the action of* Aut X *on U and W has systems of imprimitivity $\mathcal{B}$ and $\mathcal{C}$, each with blocks of length 2, and orthogonal to π^U and π^W, respectively. Then one of the following holds:*

(j) $X \cong \mathcal{T}(R, R, T, T)$ *for some disjoint subsets $R, T \subseteq \mathbb{Z}_n$; or*
(jj) for any $B \in \mathcal{B}$ and $C \in \mathcal{C}$ the induced subgraph $X[B, C]$ is either totally disconnected or isomorphic to $K_2 + 2K_1$ and moreover there are pairwise disjoint subsets $S_{00}, S_{01}, S_{10}, S_{11} \subseteq \mathbb{Z}_n$ such that X is isomorphic to the graph $\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$.

PROOF. Let $G = \text{Aut}\, X$, and let $B = \{a, b\}$ and $C = \{c, d\}$ be blocks of G^W and G^U, respectively. Furthermore, let $(S_{00}, S_{01}, S_{10}, S_{11})$ denote the symbol of the tetracirculant X with respect to the quintuple (c, d, a, b, π). Then $\mathcal{B} = \{B_0, \ldots, B_{n-1}\}$ and $\mathcal{C} = \{C_0, \ldots, C_{n-1}\}$, where $B_x = \{a^{\pi^x}, b^{\pi^x}\}$ and $C_x = \{c^{\pi^x}, d^{\pi^x}\}$ for each $x \in \mathbb{Z}_n$.

Consider those bipartite graphs $X[B_x, C_y]$, $x, y \in \mathbb{Z}_n$, which are not totally disconnected. Because of edge-transitivity of X, any two of them are isomorphic (with the corresponding isomorphism induced by an automorphism of X). The five possibilities for these graphs are shown in Figure 2. In case (i) the sets S_{ij}, $i, j \in \mathbb{Z}_2$, are all equal, which by Lemma 4.5 contradicts semisymmetry of X. In case (ii) it follows that $S_{00} = S_{01}$, $S_{10} = S_{11}$ and $S_{00} \cap S_{10} = \emptyset$, and the graph is of the desired form. In case (iii) it follows that $S_{00} = S_{10}$, $S_{01} = S_{11}$ and $S_{00} \cap S_{01} = \emptyset$, and the graph is again of the desired form. In case (iv) we have $S_{00} = S_{11}$ and $S_{10} = S_{01}$ which by Lemma 4.5 contradicts semisymmetry of X. Finally, in case (v) it is obvious that the sets S_{ij}, $i, j \in \mathbb{Z}_2$ are pairwise disjoint. □

The above result may be strengthened in the special case when n is a prime as we can see in Corollary 4.8. Let X be a graph and V_i, $i \in \mathbb{Z}_k$, be vertex orbits of a subgroup G of Aut X.

Furthermore, for each $i = \mathbb{Z}_k$ let $\mathcal{B}_i$ be a complete block system for the restriction G^{V_i} of G on V_i (possibly a trivial one). By $X/_{\mathcal{B}_0,\ldots,\mathcal{B}_{k-1}}$ we denote the *quotient graph with respect to the block systems* $\mathcal{B}_0, \ldots, \mathcal{B}_{k-1}$, that is, the graph with the vertex set $\cup_{i\in\mathbb{Z}_k}\mathcal{B}_i$, where two vertices (blocks) are adjacent if and only if there is an edge of X joining a vertex in the first block to a vertex in the second block. In the notation of the quotient graph we will conveniently omit the trivial block systems. The action of the group G on $V(X)$ induces an action of G on the set of blocks $\cup_{i\in\mathbb{Z}_k}\mathcal{B}_i$. The proof of the following proposition is straightforward and is left to the reader.

PROPOSITION 4.7. *The induced action of $G \le$ Aut X on the vertex set $\cup_{i\in\mathbb{Z}_k}\mathcal{B}_i$ of the quotient graph $X/_{\mathcal{B}_0,\ldots,\mathcal{B}_{k-1}}$ gives rise to a subgroup of the automorphism group* Aut $(X/_{\mathcal{B}_0,\ldots,\mathcal{B}_{k-1}})$.

