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Note

Box complexes, neighborhood complexes, and the chromatic number

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Abstract

Lovász's striking proof of Kneser's conjecture from 1978 using the Borsuk–Ulam theorem provides a lower bound on the chromatic number $\chi(G)$ of a graph G. We introduce the *shore subdivision* of simplicial complexes and use it to show an upper bound to this topological lower bound and to construct a strong $\mathbb{Z}_2$-deformation retraction from the box complex (in the version introduced by Matoušek and Ziegler) to the Lovász complex. In the process, we analyze and clarify the combinatorics of the complexes involved and link their structure via several "intermediate" complexes.
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1. Introduction

The topological method in graph theory was introduced by Lovász [6] to prove Kneser's conjecture [4]. The pattern to obtain a lower bound of the chromatic

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number $\chi(G)$ of a graph G is to associate a topological space and bound the chromatic number by a topological invariant of this space, e.g. connectivity or $\mathbb{Z}_2$-index. In Section 3, we present a subdivision technique that we call *shore subdivision*. We use the shore subdivision in Section 4 to show that the complex $\mathsf{L}(G)$ which Lovász used (and which we call Lovász complex for that reason) is a strong $\mathbb{Z}_2$-deformation retract of the shore subdivision of the box complex $\mathsf{B}(G)$ described by Matoušek and Ziegler [8]. Moreover, we explicitly realize (in a non-unique way) the Lovász complex $\mathsf{L}(G)$ as a $\mathbb{Z}_2$-subcomplex of the shore subdivision $\mathrm{ssd}(\mathsf{B}(G))$ of $\mathsf{B}(G)$. The advantage of the box complex is that for any graph homomorphism $f: G \to H$ one obtains an induced simplicial $\mathbb{Z}_2$-map $\mathsf{B}(f): \mathsf{B}(G) \to \mathsf{B}(H)$. This functorial property gives elegant conceptual proofs which is not the case for the Lovász complex. Walker [10] gives a rather involved construction of a non-canonical $\mathbb{Z}_2$-map $\varphi: ||\mathsf{L}(G)|| \to ||\mathsf{L}(H)||$. Using the realization of the Lovász complex as a subcomplex of $\mathrm{ssd}(\mathsf{B}(G))$, such a map can be constructed in a non-canonical but straight-forward manner.

The box complex of a graph yields a lower bound for its chromatic number: $\chi(G) \geqslant \mathrm{ind}(\mathsf{B}(G)) + 2$. It is known that this topological bound can get arbitrarily bad: Walker [10] shows that if a graph G does not contain a $\mathrm{K}_{2,2}$ then the associated invariant yields 3 as largest possible lower bound for the chromatic number $\chi(G)$. In Section 5, we generalize this result to the following statement: If G does not contain a complete bipartite graph $\mathrm{K}_{\ell,m}$ then the index of the box complex $\mathsf{B}(G)$ is bounded by $\ell + m - 3$ and this bound is sharp.

Independently from this work, de Longueville [3] used shore subdivisions to give a short and elegant proof of the fact that Bier spheres are in fact spheres.

2. Preliminaries

In this section, we recall some basic facts of graphs and simplicial complexes to fix notation. The interested reader is referred to [7] or [2] for details.

Graphs: Any graph G is assumed to be finite, simple, connected, and undirected, i.e., G is given by a finite set $V(G)$ of *nodes* (we use *vertices* for associated complexes) and a set of *edges* $\mathrm{E}(G) \subseteq \binom{V(G)}{2}$. A *graph homomorphism* f between two graphs G and H is a map that maps nodes to nodes and edges to edges. A *proper graph coloring with n colors* is a graph homomorphism $c: G \to \mathrm{K}_n$, where K_n is the complete graph on n nodes. The *chromatic number* $\chi(G)$ of G is the smallest n such that a proper graph coloring of G with n colors exists. The *neighborhood* $\mathrm{N}(u)$ of $u \in V(G)$ is the set of all nodes adjacent to u. For a set of nodes $A \subseteq V(G)$ a node v is in the *common neighborhood* $\mathrm{CN}(A)$ of A, if v is adjacent to all $a \in A$; we define $\mathrm{CN}(\emptyset) := \mathrm{V}(G)$. For $A \subseteq B \subseteq V(G)$ the common neighborhood relation satisfies

$$A \cap \mathrm{CN}(A) = \emptyset, \quad \mathrm{CN}(B) \subseteq \mathrm{CN}(A), \quad A \subseteq \mathrm{CN}^2(A), \quad \text{and} \quad \mathrm{CN}(A) = \mathrm{CN}^3(A).$$

Because of the last equality we call CN^2 a *closure operator*. For two disjoint sets of nodes $A, B \subseteq V(G)$ we define $G[A; B]$ as the (not necessarily induced) subgraph of G

with node set $\mathrm{V}(G[A;B]) = A \cup B$ and all edges $\{a,b\} \in \mathrm{E}(G)$ with $a \in A$ and $b \in B$. For a given node set A, the set $\mathrm{CN}(A)$ is the inclusion-maximal set B such that $G[A;B]$ is complete bipartite.

