doi:10.1006/aama.2001.0767, available online at http://www.idealibrary.com on IDEAL®

Shuffle Invariance of the Super-RSK Algorithm

Amitai Regev and Tamar Seeman

Department of Theoretical Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail: regev@wisdom.weizmann.ac.il, tamars@wisdom.weizmann.ac.il

Received December 4, 2000; accepted April 4, 2001

As in the (k, l)-RSK (Robinson–Schensted–Knuth) of A. Berele and A. Regev (1987, *Adv. Math.* **64**, 118–175), other super-RSK algorithms can be applied to sequences of variables from the set $\{t_1, \ldots, t_k, u_1, \ldots, u_l\}$, where $t_1 < \cdots < t_k$ and $u_1 < \cdots < u_l$. While the (k, l)-RSK is the case where $t_i < u_j$ for all i and j, these other super-RSK's correspond to all the $\binom{k+l}{k}$ shuffles of the t's and u's satisfying the above restrictions that $t_1 < \cdots < t_k$ and $u_1 < \cdots < u_l$. We show that the shape of the tableaux produced by any such super-RSK is independent of the particular shuffle of the t's and u's. © 2002 Elsevier Science (USA)

1. INTRODUCTION

We follow the tableau terminology of [7]. The classical Frobenius–Schur–Weyl theory shows how the SSYT (semistandard Young tableaux) determine the representations of $GL(m, \mathbb{C})$ (or $gl(m, \mathbb{C})$). Here $GL(m, \mathbb{C})$ ($gl(m, \mathbb{C})$) is the general linear Lie group (algebra). Also, SYT (standard Young tableaux) play an important role here. The notion of (k, l) SSYT is introduced in [1], where similar relationships between such tableaux and the representations of $pl(k, l)$ are shown. Here $pl(k, l)$ is the general linear Lie super-algebra.

The (k, l) SSYT are defined, via a (k, l)-RSK algorithm, as follows [1]. Fix integers $k, l \geq 0$, $k + l > 0$, and $k + l$ symbols $t_1, \ldots, t_k, u_1, \ldots, u_l$ such that $t_1 < \cdots < t_k < u_1 < \cdots < u_l$. Let

$$a_{k,l}(n) = \left\{ \begin{pmatrix} 1 \cdots n \\ v_1 \cdots v_n \end{pmatrix} \mid v_i \in \{t_1, \ldots, t_k, u_1, \ldots, u_l\} \right\}.$$

To map $a_{k,l}(n)$ to pairs of tableaux (P, Q), apply to each $v \in a_{k,l}(n)$ the (k, l)-RSK, in which the usual RSK insertion algorithm [7] is applied to the

0196-8858/02 $35.00

t_i's and the conjugate correspondence (see [1]) is applied to the u_j's; see the examples below. By the definitions of [1], the *insertion tableau*, $P = P(v)$, mapped from $v \in a_{k,l}(n)$, is (k, l) semistandard; that is, it satisfies the following three properties:

(a) The "t part" (i.e., the cells filled with t_i's) is a tableau.

(b) The t_i's are nondecreasing in rows, strictly increasing in columns.

(c) The u_j's are nondecreasing in columns, strictly increasing in rows.

As in the usual correspondence, the *recording tableau*, $Q = Q(v)$, indicates the order in which the new cells were added to P. Clearly, Q is SYT having the same shape as that of P.

A total order of $\{t_1, \dots, t_k, u_1, \dots, u_l\}$, which is compatible with $t_1 < \cdots < t_k$ and $u_1 < \cdots < u_l$, is called a *shuffle* (of $t_1, \dots, t_k$ and $u_1, \dots, u_l$). For example, $t_1 < u_1 < u_2 < t_2$ is such a shuffle, compatible with $t_1 < t_2$ and $u_1 < u_2$. Clearly, there are $\binom{k+l}{k}$ such shuffles; of these, Berele and Regev chose to work with $t_1 < \cdots < t_k < u_1 < \cdots < u_l$, which we call *the* (k, l) *shuffle* (see [1, 2.4]). The shuffle $t_1 < u_1 < t_2 < u_2 < \cdots < t_k < u_k$, with its corresponding SSYT, appears in Section 4 of [3].

Let $I = I(k, l)$ denote the set of all such $\binom{k+l}{k}$ shuffles. Given $A \in I$, there is a corresponding A-RSK insertion algorithm; if $v \in a_{k,l}(n)$, then $v \underset{A}{\longrightarrow} (P, Q)$ by that algorithm. $P = P_A = P(v, A)$ is the insertion tableau, and $Q = Q_A = Q(v, A)$ is the recording tableau. Here P is an A-SSYT; that is, it satisfies the following three properties:

(a) P is weakly A-increasing in both rows and columns.

(b) The t_i's are strictly increasing in columns.

(c) The u_j's are strictly increasing in rows.

EXAMPLE. Let $k = l = 2$, $A, B \in I = I(2, 2)$, where

$$A : t_1 < t_2 < u_1 < u_2 \quad \text{and} \quad B : u_1 < u_2 < t_1 < t_2.$$

Let

$$v = \begin{pmatrix} 1 \cdots\cdots 4 \\ u_2, t_1, t_2, u_1 \end{pmatrix}.$$

Then

$$v \underset{A}{\longrightarrow} \begin{array}{|c|}\hline u_2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline t_1 & u_2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline t_1 & t_2 & u_2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline t_1 & t_2 & u_2 \\ \hline u_1 \\ \cline{1-1}\end{array} = P_A,$$

while

$$v \underset{B}{\longrightarrow} \begin{array}{|c|}\hline u_2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline u_2 & t_1 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline u_2 & t_1 & t_2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline u_1 & u_2 & t_2 \\ \hline t_1 \\ \cline{1-1}\end{array} = P_B.$$

Thus $v \underset{A}{\longrightarrow} (P_A, Q)$ and $v \underset{B}{\longrightarrow} (P_B, Q)$, where

$$Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

and P_A and P_B are as above.

DEFINITION. Denote by $\mathrm{sh}(v, A) = \mathrm{sh}(P_A)$ the shape of the insertion tableau $P(v, A) = P_A$ of $v \in a_{k,l}(n)$ under the A-RSK.

Given a shuffle $A \in I$ and the pair (P, Q), where P is A-SSYT, Q is SYT, and $\mathrm{sh}(P) = \mathrm{sh}(Q)$, the A insertion algorithm can obviously be reversed. Standard arguments (see, e.g., [7, Chap. 7]) yield the following result.

THEOREM 1. *Let $A \in I$ be a shuffle. Then the A-RSK insertion algorithm $v \underset{A}{\longrightarrow} (P_A, Q_A)$ is a bijection between $a_{k,l}(n)$ and*

$$\{(P_A, Q_A) \mid P_A \text{ is } A\text{-SSYT}, Q_A \text{ is SYT}, \mathrm{sh}(P_A) = \mathrm{sh}(Q_A)\}.$$

Remark. Denote such a tableau $P = (P_{i,j})$ and denote $<_A$ by $<$. Clearly, if $P_{i,j} = t_r$, then $P_{i,j-1} \leq P_{i,j} \leq P_{i,j+1}$ and $P_{i-1,j} < P_{i,j} < P_{i,j+1}$. Similarly, if $P_{i,j} = u_r$, then $P_{i,j-1} < P_{i,j} < P_{i,j+1}$ and $P_{i-1,j} \leq P_{i,j} \leq P_{i,j+1}$.

Denote by $\mathrm{sh}(v, A)$ the shape of tableaux $P(v, A)$ and $Q(v, A)$. This brings us to our main result.

THEOREM 2. *Let $v \in a_{k,l}(n)$, $A, B \in I$, $v \underset{A}{\longrightarrow} (P_A, Q_A)$, and $v \underset{B}{\longrightarrow} (P_B, Q_B)$. Then $\mathrm{sh}(P_A) = \mathrm{sh}(P_B)$. Consequently, $Q_A = Q_B$.*

In other words, the shape of the tableau obtained through any of the (k, l)-shuffle-RSK algorithms is independent of the particular shuffle of the t's and u's.

DEFINITION. Let $A \in I$ and $\lambda \vdash n$, that is, a partition of n. Let $\Im_A(\lambda)$ denote the set of the A-SSYT of shape λ:

$$\Im_A(\lambda) = \{T \mid T \text{ is } A\text{-SSYT}, \ \mathrm{sh}(T) = \lambda\}.$$

Recall the definition of $\mathrm{type}(T)$ from [7, p. 309].

Theorem 2 implies the following.