COROLLARY 4.8. *Let X be a semisymmetric (p,π)-tetracirculant, where p is a prime, and assume that* Aut X *acts imprimitively with blocks of length 2 on both bipartition sets. Then X is isomorphic to $\mathcal{T}(R,R,T,T)$ for some subsets $R,T \subseteq \mathbb{Z}_p$.*

PROOF. If X is not isomorphic to some $\mathcal{T}(R,R,T,T)$ then case (jj) in Proposition 4.6 occurs. Let Y denote the quotient graph $X/\mathcal{B}$, where $\mathcal{B} = \{B_0,\ldots,B_{p-1}\}$ and $B_x = \{x_{10}, x_{11}\}$ for $x \in \mathbb{Z}_p$. In other words, Y is obtained from X by identifying pairs of vertices x_{10} and x_{11}. By Lemma 4.7 every automorphism of X gives rise to an automorphism of Y. By Lemma 3.4, for every $x \in \mathbb{Z}_p^*$, there exists an automorphism of X fixing 0_{00} and interchanging x_{00} with x_{01}. Applying the action of the corresponding automorphism of Y, induced by this automorphism, we get $|(S_{00}\cup S_{01})\cap(S_{00}\cup(S_{01}+x))| = |(S_{00}\cup S_{01})\cap(S_{10}\cup(S_{11}+x))|$. Taking the sum over all $x \in \mathbb{Z}_p^*$, we see that $|(S_{00}\cup S_{01})|^2 - |(S_{00}\cup S_{01})| = |S_{00}\cup S_{01}||S_{10}\cup S_{11}| - |(S_{00}\cup S_{01})\cap(S_{10}\cup S_{11})|$. Recall that the sets S_{ij}, $i,j \in \mathbb{Z}_2$, are pairwise disjoint. Furthermore, $|S_{00}| = |S_{11}|$ and $|S_{01}| = |S_{10}|$. Hence, $|(S_{00}\cup S_{01})| = |(S_{00}\cup S_{01})\cap(S_{10}\cup S_{11})|$. Therefore $(S_{10}\cup S_{11}) \subseteq (S_{00}\cup S_{01})$, which is clearly not possible, since the sets S_{ij} are nonempty and pairwise disjoint. This contradiction completes the proof of Corollary 4.8. □

Observing that the four orbits of a tetracirculant which is a generalized Folkman graph are not blocks of imprimitivity of the full automorphism group, we now turn to a considerably weaker form of Problem 4.3.

PROBLEM 4.9. Does there exist a semisymmetric (n,π)-tetracirculants with the four orbits of π being blocks of its automorphism group?

The following theorem gives a partial answer to the above problem.

THEOREM 4.10. *Let X be a semisymmetric (n,π)-tetracirculant and suppose that the orbits of π are blocks of imprimitivity of* Aut X. *Then the setwise stabilizer of the four orbits of π in* Aut X *acts imprimitively on each of these orbits.*

PROOF. Let $(S_{00}, S_{01}, S_{10}, S_{11})$ denote the symbol of the tetracirculant X with respect to the automorphism π. The graph X can then be identified with the graph $\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$. We let X_i^j denote the bipartite graph $X[V_{0i}, V_{1j}]$, $i,j \in \mathbb{Z}_2$. Since the orbits V_{ij} of π are blocks of Aut X and X is edge-transitive, the graphs X_i^j, $i,j \in \mathbb{Z}_2$, are all isomorphic and the sets S_{ij} are of the same cardinality. For $i \in \mathbb{Z}_2$ let G_i denote the setwise stabilizer $G_{V_{i0}}$ of V_{i0} in Aut X. Then $H = G_0 \cap G_1$ is the setwise stabilizer of each of the four orbits V_{ij} in Aut X. Further, H is of index 2 in both G_0 and G_1, the latter being of index 2 in Aut X.