Simplicial complexes: An *abstract simplicial complex* K is a finite hereditary set system. We denote its vertex set by $\mathrm{V}(\mathsf{K})$ and its barycentric subdivision by $\mathrm{sd}(\mathsf{K})$. For sets A, B define

$$A \uplus B := \{(a,0) \mid a \in A\} \cup \{(b,1) \mid b \in B\}.$$

An important construction in the category of simplicial complexes is the *join operation*. For two simplicial complexes K and L the join $\mathsf{K} * \mathsf{L}$ is defined as

$$\mathsf{K} * \mathsf{L} := \{F \uplus G \mid F \in \mathsf{K} \text{ and } G \in \mathsf{L}\}.$$

Any abstract simplicial complex K can be realized as a topological space $||\mathsf{K}||$ in $\mathbb{R}^d$ for some d.

$\mathbb{Z}_2$-spaces: A *$\mathbb{Z}_2$-space* is a topological space X together with a homeomorphism $v : X \to X$ that is self-inverse and free, i.e., has no fixed points. The map v is called *free $\mathbb{Z}_2$-action*. The fundamental example for a $\mathbb{Z}_2$-space is the d-sphere S^d together with the antipodal map $v(x) = -x$. A continuous map f between $\mathbb{Z}_2$-spaces (X, v) and (Y, μ) is *$\mathbb{Z}_2$-equivariant* (or a *$\mathbb{Z}_2$-map* for simplicity) if f commutes with the $\mathbb{Z}_2$-actions, i.e., if $f \circ v = \mu \circ f$. A simplicial complex (K, v) is a *simplicial $\mathbb{Z}_2$-space* if $v :\ \mathsf{K} \to \mathsf{K}$ is a simplicial map such that $||v||$ is a free $\mathbb{Z}_2$-action on $||\mathsf{K}||$. A *simplicial $\mathbb{Z}_2$-equivariant* map f is a simplicial map between two simplicial $\mathbb{Z}_2$-spaces that commutes with the simplicial $\mathbb{Z}_2$-actions.

The *index* of a $\mathbb{Z}_2$-space (X, v) is the smallest d such that there is a $\mathbb{Z}_2$-map $f : X \to S^d$, i.e., $f \circ v = -f$. The Borsuk–Ulam theorem states that there is no antipodal continuous mapping $f : S^d \to S^{d-1}$. Hence, it provides the index for spheres: $\mathrm{ind}(S^d) = d$. Since the $\mathbb{Z}_2$-actions are usually canonical, we often refer to a $\mathbb{Z}_2$-space K without explicit reference to v.

Chain notation: We denote by $\mathcal{A}$ a chain $A_1 \subset \cdots \subset A_p$ of subsets of the nodes $V(G)$ of a graph G and by $\mathcal{B}$ a chain $B_1 \subset \cdots \subset B_q$ of subsets of $V(G)$. For $1 \leqslant t \leqslant p$ we denote by $\mathcal{A}_{\leqslant t}$ the chain $A_1 \subset \cdots \subset A_t$. A similar convention is used for $\mathcal{A}_{\geqslant t}$. For chains $\mathcal{A}$, $\mathcal{B}$ satisfying $A_p \subseteq B_1$ we define a new chain, the *concatenation of $\mathcal{A}$ and $\mathcal{B}$*:

$$\mathcal{A} \sqsubset \mathcal{B} := A_1 \subset \cdots \subset A_p \subseteq B_1 \subset \cdots \subset B_q,$$

where we omit A_p or B_1 in case $A_p = B_1$. If a map f preserves (resp. reverses) inclusions, we write $f(\mathcal{A})$ instead of $f(A_1) \subseteq \ldots \subseteq f(A_p)$ (resp. $f(A_p) \subseteq \ldots \subseteq f(A_1)$). One obtains a chain of proper subsets by omitting multiple copies.

Neighborhood complex: The *neighborhood complex* $\mathsf{N}(G)$ of a graph G has vertex set $V(G)$ and the sets $A \subseteq V(G)$ with $\mathrm{CN}(A) \neq \emptyset$ as simplices.