THEOREM 3 [6]. *Let $A, B \in I, \lambda \vdash n$. Then there exists a bijection $\varphi: \Im_A(\lambda) \to \Im_B(\lambda)$ such that, for all $T \in \Im_A(\lambda)$, $\mathrm{type}(T) = \mathrm{type}(\varphi(T))$. (In fact, there exist (at least) d_λ such canonical bijections, where d_λ is the number of SYT's of shape λ.)*

Theorem 3 appears in [6], where it is proven by a different method. Our proof of the theorem is as follows.

Proof of Theorem 3. The proof is based on the following diagram:

$$v \in a_{k,l}(n), \quad v \begin{array}{l} \nearrow^{A-\mathrm{RSK}} \ (P_A, Q) \\ \searrow_{B-\mathrm{RSK}} \ (P_B, Q) \end{array}$$

Thus choose an SYT Q of shape λ. Given $P = P_A \in \Im_A(\lambda)$, we get

$$(P_A, Q) \xrightarrow[\substack{\text{inverse} \\ A\text{-RSK}}]{} v \xrightarrow[B\text{-RSK}]{} (P_B, Q).$$

This defines the bijection $\varphi = \varphi_Q : \varphi(P_A) = P_B$. Clearly, $\mathrm{type}(P_A) = \mathrm{type}(P_B)$ and, by Theorem 2, $\mathrm{sh}(P_A) = \mathrm{sh}(P_B)$. ∎

Recall from [2] the notation $w(T)$ for the weight of a tableau T. For example, let

$$T = \begin{array}{|c|c|c|c|} \hline t_1 & t_1 & u_2 & u_3 \\ \hline t_2 & t_3 & u_2 \\ \cline{1-3} u_1 & u_3 \\ \cline{1-2} u_1 \\ \cline{1-1} \end{array} .$$

Then $w(T) = x_1^2 x_2 x_3 y_1^2 y_2^2 y_3^2$. Also, recall the "hook" (or the "super") Schur function

$$HS_\lambda(x; y) = HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) \quad [1, 2].$$

When A is the shuffle $A_0 : t_1 < \cdots < t_k < u_1 < \cdots < u_l$, $HS_\lambda(x; y)$ is given by

$$HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = \sum_{T \in \Im_{A_0}(\lambda)} w(T)$$

[1, Theorem 6.10]. See also [4–6].

Theorem 3 implies the following.

COROLLARY 4. *For any $A \in I$,*

$$HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = \sum_{T \in \Im_A(\lambda)} w(T).$$

Given a shuffle $A \in I$, the A-RSK is based on A, on the regular RSK for the t_i's, and on the conjugate-regular RSK for the u_j's.

In addition to the regular RSK, there is also the dual RSK [7, p. 331]. Given the shuffle $A \in I$, this leads to four possible A insertion algorithms: either the regular or the dual for the t_i's and either the conjugate regular or the conjugate dual for the u_j's. In fact, the previous A-RSK is (t-regular, u-conjugate-regular), which we denote as the (regular, regular)-A-RSK. Similarly, (t-regular, u-dual-conjugate) is the (regular, dual)-A-RSK. Similarly for the algorithms (dual, regular)-A-RSK and (dual, dual)-A-RSK. Each of these three new insertion algorithms exhibits a similar shape invariance under all shuffles $A \in I$.

THEOREM 5. (a) *Let $v \in a_{k,l}(n)$ and $A, B \in I$ such that*

$$v \xrightarrow[\text{(regular, regular)-}A\text{-RSK}]{} (P_A^*, Q_A^*), \qquad v \xrightarrow[\text{(regular, regular)-}B\text{-RSK}]{} (P_B^*, Q_B^*).$$

Then $\mathrm{sh}(P_A^*) = \mathrm{sh}(P_B^*)$. *Consequently,* $Q_A^* = Q_B^*$.

(b) *Let $v \in a_{k,l}(n)$ and $A, B \in I$ such that*

$$v \xrightarrow[\text{(regular, dual)-}A\text{-RSK}]{} (P_A^*, Q_A^*), \qquad v \xrightarrow[\text{(regular, dual)-}B\text{-RSK}]{} (P_B^*, Q_B^*).$$

Then $\mathrm{sh}(P_A^*) = \mathrm{sh}(P_B^*)$. *Consequently,* $Q_A^* = Q_B^*$.

(c) *Let $v \in a_{k,l}(n)$ and $A, B \in I$ such that*

$$v \xrightarrow[\text{(dual, regular)-}A\text{-RSK}]{} (P_A^*, Q_A^*), \qquad v \xrightarrow[\text{(dual, regular)-}B\text{-RSK}]{} (P_B^*, Q_B^*).$$

Then $\mathrm{sh}(P_A^*) = \mathrm{sh}(P_B^*)$. *Consequently,* $Q_A^* = Q_B^*$.

(d) *Let $v \in a_{k,l}(n)$ and $A, B \in I$ such that*

$$v \xrightarrow[\text{(dual, dual)-}A\text{-RSK}]{} (P_A^*, Q_A^*), \qquad v \xrightarrow[\text{(dual, dual)-}B\text{-RSK}]{} (P_B^*, Q_B^*).$$

Then $\mathrm{sh}(P_A^*) = \mathrm{sh}(P_B^*)$. *Consequently,* $Q_A^* = Q_B^*$.

Clearly, Theorem 5(a) is Theorem 2 above. The proof of Theorem 2 is given in the next section, which is the main body of this paper. First we describe the A-RSK algorithm in detail. The main step in the proof of Theorem 2 is Lemma 2.15, which shows that a transposition of the variables in the shuffle (i.e., a single change in the order of some t_i and u_j) does not alter the shape of the resulting tableaux. In Section 3 we prove the remaining parts (b), (c), and (d) of Theorem 5, essentially by deducing them from Theorem 2.

2. INVARIANCE OF SHAPE

As in the (k, l)-RSK, the A-RSK insertion algorithm involves applying the usual RSK correspondence to the t_i's, and the conjugate correspondence to the u_j's. This is illustrated in the following example.

DEFINITION 2.1. For $i, j \in \mathbb{Z}^+$, let $c(i, j)$ denote the cell in row i and column j of a given tableau.

EXAMPLE 2.2. Under the shuffle $A = t_1 < u_1 < t_2 < u_2 < t_3$, perform the insertion

$$\begin{array}{|c|c|c|}\hline u_1 & t_2 & t_2 \\ \hline u_1 & u_2 & \multicolumn{1}{c}{} \\ \cline{1-2} t_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \leftarrow t_1.$$

(a) $t_1 < u_1 \Longrightarrow t_1$ occupies $c(1,1)$. Now, a u_i is always bumped to the next column; hence u_1 is bumped to column 2.

(b) $u_1 < t_2 \Longrightarrow u_1$ occupies $c(1,2)$. Now, a t_i is always bumped to the next row; hence t_2 is bumped to row 2.

(c) $u_1 < t_2 < u_2 \Longrightarrow t_2$ occupies $c(2,2)$, bumping u_2 to column 3.

(d) $u_2 > t_2 \Longrightarrow u_2$ settles in $c(2,3)$.

$$\text{(a)}\ \begin{array}{|c|c|c|}\hline t_1 & t_2 & t_2 \\ \hline u_1 & u_2 & \multicolumn{1}{c}{} \\ \cline{1-2} t_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \text{(b)}\ \begin{array}{|c|c|c|}\hline t_1 & u_1 & t_2 \\ \hline u_1 & u_2 & \multicolumn{1}{c}{} \\ \cline{1-2} t_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,,$$

$$\text{(c)}\ \begin{array}{|c|c|c|}\hline t_1 & t_2 & t_2 \\ \hline u_1 & t_2 & \multicolumn{1}{c}{} \\ \cline{1-2} t_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \text{(d)}\ \begin{array}{|c|c|c|}\hline t_1 & u_1 & t_2 \\ \hline u_1 & t_2 & u_2 \\ \hline t_3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,.$$

The proof of Theorem 2 follows from the following analysis of the A-RSK algorithm.

LEMMA 2.3. *Let P be an A-SSYT, $v \in \{t_1, \ldots, t_k, u_1, \ldots, u_l\}$. The insertion $P \leftarrow v$ is made of a sequence of several steps. In an intermediate mth such step, we have an A-SSYT $\widetilde{P}$ together with an element $P_{i,j}$ that was bumped from $c(i,j)$ by $\widetilde{P}_{i,j}$, $\widetilde{P}_{i,j} \underset{A}{<} P_{i,j}$, and we need to do the following insertion*:

(a) *If $P_{i,j} = t_r$, insert it into the $i+1$th row of $\widetilde{P}$.*

(b) *If $P_{i,j} = u_s$, insert it into the $j+1$th column of $\widetilde{P}$.*

We show that in both cases the result would be an A-SSYT P^* and—except for the last step—together with a new element $\widetilde{P}_{i',j'}$ (bumped from $c(i',j')$), which is to be inserted into P^*. Moreover,

(1) *If $P_{i,j} = t_r$, then $c(i',j') = c(i+1,j')$ and $j' \le j$.*

(2) *If $P_{i,j} = u_s$, then $c(i',j') = c(i',j+1)$ and $i' \le i$.*

Proof. Note that (2) is obtained from (1) by conjugation; hence it suffices to just prove (1).