Suppose first that the action of H on one of the orbits V_{ij} is 2-transitive. By Lemma 3.2 we may then assume that H acts 2-transitively on all V_{ij}. If H acts equivalently on two neighboring orbits of π, say V_{0i} and V_{1j} for some $i, j \in \mathbb{Z}_2$, then the corresponding bipartite graph X_i^j (and thus all four of them) has to be isomorphic to nK_2 or to $K_{n,n} - nK_2$. But in both cases the graph X is clearly vertex-transitive. Since by the classification of simple groups there are only two inequivalent 2-transitive actions of a given degree (see [4] or [5, Section 7.7]), it follows that the actions of H on V_{00} and V_{01} are equivalent. (The same is true for the pair of orbits V_{10} and V_{11}.) In particular, we may assume that, for some $y \in \mathbb{Z}_n$, the stabilizers of 0_{00} and y_{01} in H coincide. Because of 2-transitivity of the action of H on V_{00} we have that $r_0 = |N(0_{00}, x_{00}) \cap V_{10}| = |S_{00} \cap (S_{00} + x)|$ is independent of the choice of $x \in \mathbb{Z}_n \setminus \{0\}$. Similarly, in view of the above assumption on the stabilizers, $r_1 = |N(0_{00}, (x+y)_{01}) \cap V_{10}| = |S_{00} \cap (S_{10} + x + y)|$ is also independent of the choice of $x \in \mathbb{Z}_n \setminus \{0\}$. Subtracting the first equation from the second and taking the sum over all $x \in \mathbb{Z}_n \setminus \{0\}$, we get

$$\begin{aligned}(n-1)(r_1 - r_0) &= |S_{00}||S_{10} + y| - |S_{00} \cap (S_{10} + y)| - (|S_{00}|^2 - |S_{00}|)\\ &= |S_{00}| - |S_{00} \cap (S_{10} + y)|.\end{aligned}$$

If $r_1 \neq r_0$, then the left-hand side of the above equality is at least $n-1$ and so $|S_{00}| \geq n-1$. It follows that X is vertex-transitive, a contradiction. On the other hand, if $r_1 = r_0$ then $S_{10} + y = S_{00}$. For the same reasons as above the numbers $t_0 = |N(0_{00}, x_{00}) \cap V_{11}|$ and $t_1 = |N(0_{00}, (x+y)_{01}) \cap V_{11}|$ are independent of the choice of $x \in \mathbb{Z}_n \setminus \{0\}$. Repeating the above argument we conclude that $t_1 = t_0$ and therefore $S_{11} + y = S_{01}$. By Lemma 4.4, (i) and (iii), it follows that X is isomorphic to the graph $\mathcal{T}(S_{00}, S_{00}, S_{01}, S_{01})$. But this means that the graph X admits an automorphism interchanging two vertices from different orbits V_{ij} and fixing all other vertices, contradicting our assumption on the orbits V_{ij} being blocks of imprimitivity of Aut X.

We can now assume that the action of H on the orbits V_{ij} is simply primitive. Note that H contains a cyclic subgroup of order n. Therefore (since cyclic groups of composite order are B-groups [14, Theorem 25.3]), it follows that $n = p$ is a prime. The action of H on each of the four orbits V_{ij} is faithful for otherwise the kernel of the action of H on one block would have to be transitive on one of the adjacent blocks, giving rise to the isomorphism $X \cong K_{2p,2p}$. By the Burnside theorem on groups of prime degree [13, Theorem 7.3], there exists a subgroup S of $\mathbb{Z}_p^*$ such that for each pair $i, j \in \mathbb{Z}_2$ the action of H on V_{ij} is permutationally isomorphic with a group $\{x \mapsto ax + b | a \in S, b \in \mathbb{Z}_p\}$ of affine transformations of the field $\mathbb{Z}_p$. Since all the actions of H on the sets V_{ij} are equivalent there are $y, z, w \in \mathbb{Z}_p$, such that the stabilizers of $0_{00}, y_{10}, z_{01}$ and w_{11} in H coincide. By Lemma 4.4, it suffices to consider the case where $y = z = w = 0$. Since H acts edge-transitively on all of the graphs X_i^j, it follows that each of S_{ij} is a coset of S. Let $a_{ij} \in S_{ij}$ be such that $S_{ij} = a_{ij} S$. Consider the setwise stabilizer G_0 of V_{00} in G. Since the Sylow p-subgroup $P = \langle \pi \rangle$ is characteristic in H and H is of index 2 in G_0, it follows that P is normal in G_0. Then there exists a subgroup S' of $\mathbb{Z}_p^*$, such that the action of G_0 on V_{0i} can be identified with the group $\{x_{0i} \mapsto (ax)_{0i} + b | a \in S', b \in \mathbb{Z}_p\}$ of affine transformations. Moreover, either $S = S'$ or S is of index 2 in S'.