Lovász complex: The neighborhood complex $\mathsf{N}(G)$ has no canonical $\mathbb{Z}_2$-structure in general and can be retracted to a $\mathbb{Z}_2$-subspace, the *Lovász complex* $\mathsf{L}(G)$. This complex $\mathsf{L}(G)$ is the subcomplex of $\mathrm{sd}(\mathsf{N}(G))$ induced by the vertices that are fixed points of CN^2. The Lovász complex is

$$\mathsf{L}(G) = \{\mathcal{A} \mid \mathcal{A} \text{ a chain of node sets of } G \text{ with } \mathcal{A} = \mathrm{CN}^2(\mathcal{A})\}$$

which is a $\mathbb{Z}_2$-space with $\mathbb{Z}_2$-action CN.

Box complex: Different versions of a box complex are described by Alon et al. [1], Sarkaria [9], Kříž [5] and Matoušek and Ziegler [8]. The *box complex* $\mathsf{B}(G)$ of G in which we are interested is the one introduced by Matoušek and Ziegler and is defined by

$$\begin{aligned}\mathsf{B}(G) &:= \{A \uplus B \mid A, B \in \mathsf{N}(G) \text{ and } G[A;B] \text{ is complete bipartite}\}\\ &= \{A \uplus B \mid A, B \in \mathsf{N}(G),\ A \subseteq \mathrm{CN}(B),\ \text{and } B \subseteq \mathrm{CN}(A)\}.\end{aligned}$$

The vertex set of the box complex $\mathsf{B}(G)$ can be canonically partitioned as follows:

$$V_1 := \{\{v\} \uplus \emptyset \mid v \in V(G)\} \text{ and } V_2 := \{\emptyset \uplus \{v\} \mid v \in V(G)\}.$$

The subcomplexes of $\mathsf{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathsf{B}(G)$ that are both isomorphic to the neighborhood complex $\mathsf{N}(G)$. This follows from the definition, since $\{v_1\} \uplus \emptyset, \dots, \{v_n\} \uplus \emptyset$ are the vertices of a simplex of $\mathsf{B}(G)$ if and only if $A := \{v_1, \dots, v_n\}$ is a simplex of the neighborhood complex $\mathsf{N}(G)$, that is, if and only if $\mathrm{CN}(A) \neq \emptyset$. We refer to these two copies of $\mathsf{N}(G)$ induced by V_1 and V_2 as *shores* of the box complex. The box complex is endowed with a $\mathbb{Z}_2$-action ν which interchanges the shores.

3. Shore subdivision and useful subcomplexes

Shore subdivision: For a simplicial complex K and any partition of its vertex set V into non-empty sets V_1 and V_2, we call the simplicial subcomplexes K_1 and K_2 induced by V_1 and V_2 its *shores*. In case of the box complex we always consider the canonical partition mentioned above. The *shore subdivision* of K is

$$\mathrm{ssd}(\mathsf{K}) := \{\mathrm{sd}(\sigma \cap \mathsf{K}_1) * \mathrm{sd}(\sigma \cap \mathsf{K}_2) \mid \sigma \in \mathsf{K}\}.$$

We apply this definition to the shores of the box complex to obtain *the shore subdivision* $\mathrm{ssd}(\mathsf{B}(G))$ of $\mathsf{B}(G)$. The vertices of $\mathrm{ssd}(\mathsf{B}(G))$ are of type $A \uplus \emptyset$ and $\emptyset \uplus A$ where $\emptyset \neq A \subset V(G)$ with $\mathrm{CN}(A) \neq \emptyset$. A simplex of $\mathrm{ssd}(\mathsf{B}(G))$ is denoted by $\mathcal{A} \uplus \mathcal{B}$ (the simplex spanned by the vertices $A \uplus \emptyset$ and $\emptyset \uplus B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$).

Doubled Lovász complex: The map $\mathrm{cn}^2 : \mathrm{ssd}(\mathsf{B}(G)) \to \mathrm{ssd}(\mathsf{B}(G))$ defined on the vertices by

$$\mathrm{cn}^2(A \uplus \emptyset) := \mathrm{CN}^2(A) \uplus \emptyset \quad \text{and} \quad \mathrm{cn}^2(\emptyset \uplus A) := \emptyset \uplus \mathrm{CN}^2(A)$$

is simplicial and $\mathbb{Z}_2$-equivariant. We refer to the image of the map cn^2 as *doubled Lovász complex* $\mathsf{DL}(G)$. It is

$$\mathsf{DL}(G) = \left\{ \mathcal{A} \uplus \mathcal{B} \,\middle|\, \begin{array}{c} \mathcal{A}, \mathcal{B} \in \mathsf{L}(G), \\ G[A;B] \text{ is complete bipartite for all } A \in \mathcal{A},\ B \in \mathcal{B} \end{array} \right\}.$$

A copy of the Lovász complex can be found on each shore of $\mathsf{DL}(G) \subset \mathrm{ssd}(\mathsf{B}(G))$, but these copies do not respect the induced $\mathbb{Z}_2$-action.

