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The polytope of dual degree partitions

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Abstract

We determine the extreme points and facets of the convex hull of all dual degree partitions of simple graphs on n vertices. (This problem was raised in the Laplace Energy group of the Workshop *Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns* held at the American Institute of Mathematics Research Conference Center on October 23–27, 2006 [R.A. Brualdi, Leslie Hogben, Brian Shader, AIM Workshop – Spectra of Families of Matrices Described by Graphs, Digraphs, and Sign Patterns; Final Report: Mathematical Results, November 17, 2006].)

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1. Introduction

We deal throughout with simple graphs G (undirected, no loops, no multiple edges) on the vertex set $\{1, \ldots, n\}$. The *degree* d_i of vertex i is the number of neighbors of i, and the *degree sequence* of G is $\mathbf{d} = (d_1, \ldots, d_n)$. We assume that the vertices have been relabeled so that $n - 1 \geqslant d_1 \geqslant \cdots \geqslant d_n \geqslant 0$, and to stress this we call $\mathbf{d}$ a *degree partition*. The *dual degree partition* of G is the sequence $\mathbf{d}^* = (d_1^*, \ldots, d_n^*)$, where $d_j^* = \#\{i : d_i \geqslant j\}$, so that $n \geqslant d_1^* \geqslant \cdots \geqslant d_n^* = 0$. Both $\mathbf{d}$ and $\mathbf{d}^*$ can be conveniently pictured as a *Ferrers diagram*, which is an $n \times n$ matrix of zeros and ones, where the ones in each row are to the left of the zeros, the ones in each column

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	6	6	3	1	0	0
4	1	1	1	1	0	0
3	1	1	1	0	0	0
3	1	1	1	0	0	0
2	1	1	0	0	0	0
2	1	1	0	0	0	0
2	1	1	0	0	0	0

	5	5	2	3	1	0
4	0	1	1	1	1	0
3	1	0	1	1	0	0
3	1	1	0	1	0	0
2	1	1	0	0	0	0
2	1	1	0	0	0	0
2	1	1	0	0	0	0

Fig. 1. Ferrers diagram (left) and corrected Ferrers diagram (right) representing $\mathbf{d} = (4, 3, 3, 2, 2, 2)$, $\mathbf{d}^* = (6, 6, 3, 1, 0, 0)$, $\overline{\mathbf{d}^*} = (5, 5, 2, 3, 1, 0)$.

are above the zeros, the row sums are the d_i and the column sums are the d_i^* (such a matrix is sometimes called *maximal*). Following Berge, it is also convenient to use the *corrected Ferrers diagram*, which is obtained from the Ferrers diagram by moving every one on the main diagonal to the end of its row, replacing it with a zero. The row sums of the corrected Ferrers diagram are of course the d_i; its column sums $\overline{d_i^*}$ (which are not necessarily in non-increasing order) are called the *corrected conjugate degrees*, and we use the notation $\overline{\mathbf{d}^*} = (\overline{d_1^*}, \ldots, \overline{d_n^*})$. Fig. 1 illustrates these definitions.

Not every maximal matrix is the Ferrers diagram of the degree partition of a simple graph. If it is, we say that its row and column sums are *realizable*. There are many criteria for realizability proved in [3]. The one we use here involves the relation of majorization of sequences. We say that a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ *majorizes* a sequence $\mathbf{b} = (b_1, \ldots, b_n)$, and write $\mathbf{a} \succcurlyeq \mathbf{b}$, if the sum A_k of the largest k components of $\mathbf{a}$ and the sum B_k of the largest k components of $\mathbf{b}$ satisfy $A_k \geqslant B_k$ for $k = 1, \ldots, n$ with equality for $k = n$. We call the difference $A_k - B_k$ the kth *slack*. The following theorem is well-known; for a proof see [3].

Theorem 1.1 (Berge). *Consider an integral sequence* $\mathbf{d} = (d_1, \ldots, d_n)$ *with* $n - 1 \geqslant d_1 \geqslant \cdots \geqslant d_n \geqslant 0$. *Then* $\mathbf{d}$ *and* $\mathbf{d}^*$ *are realizable if and only if* $\sum_{i=1}^{n} d_i$ *is even and* $\overline{\mathbf{d}^*} \succcurlyeq \mathbf{d}$.

The convex hull of all realizable degree partitions of length n was studied in [1], and in particular that paper determines its extreme points and facets. In this paper, we do the same for the convex hull of all realizable dual degree partitions of length n.

2. Results

Since $d_n^* = 0$ for every realizable $\mathbf{d}^*$ of length n, we will suppress it and consider the convex hull of dual degree sequences as a subset of $\mathbb{R}^{n-1}$. We treat separately the case of n even and the somewhat harder case of n odd. We use similar techniques in both cases.