We claim that there exists an involution $\tau \in G_0$ fixing vertex 0_{00}. First we obtain an involution by taking an appropriate power of a nonidentity element in $G_0 \setminus H$. Since $|V_{00}|$ is odd this involution fixes a vertex in V_{00}. Conjugating it by a power of π gives us the desired involution τ. In view of the identification of the group G_0 with the above group of affine transformations we conclude that τ acts on V_{0i}, $i \in \mathbb{Z}_2$, either trivially or according to the rule ${x_{0i}}^\tau = (-x)_{0i}$, $x \in \mathbb{Z}_p$. We distinguish two possibilities.

Suppose first that $\tau \in H$. Then τ maps according to the rule ${x_{ij}}^\tau = (-x)_{ij}$, implying that $S = -S$. By Lemma 3.3 the setwise stabilizer of V_{0i} and V_{1j} in Aut X_i^j acts simply transitively on each of these sets and must therefore coincide with the restriction of H on X_i^j.

The mapping $\phi\colon X_0^0 \to X_0^1$, defined by the rule $x_{00}^{\phi} = a_{00}^{-1}a_{01}x_{00}$ and $x_{10}^{\phi} = a_{00}^{-1}a_{01}x_{11}$, is a graph isomorphism. By composing it with some $\alpha \in G_0 \setminus H$ we get an automorphism of the graph X_0^0 which can be extended to an automorphism of X. By composing this automorphism with α^{-1}, an extension of ϕ to some element of G_0 is obtained. But then $a_{00}^{-1}a_{01} \in S'$ and thus $(a_{00}^{-1}a_{01})^2 \in S$. Similarly $(a_{00}^{-1}a_{10})^2 \in S$, $(a_{01}^{-1}a_{11})^2 \in S$, and $(a_{10}^{-1}a_{11})^2 \in S$. It follows that all of the elements a_{ij}^2 belong to the same coset of S. Since the quotient group $\mathbb{Z}_p^*/S$ is cyclic there exists an element $b \in \mathbb{Z}_p^*$ such that each of the cosets $a_{ij}S$ is either $a_{00}S$ or $a_{00}bS$. Combining this fact with Lemma 4.4, (ii), we deduce that X is isomorphic to $\mathcal{T}(a_{00}S, a_{00}S, a_{00}bS, a_{00}bS)$ which contradicts the assumption that the orbits of π are blocks of imprimitivity for the group Aut X.

Assume now that $\tau \notin H$. Since P is normal in G_0, we have that τ normalizes π and therefore there exists $r \in \mathbb{Z}_p$ such that either $x_{10}^{\tau} = (-x + r)_{11}$ for all $x \in \mathbb{Z}_p$ or $x_{10}^{\tau} = (x+r)_{11}$ for all $x \in \mathbb{Z}_p$. Since τ fixes 0_{0i}, $i \in \mathbb{Z}_2$, we have that for each $i \in \mathbb{Z}_2$, τ interchanges the sets $N(0_{0i}) \cap V_{10}$ and $N(0_{0i}) \cap V_{11}$. It follows that either $-a_{i0}S + r = a_{i1}S$, for all $i \in \mathbb{Z}_2$, or $a_{i0}S + r = a_{i1}S$, for all $i \in \mathbb{Z}_2$. Using the fact that S is a subgroup of $\mathbb{Z}_p^*$, we have in the first case $-a_{i0}S + rS = a_{i1}S$, $i \in \mathbb{Z}_2$, and in the second case $a_{i0}S + rS = a_{i1}S$, $i \in \mathbb{Z}_2$. In both cases $r = 0$ by Lemma 2.3. We conclude that either $a_{i0}S = a_{i1}S$, for all $i \in \mathbb{Z}_2$, or $-a_{i0}S = a_{i1}S$, for all $i \in \mathbb{Z}_2$. Repeating now the above arguments (used in the previous four paragraphs for the group G_0) for the group G_1, we end up with either $a_{0j}S = a_{1j}S$, for all $j \in \mathbb{Z}_2$, or $-a_{0j}S = a_{1j}S$, for all $j \in \mathbb{Z}_2$. Checking out each of the four possibilities for the quantities a_{ij}, we have that either $X = \mathcal{T}(a_{00}S, -a_{00}S, a_{00}S, -a_{00}S)$, which is impossible since X is not vertex-transitive, or $X = \mathcal{T}(a_{00}S, a_{00}S, -a_{00}S, -a_{00}S)$, contradicting the fact that the orbits of π are blocks of imprimitivity for the group Aut X. □

In the particular case when n is a prime, we have the following consequence of the above theorem.