Halved doubled Lovász complex: We partition the vertex set of the doubled Lovász complex $\mathsf{DL}(G)$ into pairs of type $\{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\}$ to define a simplicial $\mathbb{Z}_2$-map $j\colon \mathsf{DL}(G) \to \mathsf{DL}(G)$. Our aim is to specify one vertex for every pair and map both vertices of a pair to this chosen "smaller" vertex. To do this we refine the partial order by cardinality to a linear order "$\prec$" on the vertices of the original Lovász complex $\mathsf{L}(G)$ using the lexicographic order:

$$A \prec B \quad \Leftrightarrow \quad \begin{cases} |A| < |B| \text{ or} \\ |A| = |B| \quad \text{and} \quad A <_{\mathrm{lex}} B. \end{cases}$$

In fact any refinement would work in the following. A partial order on the vertices of the doubled Lovász complex $\mathsf{DL}(G)$ is now obtained:

$$A \uplus \emptyset \prec \emptyset \uplus \mathrm{CN}(A) \;\Leftrightarrow\; A \prec \mathrm{CN}(A).$$

We define the map j on the vertices using this partial order by

$$j(\mathcal{A} \uplus \emptyset) := \min_{\prec}\{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\} \quad \text{and}$$

$$j(\emptyset \uplus B) := \min_{\prec}\ \{\emptyset \uplus B, \mathrm{CN}(B) \uplus \emptyset\}.$$

It is easy to show that j is simplicial. Let $\mathcal{A} \uplus \mathcal{B}$ be a simplex of the doubled Lovász complex $\mathsf{DL}(G)$, that is, the simplex spanned by the vertices $A \uplus \emptyset$ and $\emptyset \uplus B$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$, $A, B \in \mathsf{N}(G)$ are fixed points of CN^2, and $G[A;B]$ is complete bipartite. Suppose that $A_p \prec B_q$ holds for the largest elements A_p and B_q of $\mathcal{A}$ and $\mathcal{B}$. Then $A_p \prec \mathrm{CN}(A_p)$, since $B_q \subseteq \mathrm{CN}(A_p)$. Hence, $A \uplus \emptyset$ is a fixed point of j for each $A \in \mathcal{A}$. For some $0 \leqslant k < q$, the vertices $\emptyset \uplus B$ for $B \in \mathcal{B}_{\leqslant k}$ are fixed by j, while the vertices $\emptyset \uplus B$ are mapped to $\mathrm{CN}(B) \uplus \emptyset$ for $B \in \mathcal{B}_{>k}$. Since $A_p \prec \mathrm{CN}(B)$ for each $B \in \mathcal{B}_{>k}$, we have

$$j(\mathcal{A} \uplus \mathcal{B}) = (\mathcal{A} \sqsubset \mathrm{CN}(\mathcal{B}_{>k})) \uplus \mathcal{B}_{\leqslant k},$$

which is a simplex of $\mathsf{DL}(G)$. The argument is the same if $B_q \prec A_p$. Hence j is simplicial.

Since the image $\mathrm{Im}\, j$ has half as many vertices as $\mathsf{DL}(G)$, we refer to $\mathrm{Im}\, j$ as *halved doubled Lovász complex* $\mathsf{HDL}(G)$.

A first example: The neighborhood complex $\mathsf{N}(C_5)$ of the 5-cycle C_5 is the 5-cycle; its Lovász complex $\mathsf{L}(C_5)$ is the 10-cycle C_{10}. The box complex $\mathsf{B}(C_5)$, depicted in Fig. 1, consists of two copies of $\mathsf{N}(C_5)$ (the two shores) such that simplices of different shores are joined if and only if their vertex sets seen as node sets of the graph are common neighbors of each other. The shore subdivision $\mathrm{ssd}(\mathsf{B}(C_5))$ as illustrated in Fig. 2 is a subdivision of the box complex induced from a barycentric subdivision of the shores. The map cn^2 maps a vertex of $\mathrm{ssd}(\mathsf{B}(C_5))$ to the common neighborhood of its common neighborhood. In our example, every vertex is mapped to itself, hence $\mathrm{ssd}(\mathsf{B}(C_5)) = \mathsf{DL}(C_5)$. The partitioning of the vertex set of $\mathsf{DL}(C_5)$ into pairs of type $(A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A))$ can be visualized by edges of $\mathsf{DL}(C_5)$ that connect singletons from one shore with two-element sets from the other. The refined

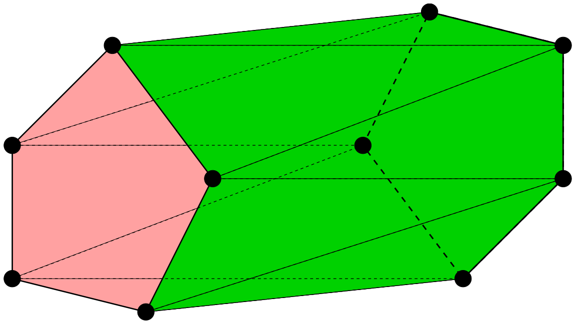

Fig. 1. The box complex $\mathsf{B}(C_5)$.