2.1. *n* Even

Consider the n points

$$\mathbf{a}^{(k)} = (\underbrace{n, \ldots, n}_{k}, \underbrace{0, \ldots, 0}_{n-1-k}), \quad k = 0, \ldots, n-1. \tag{2.1}$$

Theorem 2.1. *For even* n, *the facet-defining inequalities of the convex hull of the realizable dual degree partitions on* n *vertices are*

$$n \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1} \geqslant 0. \tag{2.2}$$

Its extreme points are the $\mathbf{a}^{(k)}$ *defined in* (2.1).

This theorem was proved independently in the Laplace Energy group of the AIM workshop [2], and by the first and third authors of this paper.

Proof. Let $P = \text{conv}\{\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(n-1)}\}$, let Q be the convex hull of the realizable dual degree partitions on n vertices, and let R be the polytope defined by (2.2). Each $\mathbf{a}^{(k)}$ is a realizable dual degree partition. This can be seen for example from the fact that by König's theorem, $K_{\frac{n}{2},\frac{n}{2}}$ is the union of $\frac{n}{2}$ edge-disjoint matchings. The union of k of these matchings is k-regular for $0 \leqslant k \leqslant \frac{n}{2}$, and the complements of these graphs are k-regular for $\frac{n}{2} - 1 \leqslant k \leqslant n-1$. It follows that $\mathbf{a}^{(k)} \in Q$ and thus $P \subseteq Q$. Obviously $Q \subseteq R$. We also have that every extreme point $\mathbf{x}$ of R is one of the $\mathbf{a}^{(k)}$ and therefore $R \subseteq P$. Indeed, since $R \subseteq \mathbb{R}^{n-1}$ is defined by the n linear inequalities (2.2), these inequalities hold with equality at $\mathbf{x}$ with at most one exception, and therefore $\mathbf{x}$ is one of the $\mathbf{a}^{(k)}$. This proves that $P = Q = R$. Furthermore, this polytope is full-dimensional because its n extreme points are affinely independent. Since none of the inequalities (2.2) is a consequence of the others, these inequalities are its unique facet-defining inequalities. □

2.2. *n* Odd

Consider the $\frac{n+1}{2}$ points

$$\mathbf{a}^{(k)} = (\underbrace{n, \dots, n}_{2k}, \underbrace{0, \dots, 0}_{n-1-2k}), \quad 0 \leqslant 2k \leqslant n-1, \tag{2.3}$$

the $\frac{n^2-1}{8}$ points

$$\mathbf{b}^{(k,l)} = (\underbrace{n, \dots, n}_{2k}, \underbrace{n-1, \dots, n-1}_{2l+1}, \underbrace{0, \dots, 0}_{n-2-2k-2l}), \quad 0 \leqslant 2k+2l \leqslant n-3, \tag{2.4}$$

and the $\frac{n^2-1}{8}$ points

$$\mathbf{c}^{(k,l)} = (\underbrace{n, \dots, n}_{2k+1}, \underbrace{1, \dots, 1}_{2l+1}, \underbrace{0, \dots, 0}_{n-3-2k-2l}), \quad 0 \leqslant 2k+2l \leqslant n-3, \tag{2.5}$$

altogether $\frac{(n+1)^2}{4}$ points.

In analogy with Theorem 2.1, we have the following result.

Theorem 2.2. *For odd n, the facet-defining inequalities of the convex hull of the realizable dual degree partitions on n vertices are the n inequalities (2.2) as well as the inequality*

$$(x_1 - x_2) + (x_3 - x_4) + \cdots + (x_{n-2} - x_{n-1}) \leqslant n-1. \tag{2.6}$$

Its extreme points are the $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k,l)}$ and $\mathbf{c}^{(k,l)}$ given in (2.3)–(2.5).

Proof. As before, let P be the convex hull of the points $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k,l)}$ and $\mathbf{c}^{(k,l)}$, let Q be the convex hull of the realizable dual degree partitions on n vertices, and let R be the polytope defined by (2.2) and (2.6).

We will show that each of the points $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k,l)}$ and $\mathbf{c}^{(k,l)}$ is a realizable dual degree partition, and consequently $P \subseteq Q$. We do this using Theorem 1.1. If $\mathbf{d}^* = \mathbf{a}^{(k)}$, then $\mathbf{d} = (\underbrace{2k, \dots, 2k}_{n})$ and $\overline{\mathbf{d}^*} = (\underbrace{n-1, \dots, n-1}_{2k}, \underbrace{2k}_{1}, \underbrace{0, \dots, 0}_{n-2k})$. When we consider the majorization inequalities $\overline{\mathbf{d}^*} \succcurlyeq \mathbf{d}$, each of the first $2k$ inequalities adds $n-1-2k \geqslant 0$ to the slack, and the next inequality leaves the slack unchanged and exhausts $\mathbf{d}^*$.