COROLLARY 4.11. *There are no semisymmetric (p, π)-tetracirculants, p a prime, with the four orbits of π being blocks of its automorphism group.*

5. SEMISYMMETRIC GRAPHS OF ORDER $4p$

The results of the previous section (Corollaries 4.8 and 4.11) together with the theorem of Burnside [13, Theorem 7.3], which characterizes simply transitive group actions of prime degree, enable us to classify semisymmetric graphs of order $4p$, where p is a prime. The first proposition of this section is an observation that every such graph is a tetracirculant.

PROPOSITION 5.1. *If X is a semisymmetric graph of order $4p$, where p is a prime, then $X \cong \mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$ for some $S_{00}, S_{01}, S_{10}, S_{11} \subseteq \mathbb{Z}_p$, such that $|S_{00}| = |S_{11}|$ and $|S_{01}| = |S_{10}|$.*

PROOF. The automorphism group $G =$ Aut X of X has two orbits of length $2p$ which form a bipartition of $V(X)$. Using [14, Theorem 3.4] we deduce that a Sylow p-subgroup P of G has four orbits A_1, A_2, B_1 and B_2 of length p, such that $\{A_1 \cup A_2, B_1 \cup B_2\}$ is a bipartition of X. If X was not a connected graph its connected components would be isomorphic semisymmetric graphs. But this is not possible since there are no semisymmetric graphs of order a proper divisor of $4p$ (see [8]). We can thus assume that X is a connected graph which implies that for any $i, j \in \{1, 2\}$, there is an edge of X joining A_i and B_j.

To prove that X is a tetracirculant it suffices to show that P contains an element without fixed vertices. Suppose that P contains a nontrivial element α fixing a vertex. With no loss of generality we may assume that α fixes every vertex in A_1 and acts regularly on B_1. Consequently, $X[A_1, B_1]$ is isomorphic to $K_{p,p}$, and since X is a regular graph, the same holds for the graph $X[A_2, B_2]$. Furthermore, if one of the graphs $X[A_i, B_j]$, $i \neq j$, is isomorphic to $K_{p,p}$, then both are. In this case $X \cong D(\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)$, contradicting the semisymmetry of X by Lemma 4.5. Consequently the graphs $X[A_i, B_j]$, $i \neq j$, are not isomorphic to $K_{p,p}$ and hence α acts trivially on B_2 and regularly on A_2. There exists $\beta \in P$ acting regularly on A_1. Since $X[A_1, B_2]$ is not isomorphic to $K_{p,p}$, it follows that β is not trivial on B_2, and must thus be regular on B_2. Moreover, the actions of β on A_1 and B_1 are simultaneously either regular or trivial and so either β or $\alpha\beta$ is an element of P without fixed vertices. □

We may now state the main result of this section.

THEOREM 5.2. *A graph X of order $4p$, where p is a prime, is semisymmetric if and only if one of the following holds.*

(i) X is isomorphic to some $\mathcal{T}(S, S, aS, aS)$, where S is a nontrivial subgroup of $\mathbb{Z}_p^*$, $a \in \mathbb{Z}_p^* \setminus S$ and $a^2 \in S$; or
(ii) $p = 5$ and X is isomorphic to $\mathcal{T}(\{0, 2, 3\}, \{0, 2, 3\}, \{0, 1, 4\}, \{0, 1, 4\})$.

The following lemma will be used in the proof of the above theorem.

LEMMA 5.3. *Let $\mathcal{T} = \mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$ be a (p, π)-tetracirculant and U, W the orbits of π. If all of $S_{ij} \subseteq \mathbb{Z}_p$ are of the same cardinality and the restriction $(\mathrm{Aut}\,\mathcal{T})^W$ of $\mathrm{Aut}\,\mathcal{T}$ to W has 2-blocks, but no p-blocks, then $S_{01} = S_{00} + y$ and $S_{11} = S_{10} + y$ for some $y \in \mathbb{Z}_p$. Consequently, there are $R, T \subseteq \mathbb{Z}_p$ such that $\mathcal{T}$ is isomorphic to $\mathcal{T}(R, R, T, T)$.*