Proof of (1). Denote the ith row of $\widetilde{P}$ by

$$a_1 \cdots\cdots\cdots a_{j-1}\widetilde{P}_{i,j}a_{j+1} \cdots\cdots\cdots a_g,$$

so $a_j = P_{i,j}$ and, by assumption, $P_{i,j} = t_r$. Thus

$$\widetilde{P} = \begin{array}{l} \vdots \\ a_1 \cdots\cdots\cdots\cdots\cdots\cdots a_{j-1}\widetilde{P}_{i,j}a_{j+1} \cdots\cdots\cdots a_g \\ b_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots b_f \\ c_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots c_h \\ \vdots \end{array}$$

and $P_{i,j} = t_r$ is inserted into the $i+1$th row $b_1 \cdots\cdots b_f$.

Let $b_{j'-1} \le P_{i,j} < b_{i,j'}$, so, in P^*, the $i+1$th row is

$$b_1 \cdots\cdots\cdots\cdots b_{j'-1}P_{i,j}b_{j'+1} \cdots\cdots\cdots\cdots b_f.$$

Since $\widetilde{P}_{i,j}$ bumped $P_{i,j}$, we have $\widetilde{P}_{i,j} < P_{i,j}$. Since $a_j = P_{i,j} = t_r$, hence $P_{i,j} < b_j$. Together with $b_{j'-1} \le P_{i,j} < b_{j'}$, this implies that $j' \le j$; hence

$$P^* = \begin{array}{l} \vdots \\ a_1 \cdots\cdots a_{j'-1}a_{j'}a_{j'+1} \cdots\cdots \widetilde{P}_{ij}a_{j+1} \cdots\cdots\cdots a_g \\ b_1 \cdots\cdots b_{j'-1}P_{ij}b_{j'+1} \cdots\cdots\cdots b_jb_{j+1} \cdots\cdots\cdots b_f \\ c_1 \cdots\cdots c_{j'-1}c_{j'}c_{j'+1} \cdots\cdots\cdots c_jc_{j+1} \cdots\cdots c_h \\ \vdots \end{array}$$

By the induction assumption on $\widetilde{P}$, we only need to verify that the part

$$\begin{array}{l} a_{j'} \\ P_{i,j} \\ c_{j'} \end{array}$$

of the j'th column is A-semistandard; that is, since $P_{i,j} = t_r$, we need to show that $a_{j'} \le \widetilde{P}_{i,j} < c_{j'}$. This follows from $a_{j'} \le \widetilde{P}_{i,j} < P_{i,j} = t_r < b_{j'} \le c_{j'}$. ∎

DEFINITION 2.4. Two shuffles $A, B \in I$ are *adjacent* if there exist t_i and u_j such that

(1) $t_i < u_j$ in A.

(2) $u_j < t_i$ in B.

(3) All other pairs have the same order relations in A and in B.

In that case, call A and B (t_i, u_j)-*adjacent*. Thus A and B differ by the transposition (t_i, u_j).

Remark 2.5. Trivially, for any $A, B \in I$, there exist $A_0, A_1, \ldots, A_n \in I$ such that $A_0 = A$, $A_n = B$, and A_r is adjacent to $A_{r+1}, 0 \le r \le n-1$. Thus, to prove Theorem 1, it suffices to show that, for all $v \in a_{k,l}(n)$ and for every pair (A, B) of adjacent shuffles, $\mathrm{sh}(v, A) = \mathrm{sh}(v, B)$. Therefore, for the rest of this section, let $A, B \in I$ be (t_i, u_j)-adjacent, with $t_i <_{_A} u_j$ and $u_j <_{_B} t_i$.

LEMMA 2.6. *Let $A \in I$, let $w \in a_{k,l}(n)$, and, for some $x \in \{t_1, \ldots, t_k, u_1, \ldots, u_l\}$, let w' be the sequence obtained by omitting from w all elements A-greater than x. Let P_A and P'_A be the insertion tableaux obtained from w and w', respectively, under shuffle A. Then P'_A is a subtableau of P_A.*

Proof. Let $w \xrightarrow[A\text{-RSK}]{} P_A$; $P : \varnothing, P_1, P_2, \ldots, P_n = P_A$, and similarly let $w' \xrightarrow[A\text{-RSK}]{} P'_A$; $P' : \varnothing, P'_1, P'_2, \ldots, P'_m = P'_A \quad (m = |w'|)$.

Assume P'_i is a subtableau of P_{j_i} and insert (a corresponding) y in w.

If $x <_{_A} y$, y is not in w' so P'_i is not affected. Also, inserting y into P_{j_i}, y does not affect the subtableau $P'_i \subseteq P_{j_i}$, since y bumps only elements that are A-greater than itself.

A similar argument applies when $y \le x$: now y is also in w', and is inserted into P'_i and into P_{j_i}. Clearly, in P_{j_i} it is also inserted into the subtableau $P'_i \subseteq P_{j_i}$, and the proof follows. ∎

COROLLARY 2.7. *Let $A, B \in I$ be (t_i, u_j)-adjacent, $v \in a_{k,l}(n)$, $v \xrightarrow[A]{} (P_A, Q_A)$, and $v \xrightarrow[B]{} (P_B, Q_B)$. Then the elements that are both A-less and B-less than t_i and u_j form identical subtableaux in P_A and P_B.*

Proof. Denote by v' the sequence obtained by omitting from v all elements (A- and B-) greater than or equal to t_i and u_j. By (t_i, u_j)-adjacency, the largest element smaller than t_i and u_j, in both A and B, is the same element x. Moreover, v' is obtained by omitting from v all elements which are (A- or B-) greater than x. Let P'_A and P'_B denote the insertion tableaux of v' under shuffles A and B, respectively. Then, by Lemma 2.6, P'_A and P'_B are subtableaux of P_A and P_B, respectively. But the elements that are A- or B-less than t_i and u_j are ordered identically in A and B, so $P'_A = P'_B$. ∎

Notation. As above, let $A, B \in I$ be two shuffles that are (t_i, u_j)-adjacent: $t_i < u_j$ in A and $u_j < t_i$ in B. Let $v \in a_{k,l}(n)$ and denote $v \xrightarrow[A]{} (P_A, Q_A)$ and $v \xrightarrow[B]{} (P_B, Q_B)$.

Notation. Given the tableau P_A (and similarly for P_B), let regions 1, 2, and 3 denote, respectively, the regions occupied (1) by elements less than t_i and u_j, (2) by t_i and u_j, and (3) by elements greater than t_i and u_j.

EXAMPLE 2.8. Let $v = u_1 t_3 t_2 u_2 t_2 u_1 t_1$ and let

$$A = t_1 < u_1 < t_2 < u_2 < t_3,$$
$$B = t_1 < u_1 < u_2 < t_2 < t_3.$$

Then A and B are (t_i, u_j)-adjacent, with $t_i = t_2$ and $u_j = u_2$, and

$$P_A = \begin{array}{|c|c|c|}\hline t_1 & u_1 & t_2 \\ \hline u_1 & t_2 & u_2 \\ \hline t_3 \\ \cline{1-1}\end{array}, \qquad P_B = \begin{array}{|c|c|c|}\hline t_1 & u_1 & u_2 \\ \hline u_1 & t_2 & t_2 \\ \hline t_3 \\ \cline{1-1}\end{array}.$$

In both tableaux, region 1 contains the elements t_1 and u_1, region 2 contains t_2 and u_2, and region 3 contains t_3. Note that, in this example, regions 1 and 3 are the same in P_A as in P_B, and region 2 is identically shaped in P_A and P_B. We shall show that this is always true.

By Lemma 2.6, both region 1 and the union of regions 1 and 2 form subtableaux in P. It is easy to check that region 2 does not contain the configuration

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}.$$

If it does, assume $d = t_i$. Then $b = u_j$, so $u_j < t_i$, and $a \neq t_i, u_j$. Similarly if $d = u_j$. It follows that region 2 forms part of the rim of the subtableaux which is the union of regions 1 and 2.

Remark 2.9. Note that (part of) region 2 in P_A (i.e., $t_i < u_j$) always looks like

$$\begin{array}{llll} & & t_i \cdots\cdots t_i \\ & & u_j \\ & & \vdots \\ t_i \cdots\cdots & t_i & u_j \\ u_j \\ \vdots \\ u_j \end{array}$$

Namely, except possibly for the rightmost element, all other elements in a row are t_i's. Similarly, except for possibly the top element, all other elements in a column are u_j's.