If $\mathbf{d}^* = \mathbf{b}^{(k,l)}$, then $\mathbf{d} = (\underbrace{2k+2l+1, \ldots, 2k+2l+1}_{n-1}, \underbrace{2k}_{1})$ and $\overline{\mathbf{d}^*} = (\underbrace{n-1, \ldots, n-1}_{2k}, \underbrace{n-2, \ldots, n-2}_{2l+1}, \underbrace{2k+2l+1}_{1}, \underbrace{0, \ldots, 0}_{n-2k-2l-3})$. Each of the first $2k$ majorization inequalities adds $n-2k-2l-2 \geqslant 1$ to the slack, each of the next $2l+1$ inequalities adds $n-2k-2l-3 \geqslant 0$ to the slack, and next inequality exhausts $\mathbf{d}^*$.

If $\mathbf{d}^* = \mathbf{c}^{(k,l)}$, then $\mathbf{d} = (\underbrace{2k+2l+2}_{1}, \underbrace{2k+1, \ldots, 2k+1}_{n-1})$ and $\overline{\mathbf{d}^*} = (\underbrace{n-1, \ldots, n-1}_{2k+1}, \underbrace{2k+1}_{1}, \underbrace{1, \ldots, 1}_{2l+1}, \underbrace{0, \ldots, 0}_{n-2k-2l-3})$. The first majorization inequality adds $n-2k-2l-3 \geqslant 0$ to the slack. Each of the next $2k$ inequalities adds $n-1-(2k+1) \geqslant 2l+1$ to the slack, for a total slack of $(n-2k-2l-3)+2k(n-1-(2k+1)) \geqslant 2k(2l+1)$. Each of the next $2l+1$ inequalities subtracts $2k$ from the slack, which keeps it non-negative, and exhausts $\overline{\mathbf{d}^*}$.

Each realizable dual degree partition $\mathbf{x}$ obviously satisfies (2.2). It also satisfies (2.6) because $(x_1-x_2)+(x_3-x_4)+\cdots+(x_{n-2}-x_{n-1})$ is the number of vertices of odd degree, which is even, and n is odd. Consequently $Q \subseteq R$.

We will show that each extreme point $\mathbf{x}$ of R is one of the points $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k,l)}$ and $\mathbf{c}^{(k,l)}$, and consequently $R \subseteq P$. Since the polytope $R \subseteq \mathbb{R}^{n-1}$ is defined by $n+1$ inequalities, at least $n-1$ of these inequalities hold with equality at $\mathbf{x}$ and at most two are strict. Obviously at least one of the inequalities (2.2) must be strict. Suppose exactly one inequality in (2.2) is strict. Then $n = x_1 = \cdots = x_p > x_{p+1} = \cdots = x_{n-1} = 0$ for some $0 \leqslant p \leqslant n-1$. By (2.6) p must be even, say $p = 2k$. Therefore $\mathbf{x} = \mathbf{a}^{(k)}$. We may thus assume that exactly two inequalities of (2.2) are strict, and (2.6) holds with equality. Then $n = x_1 = \cdots = x_p > x_{p+1} = \cdots = x_q > x_{q+1} = \cdots = x_{n-1} = 0$ for some $0 \leqslant p < q \leqslant n-1$. In other words, $\mathbf{x} = (\underbrace{n, \ldots, n}_{p}, \underbrace{x_{p+1}, \ldots, x_{p+1}}_{q-p}, \underbrace{0, \ldots, 0}_{n-1-q})$. If p is even, say $p = 2k$, then since (2.6) holds with equality, $q-p$ must be odd, say $q-p = 2l+1$, and $x_{p+1} = n-1$. Therefore $\mathbf{x} = \mathbf{b}^{(k,l)}$. If p is odd, say $p = 2k+1$, then by the same reason $q-p$ is odd, say $q-p = 2l+1$, and $x_{p+1} = 1$. Therefore $\mathbf{x} = \mathbf{c}^{(k,l)}$.

We have shown that $P \subseteq Q \subseteq R \subseteq P$, so $P = Q = R$. Once again, this polytope is full-dimensional since it contains the n affinely independent points $\mathbf{a}^{(k)}$ and $\mathbf{b}^{(0,l)}$. Since none of the inequalities (2.2) and (2.6) is a consequence of the others, they are the unique facet-defining inequalities of that polytope. □

An integral point satisfying the inequalities (2.2) and (2.6) need not be a realizable dual degree partition even if the sum of its components is even. An example for $n = 7$ is given by $\mathbf{d}^* = (5, 3, 3, 3, 3, 3)$, which satisfies (2.2) and (2.6), yet $\mathbf{d} = (6, 6, 6, 1, 1, 0, 0)$ is not realizable. Therefore to characterize realizable dual degree partitions we need nonlinear constraints.

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