PROOF. Let $G = \mathrm{Aut}\,\mathcal{T}$. There is $y \in \mathbb{Z}_p$ such that $B = \{0_{10}, y_{11}\}$ is a block of G^W. Then $\mathcal{B} = \{B_0, \ldots, B_{p-1}\}$, where $B_x = \{x_{10}, (x + y)_{11}\}$ for each $x \in \mathbb{Z}_p$, is a complete system of imprimitivity of G^W. Lemma 3.4 implies that, for each $x \in \mathbb{Z}_p^*$, there exists an element of G which fixes both 0_{10} and y_{11}, and interchanges x_{10} with $(x + y)_{11}$. It follows that

$$\begin{aligned} &|-S_{00} \cap (x - S_{00})| + |-S_{10} \cap (x - S_{10})| = |N(0_{10}, x_{10})| \\ &\quad = |N(0_{10}, (x + y)_{11})| = |-S_{00} \cap (x + y - S_{01})| + |-S_{10} \cap (x + y - S_{11})|. \end{aligned}$$

Letting $d = |S_{ij}|$ and taking the sum over all $x \in \mathbb{Z}_p^*$, we obtain

$$\begin{aligned} 2d^2 - 2d &= |-S_{00}|^2 - |-S_{00}| + |-S_{10}|^2 - |-S_{10}| \\ &= |-S_{00}||y - S_{01}| - |-S_{00} \cap (y - S_{01})| + |-S_{10}||y - S_{11}| \\ &\quad -|-S_{10} \cap (y - S_{11})| \\ &= 2d^2 - |-S_{00} \cap (y - S_{01})| - |-S_{10} \cap (y - S_{11})|. \end{aligned}$$

It follows that $2d = |-S_{00} \cap (y - S_{01})| + |-S_{10} \cap (y - S_{11})|$ and consequently $S_{01} = S_{00} + y$ and $S_{11} = S_{10} + y$. □

PROOF OF THEOREM 5.2. The fact that the graphs in (i) are semisymmetric is the content of Proposition 4.1, while the graph in (ii) is the bipartite complement of the semisymmetric graph $\mathcal{T}(\{1, -1\}, \{1, -1\}, \{2, -2\}, \{2, -2\})$ of order 20 and it may be seen that it is indeed semisymmetric.

Suppose now that X is a semisymmetric graph X of order $4p$, where p is a prime. We need to show that X is isomorphic to one of the graphs in (i) and (ii) in the statement of Theorem 5.2. By Proposition 5.1 we can identify the graph X with a (p, π)-tetracirculant

$$\begin{tikzcd} \boxed{V_{00}} \arrow[r, "S_{00}"] & \boxed{\mathcal{B}} & \boxed{V_{01}} \arrow[l, "S_{10}"'] \end{tikzcd}$$

FIGURE 3. The quotient graph $X/_{\mathcal{B}}$.

$\mathcal{T}(S_{00}, S_{01}, S_{10}, S_{11})$ for some subsets $S_{00}, S_{01}, S_{10}, S_{11} \subseteq \mathbb{Z}_p$. There are no semisymmetric graphs of orders 8 and 12 and precisely two nonisomorphic semisymmetric graphs of order 20 (see [10]). We can therefore assume that $p > 5$ for the rest of the proof. As in the previous section we let V_{00}, V_{01}, V_{10} and V_{11} denote the four orbits of π, and we let U and W denote the respective unions $V_{00} \cup V_{01}$ and $V_{10} \cup V_{11}$. Furthermore, let $G = \operatorname{Aut} X$. The proof will be carried out in two steps, each giving a further refinement of the information on the symbol $(S_{00}, S_{01}, S_{10}, S_{11})$.

STEP 1. $X \cong \mathcal{T}(R, R, T, T)$ for some $R, T \subseteq \mathbb{Z}_p$.

We show first that G is simply transitive on both U and W. Assume that the action of G on U is 2-transitive. Then $r = |N(0_{00}, x_{01})| = |S_{00} \cap (S_{10}+x)| + |S_{01} \cap (S_{11}+x)|$ is independent of the choice of $x \in \mathbb{Z}_p$. By Lemma 3.1 n

$$pr = \sum_{x=0}^{p-1} |N(0_{00}, x_{01})| = |S_{00}||S_{10}| + |S_{01}||S_{11}| = 2|S_{00}||S_{10}|,$$

and consequently $S_{00} = \mathbb{Z}_p = S_{11}$ or $S_{01} = \mathbb{Z}_p = S_{10}$. In both cases we get a contradiction in view of Lemma 4.5. Furthermore, since there are no simply primitive groups of degree $2p$, $p \geq 7$, (see [4]), the above action is imprimitive.