Similarly, in P_B (i.e., $u_j < t_i$), part of region 2 looks like

$$
\begin{array}{llll}
 & & u_j t_i \cdots\cdots t_i \\
 & & \vdots \\
u_j t_i \cdots\cdots t_i & & \\
\vdots & & \\
u_j & &
\end{array}
$$

Denote $v = v_1 \cdots v_n$. The tableau P_A is created by applying the A-RSK insertion algorithm to each of $v_1, \ldots, v_n$ successively. For each v_m, let $l_{m(A)}$ denote the length of the insertion path [7, p. 317] of v_m under shuffle A—that is, the number of insertion steps that occur when v_m is inserted while forming P_A. The total number of insertion steps involved in the formation of P_A is thus $s_A = \sum_{m=1}^{n} l_{m(A)}$. For every $r \in \{1, \ldots, s_A\}$, let P_A^r be the insertion tableau as it appears immediately after insertion step r.

Similarly, under shuffle B, the length of the insertion path of v_m into P_B is $l_{m(B)}$, and the total number of insertion steps involved in forming P_B is $s_B = \sum_{m=1}^{n} l_{m(B)}$, with P_B^r denoting the insertion tableau after insertion step r.

EXAMPLE 2.10. As in Example 2.8, let $v = v_1 \cdots v_7 = u_1 t_3 t_2 u_2 t_2 u_1 t_1$ and let $A = t_1 < u_1 < t_2 < u_2 < t_3$. Then tableau P_A is formed by the A-RSK as follows (ignore the underlines):

$$
\begin{array}{|c|}\hline \underline{u_1} \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline u_1 & \underline{t_3} \\ \hline \end{array}
\qquad
\begin{array}{|c|c|}\hline u_1 & \underline{t_2} \\ \hline \underline{t_3} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|}\hline u_1 & t_2 \\ \hline \underline{u_2} \\ \cline{1-1} \underline{t_3} \\ \cline{1-1} \end{array}
$$

$$
\begin{array}{|c|c|c|}\hline u_1 & t_2 & \underline{t_2} \\ \hline u_2 \\ \cline{1-1} t_3 \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline u_1 & t_2 & t_2 \\ \hline \underline{u_1} & \underline{u_2} \\ \cline{1-2} t_3 \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline \underline{t_1} & \underline{u_1} & t_2 \\ \hline u_1 & \underline{t_2} & \underline{u_2} \\ \hline t_3 \\ \cline{1-1} \end{array}
$$

For all $i \in \{1, \ldots, 7\}$, the underlined elements in tableau i lie in the insertion path of element v_i. Thus $l_{1(A)} = l_{2(A)} = l_{5(A)} = 1$, $l_{3(A)} = l_{4(A)} = l_{6(A)} = 2$, $l_{7(A)} = 4$, and $s_A = \sum_{i=1}^{7} l_{i(A)} = 13$. If, for example, $r = 7 = \sum_{i=1}^{5} l_{i(A)}$, then we have

$$
P^r = \begin{array}{|c|c|c|}\hline u_1 & t_2 & t_2 \\ \hline u_2 \\ \cline{1-1} t_3 \\ \cline{1-1} \end{array}\,, \qquad
P^{r+1} = \begin{array}{|c|c|c|}\hline u_1 & t_2 & t_2 \\ \hline u_1 \\ \cline{1-1} t_3 \\ \cline{1-1} \end{array}\,.
$$

EXAMPLE 2.11. Let $k = l = 1$, $A : t < u$, $B : u < t$, $v = v_1 v_2 = tu$. Then

$$P_A : \varnothing, \quad \boxed{t}\,, \quad \begin{array}{|c|}\hline t \\ \hline \underline{u} \\ \hline \end{array}\,, \qquad l_{1(A)} = l_{2(A)} = 1,$$

$$P_B : \varnothing, \quad \boxed{t}\,, \quad \begin{array}{|c|}\hline u \\ \hline \underline{t} \\ \hline \end{array}\,, \qquad l_{1(B)} = 1, l_{2(B)} = 2.$$

DEFINITION 2.12. For $p, q \in \mathbb{Z}^+$, we say that $P_A^p \sim P_B^q$ (with respect to the formations of P_A and P_B) if:

(1) Regions 1 and 3 are identical in P_A^p and P_B^q.

(2) Region 2 is identically shaped in P_A^p and P_B^q; moreover, in each connected component of that region 2, the number of t_i's (hence of u_j's) in P_A^p equals the number of t_i's (hence of u_j's) in P_B^q.

(3) Either $p = s_A$ and $q = s_B$, or both $p < s_A$ and $q < s_B$. In the latter case, the next insertion step involves inserting the same element into the same row (or column) in both tableaux.

EXAMPLE 2.13. The tableaux of Example 2.8 satisfy $P_A \sim P_B$. Regions 1 and 3 in the two tableaux are identical, satisfying Definition 2.12(1). Region 2 consists of one component which is identically shaped and contains exactly one t_i and one u_j in both tableaux. This verifies Definition 2.12(2). Since both tableaux correspond to $p = s_A$ and $q = s_B$, Definition 2.12(3) is satisfied as well.

LEMMA 2.14. *For any shuffle $A \in I$ and for all $p \in \{2, \dots, s_A\}$ and $r, s \in \mathbb{Z}^+$, if $c(r, s)$ contains some w in P_A^{p-1}, then $c(r, s)$ contains some $z \leq_A w$ in P_A^p.*

Conversely, if $c(r, s)$ contains some element z in P_A^p, then $c(r, s)$ was either empty or contained some $w \geq_A z$ in P_A^{p-1}.

Proof. Follows from the A-RSK algorithm. ∎

The Proof of Theorem 2 clearly follows from the next result.

LEMMA 2.15. *Let $A, B \in I$ be (t_i, u_j)-adjacent, $v \in a_{k,l}(n)$, $v \xrightarrow[A\text{-RSK}]{} (P_A, Q_A)$, and $v \xrightarrow[B\text{-RSK}]{} (P_B, Q_B)$. Then $P_A \sim P_B$.*

Proof. We prove that $P_A \sim P_B$ by induction on the insertion steps of P_A and P_B. Trivially, $P_A^1 = P_B^1$. Now let $p \in \{1, \dots, s_A - 1\}$, $q \in \{1, \dots, s_B - 1\}$ and assume that (1) $P_A^p \sim P_B^q$ and also (2) $P_A^{p-1} \sim P_B^{q-1}$ or $P_A^{p-1} \sim P_B^{q-2}$ or $P_A^{p-2} \sim P_B^{q-1}$. We show that this implies that $P_A^{p+1} \sim P_B^{q+1}$ or $P_A^{p+1} \sim P_B^{q+2}$ or $P_A^{p+2} \sim P_B^{q+1}$. This clearly implies the proof of the lemma (by induction on $p + q$).

Note that if $P_A^p \sim P_B^q$, then, by Definition 2.12(3), step $p+1$ in P_A and step $q+1$ in P_B are identical; that is, the same element, x, is inserted into the same row (or column) in both tableaux. We assume that x is a t-element and therefore enters some row, denoted *row* r; the case where x is a u-element is analogous. Since $P_A^p \sim P_B^q$, row r is empty in P_A^p if and only if it is empty in P_B^q. The case where row r is empty is trivial, so we assume throughout that row r is nonempty in P_A^p and P_B^q.

Case 1. Suppose that, under both shuffles A and B, $x > t_i$ and u_j. Since $P_A^p \sim P_B^q$, the last nonempty cell in row r must be in the same region in both P_A^p and P_B^q, and if it is in region 3, then it must be occupied by the same element in both tableaux.

Case 1.1. Row r in P_A^p (and in P_B^q) terminates with an element less than or equal to x. In this case, x is affixed to the end of the row in both tableaux, so P_A^{p+1} and P_B^{q+1} have the same shape and clearly satisfy properties (1) and (2) of Definition 2.12. Let m denote the size of P_A^{p+1} and P_B^{q+1}. If $m = n$, which is the size of P_A and P_B, then the insertion algorithm terminates here. Otherwise, the next step is to begin v_{m+1}'s insertion path by inserting v_{m+1} into either the first row or the first column in both tableaux. This verifies Definition 2.12(3) and we have $P_A^{p+1} \sim P_B^{q+1}$.

Case 1.2. Row r in P_A^p contains an element $z > x$ (under both A and B). Since $P_A^p \sim P_B^q$, the same is true in P_B^q. In this case, x bumps an element greater than itself—a region-3 element—and occupies its cell in both tableaux. Thus both the cell occupied by x and the element bumped by x are identical in the two tableaux, which verifies Definition 2.12(3). Since Definition 2.12(1) and (2) clearly hold, it follows that $P_A^{p+1} \sim P_B^{q+1}$.