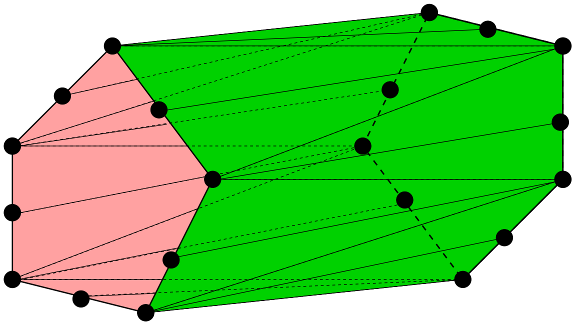

Fig. 2. Here the shore subdivision $\mathrm{ssd}(\mathsf{B}(C_5))$ coincides with $\mathsf{DL}(C_5)$.

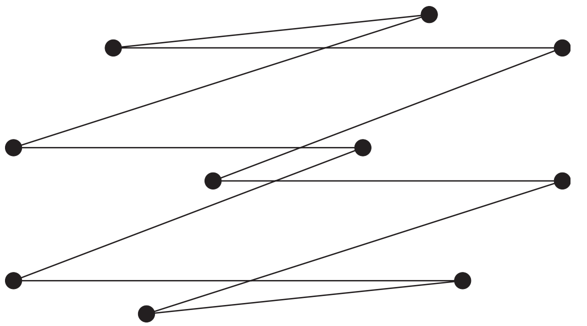

Fig. 3. The halved doubled Lovász complex of C_5.

lexicographic order determines the image of such an edge under j: the smaller vertex is a singleton. Hence, the map j collapses all edges of type $(A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A))$, which yields the halved doubled Lovász complex $\mathsf{HDL}(G)$ as shown in Fig. 3.

A second example: Let us first describe the neighborhood complex and the Lovász complex of the complete graph K_n on n nodes. The neighborhood complex of K_n is the boundary of a simplex on n vertices. This follows from the fact that every set of $n-1$ nodes has a common neighbor but the set $[n]$ has empty common neighborhood. The neighborhood complex $\mathsf{N}(\mathrm{K}_n)$ is therefore a pure abstract simplicial complex of dimension $n-2$, the set of facets is $\binom{[n]}{n-1}$. The Lovász complex $\mathsf{L}(\mathrm{K}_n)$ is its barycentric subdivision, since $\mathrm{CN}(A) = [n]\backslash A$ and therefore $\mathrm{CN}^2(A) = A$ for each $A \subset [n]$. The $\mathbb{Z}_2$-action of $\mathsf{L}(\mathrm{K}_n)$ maps a vertex $A \in \mathrm{V}(\mathsf{L}(\mathrm{K}_n))$ to its complement in $[n]$. We now describe the box complex $\mathsf{B}(\mathrm{K}_n)$. It is the subcomplex of the join $\mathsf{N}(\mathrm{K}_n) * \mathsf{N}(\mathrm{K}_n)$ that has facets $A \uplus ([n]\backslash A)$ for each non-empty set $A \subset [n]$. The box complex $\mathsf{B}(G)$ can also be interpreted as the boundary of an n-dimensional crosspolytope where a pair of opposite facets is removed. The $\mathbb{Z}_2$-action maps a simplex $A \uplus B$ to the simplex $B \uplus A$. The shore subdivision $\mathrm{ssd}(\mathsf{B}(\mathrm{K}_n))$ is a subcomplex of the join $\mathsf{L}(\mathrm{K}_n) * \mathsf{L}(\mathrm{K}_n)$. Its facets can be described as follows. Consider a non-empty set $A \subset [n]$ and a maximal chain $\mathcal{A}$ of non-empty subsets of A. Such a chain represents a $(|A|-1)$-dimensional face of $\mathsf{L}(\mathrm{K}_n)$. Consider a *complementary simplex* $\mathcal{B}$, that is, a maximal chain of non-empty subsets of $[n]\backslash A$. Then $\mathcal{A} \uplus \mathcal{B}$ is a facet of $\mathrm{ssd}(\mathsf{B}(\mathrm{K}_n))$. The $\mathbb{Z}_2$-action maps $\mathcal{A} \uplus \mathcal{B}$ to $\mathcal{B} \uplus \mathcal{A}$. Since every vertex of $\mathrm{sd}(\mathsf{N}(\mathrm{K}_n))$ is a fixed point of CN^2, the shore subdivision $\mathrm{ssd}(\mathsf{B}(\mathrm{K}_n))$ coincides with the doubled Lovász complex $\mathsf{DL}(\mathrm{K}_n)$. To define the map j, we consider the following partitioning of the vertices of $\mathsf{DL}(\mathrm{K}_n)$ into pairs formed by $A \uplus \emptyset$ and $\emptyset \uplus \mathrm{CN}(A)$. The map j maps both vertices to the smaller one of $A \uplus \emptyset$ and $\emptyset \uplus \mathrm{CN}(A)$ with respect to $\prec$. The image of j is the halved doubled Lovász complex. Its $\mathbb{Z}_2$-action maps $A \uplus \emptyset$ to $\emptyset \uplus A$.