If G has 2-blocks on both U and W then Corollary 4.8 implies that the symbol of X is of the desired form. Therefore, in view of Corollary 4.11 we may assume that G has p-blocks on precisely one of the sets U and W.

Assume that G^U has p-blocks (and consequently G^W has 2-blocks but no p-blocks). Then we can easily show that the cardinalities of the sets S_{00}, S_{01}, S_{10} and S_{11} are all equal. By Lemma 5.3, we conclude that X is isomorphic to $\mathcal{T}(R, R, T, T)$ for some $R, T \subseteq \mathbb{Z}_p$.

STEP 2. $X \cong \mathcal{T}(S, S, aS, aS)$ where S is a subgroup of $\mathbb{Z}_p^*$, $a \in \mathbb{Z}_p^* \setminus S$ and $a^2 \in S$.

By Step 1, we may identify the graph X with $\mathcal{T}(R, R, T, T)$ for some $R, T \subseteq \mathbb{Z}_p$. First, since X is semisymmetric (and therefore regular and not vertex-transitive) we have $1 < |R| = |T| < p - 1$. If the action of G on U had blocks of length 2, then by Lemma 5.3, X would be isomorphic to the graph $\mathcal{T}(R, R, R, R)$ and hence not semisymmetric by Lemma 4.5. Therefore the action of G on U has blocks of length p. Also, the action of G on U is unfaithful with the corresponding kernel K isomorphic to $\mathbb{Z}_2^p$ and acting on W with orbits $B_x = \{x_{10}, x_{11}\}$.

Let $Y = X/\mathcal{B}$ be the quotient graph of X relative to the imprimitivity block system $\mathcal{B} = \{B_x \mid x \in \mathbb{Z}_p\}$ of G^W, that is the graph of order $3p$, obtained from X by identifying x_{10} with x_{11} for each $x \in \mathbb{Z}_p$ (see Figure 3). By Proposition 4.7, every automorphism of X induces an automorphism of Y with all the elements of K giving rise to the identity automorphism of Y. It is therefore clear that, because of edge-transitivity of X, the action of $\bar{G} = G/K$ on Y is also edge-transitive, having two orbits on vertices: U and $\mathcal{B}$. The sets V_{00} and V_{01} are blocks of the action of $\bar{G}$ on U. Let $\bar{H}$ denote the corresponding setwise stabilizer of V_{00} and V_{01} in $\bar{G}$, fixing V_{00} and V_{01} setwise. The actions of $\bar{H}$ on $\mathcal{B}$, V_{00} and V_{01} are faithful for otherwise $R = \mathbb{Z}_p = T$, contradicting the semisymmetry of X.

Suppose first that all of these actions are 2-transitive. If the actions of $\bar{H}$ on $\mathcal{B}$ and on one of the sets V_{0i} were equivalent, then $|R|$ would be either 1 or $p - 1$, which is impossible. Since by the classification of simple groups there are only two inequivalent 2-transitive actions of a given degree (see [4] or [5, Section 7.7]) we can conclude that the actions of $\bar{H}$ on V_{00} and V_{01} are equivalent. Therefore there exists an element $y \in \mathbb{Z}_p$, such that $\bar{H}_{0_{00}} = \bar{H}_{y_{01}}$. We let $r_0 = |N(0_{00}, x_{00})| = |R \cap (R+x)|$, $r_1 = |N(0_{00}, (x+y)_{01})| = |R \cap (T+x+y)|$. Subtracting the two equations and taking the sum over all $x \in \mathbb{Z}_p^*$, we see that $(p-1)(r_1 - r_0) = |R| - |R \cap (T+y)|$. If $r_1 \neq r_0$ then we must have $|R| \geq p-1$, contradicting the assumption on R. If $r_1 = r_0$ we have $R = T + y$ and, by Lemma 4.4, X is isomorphic to $\mathcal{T}(R, R, R, R)$ and hence not semisymmetric, a contradiction.