Case 2. Suppose that $x = t_i$. During step $P_B^q \to P_B^{q+1}$, $x = t_i >_B u_j$ bumps the first region-3 element in row r, or if no such element exists, x occupies the first empty cell in that row. Let $c(r, s)$ be the cell occupied by x in P_B^{q+1}.

Case 2.1. In row r of P_A^p, region 2 either terminates with t_i or does not appear at all in that row. Then x occupies $c(r, s)$ also in P_A^{p+1} (and bumps the same element as in P_B^{q+1}), so $P_A^{p+1} \sim P_B^{q+1}$.

Case 2.2. In P_A^p, the last region-2 element in row r is u_j. Let this u_j be in $c(r, s')$. Since $P_A^p \sim P_B^q$, $c(r, s')$ is the last region-2 cell in row r in both tableaux. Since, in $P_B^q \to P_B^{q+1}$, x was inserted into $c(r, s)$, we have $s = s' + 1$. Thus u_j is in $c(r, s-1)$ and is bumped by $x = t_i$ to column s during $P_A^p \to P_A^{p+1}$. We prove that, in such a case, $P_A^{p+2} \sim P_B^{q+1}$. To do so, we show that

2.2.1. In $P_A^{p+1} \to P_A^{p+2}$, u_j settles in $c(r, s)$, to the immediate right of x.

2.2.2. This implies that Definition 2.12(2) for $P_A^{p+2} \sim P_B^{q+1}$ is satisfied.

2.2.3. Both (1) and (3) of Definition 2.12 for $P_A^{p+2} \sim P_B^{q+1}$ are satisfied.

Proof of 2.2.1. If $r = 1$, then u_j clearly settles in $c(r, s)$ in P_A^{p+2}. We therefore assume that $r > 1$.

To prove that u_j settles in $c(r, s)$ in P_A^{p+2}, we need only to show that $c(r-1, s)$ in P_A^{p+1} contains an element $b \le u_j$, since $c(r, s)$ in P_A^{p+1} contains some element $z >_A u_j$. Now, since $r > 1$, $x = t_i$ arrived at row r in P_A^p (and similarly in P_B^q) after being bumped from row $r-1$ of P_A^{p-1}. Let $c(r-1, h)$ be the cell occupied by x in P_A^{p-1}, before it was bumped from row $r-1$.

$$P_A^{p-1} \xrightarrow[\text{from } c(r-1,\,h)]{x \text{ is bumped}} P_A^{p} \xrightarrow[\text{into } c(r,\,s-1)]{x \text{ is inserted}} P_A^{p+1} \xrightarrow[\text{into column } s]{u_j \text{ is inserted}} P_A^{p+2}.$$

Since x is inserted into $c(r, s-1)$ of P_A^{p+1}, Lemma 2.3 implies that $s - 1 \le h$. If $s \le h$, then $c(r-1, s)$ was occupied by an element less than or equal to x in P_A^{p-1}, and this continues to be true in P_A^p and P_A^{p+1}, which implies our claim (that $b \le u_j$). On the other hand, suppose that $h = s - 1$. In this case, we prove that $c(r-1, s)$ contains an element less than or equal to u_j by showing that otherwise we would have a contradiction to the induction assumptions. Our proof of this is illustrated by the following figures, each of which consists of the block of cells in rows $r-1$ and r and columns $s-1$ and s in the corresponding tableaux.

P_A^{p-1}:

		s
	x	b
r …		

,

P_A^{p}:

		s
		b
r …	u_j	

,

P_A^{p+1}:

		s
		b
r …	x	

,

P_B^{q-2}:

		s
		z
r …		

,

P_B^{q-1}:

		s
		e
r …		

.

So, assume that, in P_A^{p-1}, x occupied $c(r-1, s-1)$, and $c(r-1, s)$ was either empty or contained an element $b > u_j$ (a region-3 element). We show that this contradicts the claim of the induction hypothesis, that $P_A^{p-1} \sim P_B^{q-1}$, $P_A^{p-1} \sim P_B^{q-2}$, or $P_A^{p-2} \sim P_B^{q-1}$.

$$P_B^{q-2} \xrightarrow[\text{into } c(r-1,s)]{e \text{ is inserted}} P_B^{q-1} \xrightarrow[\text{from } c(r-1,s')]{x \text{ is bumped}} P_B^{q} \xrightarrow[\text{into } c(r,s)]{x \text{ is inserted}} P_B^{q+1}.$$

Now, in P_B^{q+1}, $x = t_i$ occupies $c(r, s)$, so $c(r-1, s)$ is occupied by an element less than x. Lemma 2.3 implies that, in P_B^q, x occupied $c(r-1, s')$, where $s \leq s'$, and this implies that, in P_B^{q-1}, $c(r-1, s)$ contained an element $e \leq x$, a region-1 or -2 element. But the same cell in P_A^{p-1} was either empty or contained a region-3 element, and by Lemma 2.14, the same must have been true in P_A^{p-2}. Thus the above assumption, that $h = s-1$, implies that neither $P_A^{p-1} \sim P_B^{q-1}$ nor $P_A^{p-2} \sim P_B^{q-1}$ is satisfied. We show now that it also implies that $P_A^{p-1} \nsim P_B^{q-2}$.

Suppose that $P_A^{p-1} \sim P_B^{q-2}$ is satisfied. Denote by m the size of P_A^{p-1} and P_B^{q-2}. By assumption, in P_A^{p-1}—hence also in $P_B^{q-2} \sim P_A^{p-1}$—x occupies $c(r-1, s-1)$ and $c(r-1, s)$ is either empty or contains a region-3 element $z > x$. But we saw above that, in P_B^{q-1}, $c(r-1, s)$ contained $e \leq x$. It follows that insertion step $P_B^{q-2} \to P_B^{q-1}$ consisted of e being inserted into $c(r-1, s)$ and—if $c(r-1, s)$ were previously occupied—of bumping from it some $z > x$. If e bumped some z, then, in $P_B^{q-1} \to P_B^q$, z would have been inserted into either row r or column $s+1$. But we saw earlier that x was bumped to row r in $P_B^{q-1} \to P_B^q$, which implies that z must have bumped x during this step. This leads to a contradiction, since z could not have bumped $x < z$. Thus, when e occupied $c(r-1, s)$ during $P_B^{q-2} \to P_B^{q-1}$, it did not bump any element; the cell was previously empty. By occupying an empty cell, e increased the size of the tableau from m to $m+1$: $|P_B^{q-2}| = m$, $|P_B^{q-1}| = m+1$. Hence $|P_B^q| \geq m+1$.

On the other hand, P_A^{p-1} is of size m, and during $P_A^{p-1} \to P_A^p$, x was bumped from row $r-1$ by some smaller element, so the size of the tableau did not change. It follows that $|P_A^p| = m < |P_B^q|$, contradicting $P_A^p \sim P_B^q$.

It follows that $c(r-1, s)$ in P_A^{p+1} contains an element $b \leq u_j$, and this implies that, when u_j enters column s in $P_A^{p+1} \to P_A^{p+2}$, it settles in $c(r, s)$, to the right of x. This completes the proof of 2.2.1.

Proof of 2.2.2. By 2.2.1, u_j settles in $c(r, s)$ in P_A^{p+2}. We show that this implies that Definition 2.12(2) for $P_A^{p+2} \sim P_B^{q+1}$ is satisfied. In the diagrams below the proof, the cells outside of region 2 are marked with a $\star$.

Recall that, by Case 2.2, u_j is located in $c(r, s-1)$ in P_A^p, so $c(r, s-1)$ is part of a connected component of region 2. Let τ be the number of t_i's and μ be the number of u_j's in this connected component. By Definition 2.12(2) of $P_A^p \sim P_B^q$, region 2 of P_B^q contains a corresponding connected component (in the same cells as that of P_A^p), with τ t_i's and μ u_j's.

Since u_j is located in $c(r, s-1)$ in P_A^p, by strict row inequality, $c(r, s)$ is either empty or in region 3. Similarly, since $x = t_i$ occupies $c(r, s-1)$ in $P_A^p \to P_A^{p+1}$, $c(r-1, s-1)$ must contain a region-1 element in P_A^{p+1} (and in P_A^p):

$$P_A^p : \quad \begin{array}{rcc} & & s \\ & & \vdots \\ & \multicolumn{2}{c}{\begin{array}{|c|c|}\hline \star & \\ \hline u_j & \star \\ \hline \end{array}} \end{array} \quad (\text{row } r \cdots \text{ at the lower row}).$$

Thus, if $c(r-1, s)$ is in region 2 in P_A^p and P_B^q, then in both tableaux it is part of a connected component distinct from that of $c(r, s-1)$. This other component consists of τ' t_i's and μ' u_j's. (If $c(r-1, s)$ is not in region 2, then we let $\tau' = \mu' = 0$.)