4. $\mathsf{L}(G)$ as a $\mathbb{Z}_2$-deformation retract of $\mathsf{B}(G)$

Theorem 1. *The Lovász complex* $\mathsf{L}(G)$ *and the halved doubled Lovász complex* $\mathsf{HDL}(G)$ *are* $\mathbb{Z}_2$*-isomorphic.*

The proof makes use of the *chain notation* introduced in Section 2.

Proof. Since each shore of $\mathsf{DL}(G)$ is isomorphic (but not $\mathbb{Z}_2$-isomorphic) to $\mathsf{L}(G)$, we have $|\mathrm{V}(\mathsf{L}(G))| = |\mathrm{V}(\mathsf{HDL}(G))|$. To define a simplicial $\mathbb{Z}_2$-map $f: \mathsf{L}(G) \to \mathsf{HDL}(G)$, we partition $\mathrm{V}(\mathsf{L}(G))$ into

$$S := \left\{ A \,\middle|\, \begin{array}{l} A \in \mathrm{V}(\mathsf{L}(G)) \text{ and} \\ j(A \uplus \emptyset) = A \uplus \emptyset \end{array} \right\} \quad \text{and} \quad J := \left\{ A \,\middle|\, \begin{array}{c} A \in \mathrm{V}(\mathsf{L}(G)) \text{ and} \\ j(A \uplus \emptyset) = \emptyset \uplus \mathrm{CN}(A) \end{array} \right\}$$

(where "S" and "J" denote the vertices that Stay fixed or Jump to their neighbor), and set

$$f(A) := \begin{cases} A \uplus \emptyset & \text{if } A \in S \\ \emptyset \uplus \mathrm{CN}(A) & \text{if } A \in J. \end{cases}$$

This map is a bijection between the vertex sets $\mathrm{V}(\mathsf{L}(G))$ and $\mathrm{V}(\mathsf{HDL}(G))$ that commutes on vertex level with the $\mathbb{Z}_2$-actions. We now show that it is also surjective and simplicial. For simpliciality, consider a simplex $\mathcal{A}$ in $\mathsf{L}(G)$. Let t denote the largest index k such that A_k is mapped onto the first shore. The image of $\mathcal{A}$ under f is $\mathcal{A}_{\leqslant t} \uplus \mathrm{CN}(\mathcal{A}_{\geqslant t+1})$. This is a simplex since $G[A_t; \mathrm{CN}(A_{t+1})]$ is complete bipartite. For surjectivity consider a simplex $\mathcal{A} \uplus \mathcal{B}$ of $\mathsf{HDL}(G)$, i.e., $G[A; B]$ is complete bipartite for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This simplex is the image of the simplex $\mathcal{A} \sqsubset \mathrm{CN}(\mathcal{B})$ of $\mathsf{L}(G)$. $\square$

Theorem 2. *The halved doubled Lovász complex* $\mathsf{HDL}(G)$ *is a strong* $\mathbb{Z}_2$*-deformation retract of the box complex* $\mathsf{B}(G)$.

Proof. First, we observe that $||\mathsf{DL}(G)||$ is a strong $\mathbb{Z}_2$-deformation retract of $||\mathsf{B}(G)|| = ||\mathrm{ssd}(\mathsf{B}(G))||$. This follows from the fact that a closure operator induces a strong deformation retraction from its domain to its image [2, Corollary 10.12 and the following remark]. Explicitly, this map is obtained by sending each point $p \in ||\mathrm{ssd}(\mathsf{B}(G))||$ towards $||\mathrm{cn}^2||(p)$ with uniform speed, which is $\mathbb{Z}_2$-equivariant at any time of the deformation.