By Lemma 3.2, we may assume that the action of $\bar{H}$ is simply transitive on one, and therefore on all of the sets V_{00}, V_{01} and $\mathcal{B}$. The argument below is just a slight modification of the argument used in the proof of Theorem 4.10. By the Burnside theorem on groups of prime degree [13, Theorem 7.3] there exists a subgroup S of $\mathbb{Z}_p^*$ such that the actions of $\bar{H}$ on V_{00}, V_{01} and $\mathcal{B}$ are permutationally isomorphic with a group $\{x \mapsto ax + b | a \in S, b \in \mathbb{Z}_p\}$ of affine transformations of the field $\mathbb{Z}_p$. Therefore there are $y, z \in \mathbb{Z}_p$, such that the stabilizers of $0_{00}, y_{01}, B_z$ in $\bar{H}$ coincide. By Lemma 4.4, it suffices to consider the case $y = 0 = z$. Because of edge-transitivity of the action of $\bar{H}$ on the graphs $Y[V_{0j}, \mathcal{B}]$, $j \in \mathbb{Z}_2$, it follows that both R and T are cosets of S. By Lemma 4.4, we can assume that $R = S$ and $T = aS$ for some $a \in \mathbb{Z}_p^* \setminus S$. To complete the proof of Step 2 we need to show that a^2 belongs to S.

Let $\bar{\pi}$ denote the automorphism of Y induced by the element π of order p with orbits V_{ij}, $i, j \in \mathbb{Z}_2$. Since $\bar{P} = \langle \bar{\pi} \rangle$ is characteristic in $\bar{H}$ and the latter is normal in $\bar{G}_1 = \bar{G}$, we have that $\bar{P}$ is normal in $\bar{G}$. It follows that there exists a subgroup S' of $\mathbb{Z}_p^*$ such that the action of $\bar{G}$ on $\mathcal{B}$ can be identified with the group $\{B_x \mapsto B_{ax+b} | a \in S', b \in \mathbb{Z}_p\}$ of affine transformations. Moreover, either $S = S'$ or S is of index 2 in S'.

Using a similar argument as in the proof of Theorem 4.10 we obtain an involution $\bar{\tau} \in \bar{G}$ fixing the vertex B_0 acting on $\mathcal{B}$ either trivially or according to the rule $B_x \mapsto B_{-x}$, $x \in \mathbb{Z}_p$. We distinguish two possibilities.

If $\bar{\tau} \in \bar{H}$ then $\bar{\tau}$ interchanges B_x with B_{-x} and so $S = -S$. As in the proof of Theorem 4.10, using Lemma 3.3, we conclude that every automorphism of $Y[V_{00}, \mathcal{B}]$ can be extended to an automorphism of Y. The mapping $\phi : Y[V_{00}, \mathcal{B}] \to Y[V_{01}, \mathcal{B}]$, defined by the rule $x_{00}^{\phi} = (ax)_{01}$, $B_x^{\phi} = B_{ax}$, for all $x \in \mathbb{Z}_p$, is a graph isomorphism, which can be extended to an element of $\bar{G}$. But then $a \in S'$ and so $a^2 \in S$.

Suppose now that $\bar{\tau} \notin \bar{H}$. Note that $\bar{\tau}$ normalizes $\bar{\pi}$. Therefore there exists $r \in \mathbb{Z}_p$ such that $\bar{\tau}$ acts on U by interchanging x_{00} either with $(-x+r)_{01}$ for each $x \in \mathbb{Z}_p$ or with $(x+r)_{01}$ for each $x \in \mathbb{Z}_p$. Recall that $\bar{\tau}$ fixes B_0 and so it must interchange the sets of neighbors $N(B_0) \cap V_{00}$ and $N(B_0) \cap V_{01}$. It may be deduced that either $S + r = -aS$ or $-S + r = aS$. Multiplying these two equalities by S and using Lemma 2.3 we see that $r = 0$, and consequently $aS = -S$ and so $a^2 \in S$, as required. □

REMARK 5.4. Theorem 5.2 solves the isomorphism problem for semisymmetric graphs of order $4p$, p a prime. Namely, for a given prime $p \geq 7$ and a positive integer d there exists at most one semisymmetric graph of order $4p$ and valency d. More precisely, such a graph exists (and is unique) if and only if d is an even number dividing $p - 1$.

While preparing the final version of this article it came to our notice that these graphs were independently found by Du and Xu [7]. Using the classification of finite simple groups they prove their existence as part of a larger family of semisymmetric graphs of order $2pq$, where p, q are primes. However, they provide neither the explicit labeling for these graphs nor their connection to the original Folkman's construction.

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