Since $c(r, s)$ is either empty or in region 3 in P_A^p (and thus in P_B^q), it follows that the same is true for $c(r, s+1)$ and $c(r+1, s)$. By the assumption at the beginning of Case 2, during $P_B^q \to P_B^{q+1}$, $x = t_i$ bumps some region-3 element z from $c(r, s)$, thereby adding $c(r, s)$ to region 2. But $c(r, s)$ is adjacent to both $c(r, s-1)$ and $c(r-1, s)$, so when it joins region 2, it combines their respective connected components into a single larger one. Since neither $c(r, s+1)$ nor $c(r+1, s)$ is in region 2, it follows that, in P_B^{q+1}, $c(r, s)$ becomes part of a connected component of region 2, consisting of $\tau + \tau' + 1$ t_i's and $\mu + \mu'$ u_j's:

$$P_B^{q+1} : \quad \begin{array}{ccc} & s & \\ & \vdots & \end{array} \atop r \cdots \begin{array}{|c|c|c|}\hline \star & & \\ \hline & x & \star \\ \hline & \star & \\ \hline \end{array}.$$

Similarly, during $P_A^p \to P_A^{p+1} \to P_A^{p+2}$, $x = t_i$ bumps u_j from $c(r, s-1)$, and u_j bumps z from $c(r, s)$, so the only change in the shape of region 2 in P_A^{p+2} is the addition of $c(r, s)$. Thus $c(r, s)$ of P_A^{p+2} is part of a connected component of region 2, also containing $\tau + \tau' + 1$ t_i's and $\mu + \mu'$ u_j's, and this component is identically shaped to the corresponding component of P_B^{q+1}, so Definition 2.12(2) of $P_A^{p+2} \sim P_B^{q+1}$ is satisfied. This completes the proof of 2.2.2.

Proof of 2.2.3. Definition 2.12(1) is satisfied for $P_A^{p+2} \sim P_B^{q+1}$, since, in both $P_B^q \to P_B^{q+1}$ and $P_A^p \to P_A^{p+1} \to P_A^{p+2}$, region 1 is unchanged, and the only change in region 3 is the elimination of $c(r, s)$.

Since the same element z is bumped from $c(r, s)$ in $P_A^{p+1} \to P_A^{p+2}$ and $P_B^q \to P_B^{q+1}$, Definition 2.12(3) is satisfied, and the proof of 2.2.3 is complete.

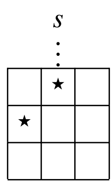
It follows that $P_A^{p+2} \sim P_B^{q+1}$.

Case 3. Suppose that $x = t_a < t_i$.

Case 3.1. Row r in P_A^p terminates with $z \leq x$. Thus z is in region 1, so, by Definition 2.12(1) of $P_A^p \sim P_B^q$, row r in P_B^q terminates with z. In such a case, x is affixed to the end of row r in both tableaux, so P_A^{p+1} and P_B^{q+1} have the same shape, and clearly satisfy Definition 2.12(1) and (2). Let m denote the size of P_A^{p+1} and P_B^{q+1}. If $m = n$, which is the size of P_A and P_B, then the insertion algorithm terminates here. Otherwise, the next step is to begin v_{m+1}'s insertion path by inserting v_{m+1} into either the first row or the first column in both tableaux. This verifies Definition 2.12(3) and we have $P_A^{p+1} \sim P_B^{q+1}$.

Case 3.2. Row r in P_A^p (and hence in P_B^q) contains an element greater than x. In both tableaux, x bumps from row r the leftmost element greater than itself. By Definition 2.12(1) and (2) of $P_A^p \sim P_B^q$, the same cell—denoted $c(r, s)$—becomes occupied by x in both tableaux. Thus, if the element bumped by x is identical in the two tableaux, then $P_A^{p+1} \sim P_B^{q+1}$.

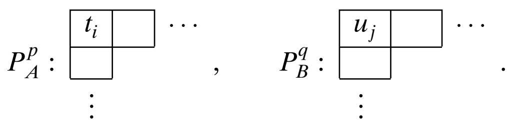
Suppose, however, that x bumps different elements from the cell(s) $c(r, s)$ of P_A^p and P_B^q. By $P_A^p \sim P_B^q$, these must be t_i and u_j. Since $x = t_a$ occupies $c(r, s)$ in P_A^{p+1}, Lemma 2.3 implies that, in P_A^{p-1}, x occupied $c(r-1, s')$ with $s \leq s'$. Thus the $c(r-1, s)$ element g in P_A^{p-1} was $g \leq x < u_j, t_i$, so $c(r-1, s)$ was a region-1 cell. Since x was subsequently bumped from row $r-1$ by an element smaller than itself, it follows that $c(r-1, s)$ is a region-1 cell also in P_A^p (and P_B^q). Similarly, since $x = t_a < t_i$ settles in $c(r, s)$ in $P_A^p \to P_A^{p+1}$, $c(r, s-1)$ is a region-1 cell in P_A^{p+1} (and P_B^{q+1}), and in P_A^p (and P_B^q),

$P_A^p, P_B^q :$ $r \ldots$  .

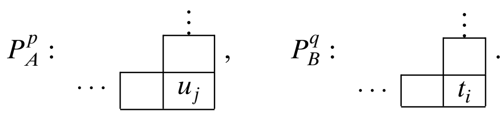
(the stars represent region-1 elements).

Now, since $c(r, s)$ is in region 2 in both P_A^p and P_B^q, by Definition 2.12(2), it is part of a connected component of region 2 which is identically shaped and contains the same number of t_i's and u_j's in both tableaux. But $c(r, s)$ contains t_i in one tableau and u_j in the other, so it follows that at least one of $c(r, s+1)$ and $c(r+1, s)$ is in region 2 in P_A^p and P_B^q. By strict row and column inequality, this implies that $c(r, s)$ contains t_i in P_A^p and u_j in P_B^q.

Denote by C the connected component of region 2 containing $c(r, s)$. Consider the subcomponent C_1, consisting of all cells in C which are to the right of or above $c(r, s)$. In P_A^p, let $\alpha_A = \#t_i$'s and $\beta_A = \#u_j$'s in C_1; define α_B and β_B similarly in P_B^q. If $c(r, s+1)$ is not in region 2, then C_1 is empty and $\alpha_A = \beta_A = \alpha_B = \beta_B = 0$. On the other hand, if C_1 is nonempty, then by strict row and column inequality, every northwest proper corner cell of C_1 contains t_i in P_A^p and u_j in P_B^q:



Similarly, every southeast proper corner cell of C_1 contains u_j in P_A^p and t_i in P_B^q:



Consider the top row of C_1. If it contains more than one cell, then its leftmost cell is a northwest corner. Thus the structure of C_1 is as in the following diagram, where, for example, a cell marked t_i/u_j contains t_i in P_A^p and u_j in P_B^q (a question mark denotes that a cell may contain either t_i or u_j) and elements:

$$\begin{array}{ll} & t_i/u_j \cdots\cdots ?/t_i \\ & \vdots \\ \cdots\cdots u_j/t_i & \end{array}$$

On the other hand, if the top row of C_1 contains only one cell, then the structure of C_1 is

$$\begin{array}{lll} & & ?/u_j \\ & & \vdots \\ & t_i/u_j \cdots\cdots & u_j/t_i \\ & \vdots & \\ \cdots\cdots u_j/t_i & & \end{array}$$

In both cases, it follows that $\alpha_B - \alpha_A = \beta_A - \beta_B \in \{0, 1\}$. We prove that

3.2.1. $\alpha_B - \alpha_A = \beta_A - \beta_B = 1 \Longrightarrow P_A^{p+1} \sim P_B^{q+2}$.

3.2.2. $\alpha_B - \alpha_A = \beta_A - \beta_B = 0 \Longrightarrow P_A^{p+2} \sim P_B^{q+1}$.