To show that $||\mathsf{HDL}(G)||$ is a strong $\mathbb{Z}_2$-deformation retract of $||\mathsf{DL}(G)||$, we define simplicial complexes and simplicial $\mathbb{Z}_2$-maps

$$\mathsf{DL}(G) =: S_0 \xrightarrow{f_0} S_1 \xrightarrow{f_1} \cdots \xrightarrow{f_N} S_{N+1} := \mathsf{HDL}(G)$$

such that S_{i+1} is a $\mathbb{Z}_2$-subcomplex of S_i and S_{i+1} is a strong $\mathbb{Z}_2$-deformation retract of S_i. The composition of the f_i yields the earlier defined map j, i.e.,

$$j = f_N \circ \cdots \circ f_1 \circ f_0.$$

To construct S_{i+1} inductively from S_i, we consider

$$X := \max_{\prec}\{ Y \in J \mid Y \uplus \emptyset \in S_i\}$$

and obtain S_{i+1} from S_i by deleting each simplex of S_i that contains $X \uplus \emptyset$ or its $\mathbb{Z}_2$-partner $\emptyset \uplus X$, that is,

$$S_{i+1} := \{\sigma \mid \sigma \in S_i \text{ and } X \uplus \emptyset \notin \sigma \text{ and } \emptyset \uplus X \notin \sigma\}.$$

The maximality of X implies that a maximal simplex which contains $X \uplus \emptyset$ (resp. $\emptyset \uplus X$) does also contain $\emptyset \uplus \mathrm{CN}(X)$ (resp. $\mathrm{CN}(X) \uplus \emptyset$). Hence, the map f_i defined on the vertices $v \in \mathrm{V}(S_i)$ via

$$f_i(v) := \begin{cases} \emptyset \uplus \mathrm{CN}(X) & \text{if } v = X \uplus \emptyset, \\ \mathrm{CN}(X) \uplus \emptyset & \text{if } v = \emptyset \uplus X, \\ v & \text{otherwise} \end{cases}$$

is simplicial and $\mathbb{Z}_2$-equivariant.

Thus $F : ||S_i|| \times [0,1] \to ||S_i||$ given by $F(x,t) := tx + (1-t) \cdot ||f_i||(x)$ is a well-defined $\mathbb{Z}_2$-homotopy from $||f_i||$ to $\mathrm{Id}_{||S_i||}$ that fixes $||S_{i+1}||$. $\square$

We end this section with a construction of a $\mathbb{Z}_2$-map $\mathsf{HDL}(f)$ between $\mathsf{HDL}(G)$ and $\mathsf{HDL}(H)$ if we are given a graph homomorphism $f : G \to H$. Once we have chosen the partial orders that define the maps j_G and j_H that give $\mathsf{HDL}(G)$ and $\mathsf{HDL}(H)$, we simply compose the following simplicial $\mathbb{Z}_2$-maps:

- The inclusion $\imath : \mathsf{HDL}(G) \to \mathrm{ssd}(\mathsf{B}(G))$,
- the map $\mathrm{ssd}(\mathsf{B}(f)) : \mathrm{ssd}(\mathsf{B}(G)) \to \mathrm{ssd}(\mathsf{B}(H))$ canonically induced from f,
- the map $\mathrm{cn}^2 : \mathrm{ssd}(\mathsf{B}(H)) \to \mathsf{DL}(H)$, and
- the map $j_H : \mathsf{DL}(H) \to \mathsf{HDL}(H)$.

More precisely, the simplicial $\mathbb{Z}_2$-map $\Psi : \mathsf{HDL}(G) \to \mathsf{HDL}(H)$ is defined by

$$\Psi \coloneqq j_H \circ \mathrm{cn}^2 \circ \mathrm{ssd}(\mathsf{B}(f)) \circ \imath.$$

Since the halved doubled Lovász complex $\mathsf{HDL}(G)$ is $\mathbb{Z}_2$-isomorphic to the original Lovász complex $\mathsf{L}(G)$, this map can be interpreted as a simplicial $\mathbb{Z}_2$-map $\mathsf{L}(f)$ between $\mathsf{L}(G)$ and $\mathsf{L}(H)$. This construction is significantly simpler than the construction of the $\mathbb{Z}_2$-map $\mathsf{L}(f) : \mathsf{L}(G) \to \mathsf{L}(H)$ described by Walker [10].

5. The $\mathrm{K}_{l,m}$-theorem

Theorem 3. *If a graph G does not contain a complete bipartite subgraph* $\mathrm{K}_{\ell,m}$ *then the index of its box complex is bounded by* $\mathrm{ind}(\mathsf{B}(G)) \leqslant \ell + m - 3$.