Let C_2 be the subcomponent of C consisting of all cells below or to the left of $c(r, s)$. Let $\gamma_A = \#t_i$'s and $\delta_A = \#u_j$'s in C_2 of P_A^p; define γ_B and δ_B similarly for P_B^q. Since neither $c(r-1, s)$ nor $c(r, s-1)$ is in region 2, it follows that $C = C_1 + C_2 + c(r, s)$. By Definition 2.12(2) of $P_A^p \sim P_B^q$, C contains the same number of t_i's and u_j's in P_A^p as in P_B^q. In both tableaux, let τ be the number of t_i's and let μ be the number of u_j's in C. Since $c(r, s)$ contains t_i in P_A^p and u_j in P_B^q, it follows that

$$\tau = \alpha_A + \gamma_A + 1 = \alpha_B + \gamma_B, \qquad \mu = \beta_A + \delta_A = \beta_B + \delta_B + 1. \qquad (\star)$$

Proof of 3.2.1. Suppose that $\alpha_B - \alpha_A = \beta_A - \beta_B = 1$. Then $\gamma_A - \gamma_B = \delta_B - \delta_A = 0$, so C_2 is either empty or contains an equal number of t_i's and u_j's in P_A^p as in P_B^q. Also, C_1 is nonempty, so $c(r, s+1)$ is in region 2 in P_A^p and in P_B^q. In P_B^q, $c(r, s)$ contains u_j, so, by strict row inequality, $c(r, s+1)$ contains t_i. The subsequent insertion steps are therefore

$$P_A^p \xrightarrow[\text{from } c(r,s)]{x \text{ bumps } t_i} P_A^{p+1} \xrightarrow[\text{row } r+1]{t_i \text{ enters}},$$

$$P_B^q \xrightarrow[\text{from } c(r,s)]{x \text{ bumps } u_j} P_B^{q+1} \xrightarrow[\text{from } c(r,s+1)]{u_j \text{ bumps } t_i} P_B^{q+2} \xrightarrow[\text{row } r+1]{t_i \text{ enters}}.$$

Thus, in both $P_A^p \to P_A^{p+1}$ and $P_B^q \to P_B^{q+1} \to P_B^{q+2}$, $c(r, s)$ is eliminated from region 2, and we are left with two separate components C_1 and C_2 (and with t_i to be inserted into row $r+1$). No change occurs in C_2, so, in both P_A^{p+1} and P_B^{q+2}, C_2 has $\gamma_A = \gamma_B$ t_i's and $\delta_A = \delta_B$ u_j's. Similarly, no change occurs in C_1 in $P_A^p \to P_A^{p+1}$, so C_1 of P_A^{p+1} contains α_A t_i's and β_A u_j's. On the other hand, in $P_B^q \to P_B^{q+1} \to P_B^{q+2}$, a single change occurs in C_1, when the t_i in $c(r, s+1)$ is replaced with u_j. Thus C_1 of P_B^{q+2} contains $\alpha_B - 1$ t_i's and $\beta_B + 1$ u_j's. But, by 3.2.1, $\alpha_B - 1 = \alpha_A$ and $\beta_B + 1 = \beta_A$, so C_1 contains the same number of t_i's and u_j's in P_A^{p+1} as in P_B^{q+2}, and Definition 2.12(2) is satisfied for $P_A^{p+1} \sim P_B^{q+2}$.

Now, in both P_A^{p+1} and P_B^{q+2}, the only change that occurs in region 1 is that the same element x is added to $c(r, s)$, so Definition 2.12(1) is satisfied. Similarly, as was already mentioned, both $P_A^{p+1} \to P_A^{p+2}$ and $P_B^{q+2} \to P_B^{q+3}$ consist of t_i entering row $r+1$, so Definition 2.12(3) is satisfied. It follows that $P_A^{p+1} \sim P_B^{q+2}$. This completes the proof of 3.2.1.

Proof of 3.2.2. The proof of 3.2.2 is dual, in a sense, to the proof of 3.2.1. Here are the details.

Suppose that $\alpha_B - \alpha_A = \beta_A - \beta_B = 0$. Then C_1 is either empty or contains an equal number of t_i's and u_j's in P_A^p as in P_B^q. Thus, by ($\star$), $\gamma_B - \gamma_A = \delta_A - \delta_B = 1$, so, C_2 is nonempty, which implies that $c(r+1, s)$ is in region 2 in P_A^p and in P_B^q. In P_A^p, $c(r, s)$ contains t_i, so, by strict column inequality, $c(r+1, s)$ contains u_j. The subsequent insertion steps are therefore

$$P_A^p \xrightarrow[\text{from } c(r,s)]{x \text{ bumps } t_i} P_A^{p+1} \xrightarrow[\text{from } c(r+1,s)]{t_i \text{ bumps } u_j} P_A^{p+2} \xrightarrow[\text{column } s+1]{u_j \text{ enters}},$$

$$P_B^q \xrightarrow[\text{from } c(r,s)]{x \text{ bumps } u_j} P_B^{q+1} \xrightarrow[\text{column } s+1]{u_j \text{ enters}} .$$

Thus, in both $P_A^p \to P_A^{p+1} \to P_A^{p+2}$ and $P_B^q \to P_B^{q+1}$, $c(r, s)$ is eliminated from region 2, and we are left with two separate components C_1 and C_2 (and with u_j to be inserted into column $s+1$). No change occurs in C_1, so, in both P_A^{p+2} and P_B^{q+1}, C_1 has $\alpha_A = \alpha_B$ t_i's and $\beta_A = \beta_B$ u_j's. Similarly, no change occurs in C_2 in $P_B^q \to P_B^{q+1}$, so C_2 of P_B^{q+1} contains γ_B t_i's and δ_B u_j's. On the other hand, in $P_A^p \to P_A^{p+1} \to P_A^{p+2}$, a single change occurs in C_2, when the u_j in $c(r+1, s)$ is replaced with t_i. Thus C_2 of P_A^{p+2} contains $\gamma_A + 1$ t_i's and $\delta_A - 1$ u_j's. But 3.2.2 and ($\star$) imply that $\gamma_A + 1 = \gamma_B$ and $\delta_A - 1 = \delta_B$, so C_2 contains the same number of t_i's and u_j's in P_A^{p+2} as in P_B^{q+1}, and Definition 2.12(2) is satisfied for $P_A^{p+2} \sim P_B^{q+1}$.

Now, in both P_A^{p+2} and P_B^{q+1}, the only change that occurs in region 1 is that the same element x is added to $c(r, s)$, so Definition 2.12(1) is satisfied. Similarly, as was already mentioned, both $P_A^{p+2} \to P_A^{p+3}$ and $P_B^{q+1} \to P_B^{q+2}$ consist of u_j entering column $s+1$, so Definition 2.12(3) is satisfied. It follows that $P_A^{p+2} \sim P_B^{q+1}$. This completes the proof of 3.2.2. ∎

3. PROOF OF THEOREM 5

Here we prove, for example, Theorem 5(b). The proofs of parts (c) and (d) of the theorem are similar.

Given $v \in a_{k,l}(n)$ and shuffle A, the (regular, dual)-A-RSK forms the tableau pair $(P^*, Q^*) = (P^*(v, A), Q^*(v, A))$ by applying the regular RSK to the t_i's and the dual conjugate RSK to the u_j's of v under shuffle A. For simplicity, we refer to this algorithm as the dual-A-RSK. As in the A-RSK, P^* is the insertion tableau, and Q^* is the recording tableau of v under A. Here P^* is what we call a dual-A-SSYT; that is, it is weakly A-increasing in rows and strictly A-increasing in columns.

EXAMPLE 3.1. Let $k = 2$, $l = 1$, and $A : u_1 < u_2 < t_1 < t_2$. Let

$$v = \begin{pmatrix} 1 \cdots\cdots 4 \\ u_1, t_1, t_2, u_1 \end{pmatrix}.$$

Then

$$v \xrightarrow[\text{dual-}A\text{-RSK}]{} \begin{array}{|c|}\hline u_1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline u_1 & t_1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline u_1 & t_1 & t_2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline u_1 & u_1 & t_2 \\ \hline t_1 \\ \cline{1-1} \end{array} = P^*$$

and

$$Q^* = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

LEMMA 3.2. *Let $v \in a_{k,l}(n)$, $A \in I$, and*

$$v \xrightarrow[A\text{-RSK}]{} (P, Q), \qquad v \xrightarrow[\text{dual-}A\text{-RSK}]{} (P^*, Q^*).$$

If v is nonrepeating in its u-elements, then $P = P^*$ and $Q = Q^*$.

Proof. The A-RSK and the dual-A-RSK differ in only one rule: When some u_j enters a column under the A-RSK, it bumps the first element w_m such that $w_m > u_j$ (or if no such w_m exists, it settles at the end of the column). On the other hand, under the dual-A-RSK, u_j bumps the first element w_r such that $w_r \geq u_j$ (or settles at the end of the column). But u_j may appear only once in v, which implies that $w_r > u_j$, so this step is the same as that of the A-RSK. The proof now follows. ∎

Notation. $v \in a_{k,l}(n)$ is said to be of type $(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l)$ if it is a permutation of $t_1^{\alpha_1} \cdots t_k^{\alpha_k} u_1^{\beta_1} \cdots u_l^{\beta_l}$.