From Example 2 we know that $\mathrm{ind}(\mathsf{B}(\mathrm{K}_{\ell+m-1})) = \ell + m - 3$, since $\mathsf{B}(\mathrm{K}_{\ell+m-1})$ is the boundary of a crosspolytope with an opposite pair of facets removed, that is, homotopy equivalent to a sphere. Therefore, the statement of the theorem is best possible. On the other hand, we obtain $\mathrm{ind}(\mathsf{B}(\mathrm{K}_{k,k})) \leqslant k - 1$ from the theorem, since $\mathrm{K}_{1,k+1}$ is not a subgraph of $\mathrm{K}_{k,k}$. But $\mathrm{ind}(\mathsf{B}(\mathrm{K}_{k,k})) = 0$, since $\mathrm{K}_{k,k}$ is bipartite. So the gap in the inequality can be arbitrarily large.

We give two proofs for this theorem. The first one uses the shore subdivision and the halved doubled Lovász complex, the other is a direct argument on $\mathsf{L}(G)$ along the lines of Walker [10].

Proof (using shore subdivision). Let $\Phi : \mathrm{ssd}(\mathsf{B}(G)) \to \mathrm{ssd}(\mathsf{B}(G))$ be the simplicial $\mathbb{Z}_2$-map defined by $j \circ \mathrm{cn}^2$. Using that the index is dominated by dimension, it suffices to show the last inequality of

$$\mathrm{ind}(\mathsf{B}(G)) = \mathrm{ind}(\mathrm{ssd}(\mathsf{B}(G))) \leqslant \mathrm{ind}(\mathrm{Im}\,\Phi) \leqslant \dim\,(\mathrm{Im}\,\Phi) \leqslant \ell + m - 3.$$

To estimate the dimension of $\mathrm{Im}\,\Phi = \mathsf{HDL}(G)$, we use that the graph G does not contain a subgraph of type $\mathrm{K}_{\ell,m}$ and assume without loss of generality that $\ell \leqslant m$. A vertex of $\mathsf{HDL}(G)$ or $\mathsf{DL}(G)$ of the form $A \uplus \emptyset$ or $\emptyset \uplus A$ is called *small* if $|A| < \ell$, *medium* if $\ell \leqslant |A| < m$, and *large* if $m \leqslant |A|$. For $\ell = m$ there are no medium vertices. Let $\sigma = \mathcal{A} \uplus \mathcal{B}$ be a simplex of $\mathsf{HDL}(G)$ and consider the set of vertices

$$M_\sigma \coloneqq \mathrm{V}(j^{-1}(\sigma)) = \bigcup_{A \in \mathcal{A}} \{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\} \cup \bigcup_{B \in \mathcal{B}} \{\mathrm{CN}(B) \uplus \emptyset, \emptyset \uplus B\}.$$

Clearly, $|M_\sigma|$ is at most twice $|V(\sigma)|$. If σ has a large vertex $A \uplus \emptyset$, then the vertex $\emptyset \uplus \mathrm{CN}(A)$ must be small, otherwise G would contain a $\mathrm{K}_{\ell,m}$. Hence there are at most $4(\ell - 1)$ many vertices in M_σ that are large or small. Since the number of medium vertices is at most $2(m - \ell)$, we have

$$|M_\sigma| \leqslant 4(\ell - 1) + 2(m - \ell) = 2(\ell + m - 2).$$

Hence $|\mathrm{V}(\sigma)| \leqslant \ell + m - 2$ for all σ, and therefore $\dim(\mathsf{HDL}(G))$ is at most $\ell + m - 3$. $\square$

Proof (using Lovász complex). It suffices to prove $\dim(\mathsf{L}(G)) \leqslant \ell + m - 3$ since

$$\mathrm{ind}(\mathsf{B}(G)) = \mathrm{ind}(\mathsf{L}(G)) \leqslant \dim(\mathsf{L}(G)),$$

[8] or use that $\mathrm{Im}\,\Phi = \mathsf{HDL}(G) \simeq_{\mathbb{Z}_2} \mathsf{L}(G)$ by Section 4. Without loss of generality let $\ell \leqslant m$ and consider a simplex $\mathcal{A} = A_1 \subset \cdots \subset A_p$ of $\mathsf{L}(G)$ of maximal dimension $p - 1$. If $p < \ell$ we are done. Suppose that $p \geqslant \ell$. Then $G[A_\ell; \mathrm{CN}(A_\ell)]$ is a bipartite subgraph of G and we have $|A_\ell| \geqslant \ell$ as well as $|\mathrm{CN}(A_\ell)| \geqslant p - \ell + 1$. The assumption that G does not contain a $\mathrm{K}_{\ell,m}$ implies that $m > p - \ell + 1$, i.e., $\dim(\mathcal{A}) \leqslant \ell + m - 3$. $\square$

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