LEMMA 3.3. *Let $v \in a_{k,l}(n)$ be of type $(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l)$ and denote $\beta = \sum_{i=1}^{l} \beta_i$. Then there exists $w \in a_{k,\beta}(n)$ such that*

(1) *The u-elements of w are nonrepeating.*

(2) *For every shuffle A, if $v \xrightarrow[\text{dual-}A\text{-RSK}]{} (P_v^*, Q_v^*)$, then there exists a corresponding shuffle A' of the elements of w such that $w \xrightarrow[\text{dual-}A\text{-RSK}]{} (P_w^*, Q_w^*)$, where P_w^* is identical to P_v^* but with every v_i changed to w_i for all $i \leq n$. Consequently,* $\mathrm{sh}(P_v^*) = \mathrm{sh}(P_w^*)$.

Proof. To avoid confusion between the elements of v and of w, we let $u'_1, \ldots, u'_l$ denote the u-elements of v.

Form the sequence w from v as follows. Replace the u'_1's in v with $u_1, \ldots, u_{\beta_1}$, moving from right to left. Replace the u'_2's with $u_{\beta_1+1}, \ldots, u_{\beta_1+\beta_2}$, moving from right to left. Continue in this way until u'_l, including the u'_l's.

Clearly, the u-elements of w are nonrepeating, satisfying Lemma 3.3(1).

Given some shuffle A of the elements of v, define the shuffle A' of the elements of w as follows. For every $i \in \{1, \ldots, k\}$,

$$\begin{aligned}
t_i <_A u'_1 &\Longrightarrow t_i <_{A'} u_1 <_{A'} \cdots <_{A'} u_{\beta_1}, \\
t_i <_A u'_2 &\Longrightarrow t_i <_{A'} u_{\beta_1+1} <_{A'} \cdots <_{A'} u_{\beta_1+\beta_2}, \\
&\vdots \\
t_i <_A u'_l &\Longrightarrow t_i <_{A'} u_{\beta_1+\cdots+\beta_{l-1}+1} <_{A'} \cdots <_{A'} u_{\beta}, \\
u'_1 <_A t_i &\Longrightarrow u_1 <_{A'} \cdots <_{A'} u_{\beta_1} <_{A'} t_i, \\
&\vdots \\
u'_l <_A t_i &\Longrightarrow u_{\beta_1+\cdots+\beta_{l-1}+1} <_{A'} \cdots <_{A'} u_\beta <_{A'} t_i.
\end{aligned}$$

We compare the A-RSK insertion of the v's with the A'-RSK insertion of the w's. Note that the shuffle A and its derived shuffle A' are similar in that $v_i <_A v_j \Longrightarrow w_i <_{A'} w_j$, but they differ in one fundamental way: For $i < j$ such that v_i, v_j, w_i, and w_j are u-elements, $v_i =_A v_j \Longrightarrow w_i >_{A'} w_j$. Now, if w_j reaches a cell inhabited by $w_i >_{A'} w_j$, then it bumps w_j to the next column, just as v_j would bump $v_i =_A v_j$ to the next column under the dual-A-RSK. On the other hand, if w_i reaches a cell inhabited by $w_j <_{A'} w_i$, it settles below w_j, whereas v_i would bump $v_j =_A v_i$ to the next column. However, such a situation never occurs, since $i < j$ and $v_i =_A v_j$ implies that every column reached by w_j is first reached by w_i. The proof of this is as follows.

Suppose that, for some x, $w_i = u_{x+1}$ and $w_j = u_x$. Then every column reached by w_j is first reached by w_i, by induction on the columns of P^*_w. Trivially, w_i reaches column 1 before w_j. By the induction assumption, $w_i = u_{x+1}$ is in column c', $c' \geq c$. If $c' > c$, then we are done. Assume $c' = c : w_i = u_{x+1}$ is already in column c, and $w_j = u_x$ is inserted into column c. It bumps the first w_d such that $w_d \geq w_j = u_x$. Now $v_i =_A v_j$ implies that there does not exist any t_z such that $w_j <_{A'} t_z <_{A'} w_i$. Hence $w_d = u_{x+1} = w_i$ is bumped to column $c+1$.

This clearly extends to the general case $i < j$, $v_i = v_j$, $w_i = u_y$, $w_j = u_x$, for general $y > x$.

Hence the steps of the dual-A'-RSK on w are identical to the steps of the dual-A-RSK on v, but with every v_i, $i \leq n$, changed to w_i. This implies that Lemma 3.3(2) is satisfied for w. ∎

EXAMPLE 3.4. Let $v = t_2 u_2 u_1 u_1 t_1$ and $A = t_1 < t_2 < u_1 < u_2$. The sequence $w = t_2 u'_3 u'_2 u'_1 t_1$ clearly satisfies Lemma 3.3(1); we show that it satisfies Lemma 3.3(2) for shuffle A, by letting $A' = t_1 < t_2 < u'_1 < u'_2 < u'_3$.

Under shuffles A and A',

$$v \xrightarrow[A\text{-RSK}]{} (P_v^*, Q_v^*) \quad \text{and} \quad w \xrightarrow[\text{dual-}A'\text{-RSK}]{} (P_w^*, Q_w^*),$$

where

$$P_v^* = \begin{array}{|c|c|c|c|}\hline t_1 & u_1 & u_1 & u_2 \\ \hline t_2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline v_5 & v_4 & v_3 & v_2 \\ \hline v_1 \\ \cline{1-1}\end{array},$$

$$P_w^* = \begin{array}{|c|c|c|c|}\hline t_1 & u_1' & u_2' & u_3' \\ \hline t_2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline w_5 & w_4 & w_3 & w_2 \\ \hline w_1 \\ \cline{1-1}\end{array}.$$

Thus Lemma 3.3(2) is satisfied for shuffle A.

We can now give the following proof.

Proof of Theorem 5(b). Let v be of type $(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l)$ and denote $\beta = \sum_{i=1}^{l} \beta_i$. Lemma 3.3 implies that there exists a sequence $w \in a_{k,\beta}(n)$ with no repeating u-elements and with shuffles A', B' such that

$$w \xrightarrow[\text{dual-}A'\text{-RSK}]{} (P_{A'}^*, Q_{A'}^*), \qquad w \xrightarrow[\text{dual-}B'\text{-RSK}]{} (P_{B'}^*, Q_{B'}^*),$$

where $\text{sh}(P_{A'}^*) = \text{sh}(P_A^*)$ and $\text{sh}(P_{B'}^*) = \text{sh}(P_B^*)$. Since w contains no repetitions in its u-elements, Lemma 3.3 implies that

$$w \xrightarrow[A'\text{-RSK}]{} (P_{A'}^*, Q_{A'}^*), \qquad w \xrightarrow[B'\text{-RSK}]{} (P_{B'}^*, Q_{B'}^*).$$

Thus, by Theorem 2, $\text{sh}(P_{A'}^*) = \text{sh}(P_{B'}^*)$, which implies our result. ∎

The proofs of parts (c) and (d) of Theorem 5 are similar to that of Theorem 5(b), since Lemma 3.3 can also be applied to the (dual, regular)-A-RSK and the (dual, dual)-A-RSK. Both algorithms are t-dual; for simplicity, let $t_1', \ldots, t_k'$ denote the t-elements of v. The t's of the sequence w of Lemma 3.3 for parts (c) and (d) are set as follows. Replace the t_1''s in v with $t_1, \ldots, t_{\alpha_1}$, moving from left to right. Replace the t_2''s with $t_{\alpha_1+1}, \ldots, t_{\alpha_1+\alpha_2}$, moving from left to right. Continue in this way until t_k', including t_k'.

Since the (dual, regular)-A-RSK of part (c) is u-regular, the u's of w are identical to those of v. However, the (dual, dual)-A-RSK of part (d) is u-dual, so, in this case, the u's of w are derived the same way as in the proof of Lemma 3.3. Finally, shuffle A' is derived from A in parts (c) and (d) by methods analogous to that of part (b).

REFERENCES

1. A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* **64** (1987), 118–175.
2. A. Berele and J. B. Remmel, Hook flag characters and combinatorics, *J. Pure Appl. Algebra* **35** (1985), 245.
3. G. Olshanski, A. Regev, and A. Vershik, "Frobenius–Schur Functions," Studies in Memory of I. Schur, Birkhauser, Boston, in press.
4. J. B. Remmel, The combinatorics of (k, l)-hook Schur functions, *Contemp. Math.* **34** (1984), 253–287.
5. J. B. Remmel, Permutation statistics and (k, l)-hook Schur functions, *Discrete Math.* **67** (1987), 271–298.
6. J. B. Remmel, A bijective proof of a factorization theorem for (k, l)-hook Schur functions, *Linear and Multilinear Algebra* **28** (1990), 119–154.
7. R. Stanley, "Enumerative Combinatorics," Vol. 2, Cambridge Univ. Press, Cambridge, UK, 1999.