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Gallai-type theorems and domination parameters

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Abstract

Let $\gamma(G)$ denote the minimum cardinality of a dominating set of a graph $G = (V, E)$. A longstanding upper bound for $\gamma(G)$ is attributed to Berge: For any graph G with n vertices and maximum degree $\Delta(G)$, $\gamma(G) \leqslant n - \Delta(G)$. We characterise connected bipartite graphs which achieve this upper bound. For an arbitrary graph G we furnish two conditions which are necessary if $\gamma(G) + \Delta(G) = n$ and are sufficient to achieve $n - 1 \leqslant \gamma(G) + \Delta(G) \leqslant n$.

We further investigate graphs which satisfy similar equations for the independent domination number, $i(G)$, and the irredundance number $\mathrm{ir}(G)$. After showing that $i(G) \leqslant n - \Delta(G)$ for all graphs, we characterise bipartite graphs which achieve equality.

Lastly, we show for the upper irredundance number, $\mathrm{IR}(G)$: For a graph G with n vertices and minimum degree $\delta(G)$, $\mathrm{IR}(G) \leqslant n - \delta(G)$. Characterisations are given for classes of graphs which achieve this upper bound for the upper irredundance, upper domination and independence numbers of a graph.

1. Introduction

Let $G = (V, E)$ be a graph. For any vertex $x \in V$ we define the *neighbourhood of* x, denoted $N(x)$, as the set of all vertices adjacent to x. The *closed neighbourhood of* x, denoted $N[x]$, is the set $N(x) \cup \{x\}$. For a set of vertices S, we define $N(S)$ as the union of $N(x)$ for all $x \in S$, and $N[S] = N(S) \cup S$. If $x \in S$, a *private neighbour of* x *with respect to* S is a vertex $v \in N[S] - N[S - \{x\}]$. The *degree* of a vertex is the size of its neighbourhood. The *maximum degree* of a graph G is denoted $\Delta(G)$ and the *minimum degree* is denoted by $\delta(G)$. In this paper, n will denote the number of vertices in a graph.

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A set $S \subseteq V$ is said to be *independent* if every pair of vertices in S is nonadjacent. Let $i(G)$ denote the size of a smallest maximal independent set and let $\beta(G)$ denote the size of a largest independent set. Equivalently, $i(G)$ is the size of a smallest independent dominating set. The number $i(G)$ is called the *independent dominating number* and $\beta(G)$ is called the *independence number.*

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. In other words, every vertex in V is either in S or adjacent to a vertex of S. Let $\gamma(G)$ and $\Gamma(G)$ denote the sizes of smallest and largest minimal dominating sets of a graph G, respectively. The number $\gamma(G)$ is called the *domination number* and $\Gamma(G)$ is called the *upper domination number*. Note that any maximal independent set is a dominating set. For a dominating set S to be minimal, each vertex $x \in S$ must have a private neighbour, otherwise the smaller set $S - \{x\}$ is dominating.

A set S is *irredundant* if for all $v \in S$, v has a private neighbour with respect to S. That is, for all $v \in S$, $N[v] - N[S - \{v\}] \neq \emptyset$. Any minimal dominating set is therefore irredundant. Moreover, an irredundant set which is dominating is a minimal dominating set. Let $\mathrm{ir}(G)$ and $\mathrm{IR}(G)$ denote the sizes of smallest and largest maximal irredundant sets of a graph G, respectively. The number $\mathrm{ir}(G)$ is called the *irredundance number* and $\mathrm{IR}(G)$ is called the *upper irredundance number*. Cockayne et al. [3] proved the following inequality:

Theorem 1 (Cockayne [3]). *For any graph G,*

$$\mathrm{ir}(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \beta(G) \leqslant \Gamma(G) \leqslant \mathrm{IR}(G).$$

The parameters $\mathrm{ir}(G), \gamma(G)$ and $i(G)$ are collectively known as the *lower domination parameters*. The parameters $\beta(G), \Gamma(G)$ and $\mathrm{IR}(G)$ are known as the *upper domination parameters*.

A classical theorem in graph theory is due to Gallai [4]. Here, $\alpha_0(G)$ is the *vertex covering number*, the smallest size of a set of vertices needed so that every edge has at least one end vertex in the set.

Theorem 2 (Gallai [4]). *For any graph G,*

$$\beta(G) + \alpha_0(G) = n.$$

A *spanning forest* of a graph G is a spanning subgraph which contains no cycles. Let $\varepsilon(G)$ denote the maximum number of pendant edges in a spanning forest of G. In [6], Nieminen proved the following:

Theorem 3 (Nieminen [6]). *For any nontrivial connected graph G,*

$$\gamma(G) + \varepsilon(G) = n.$$

A *Gallai-type Theorem* has the form $x(G) + y(G) = n$ where $x(G), y(G)$ are parameters defined on the graph G. In [2], Cockayne et al. survey Gallai-type theorems. In this spirit, we will investigate the lower domination parameters and combine them with the maximum degree, then look at the upper domination parameters combined with the minimum degree.

2. Graphs which satisfy $i(G) + \Delta(G) = n$

The first parameter we will consider is $i(G)$, the independent domination number.

Theorem 4. *For any graph G, $i(G) + \Delta(G) \leqslant n$.*

Proof. Let x be a vertex of degree $\Delta(G)$. Let the set S be the vertex x together with any independent dominating set of $V - N[x]$. Then S will independently dominate the graph and $i(G) + \Delta(G) \leqslant |S| + \Delta(G) \leqslant n$. □

From this theorem and the inequality in Theorem 1, we get the following corollary:

Corollary 1. *For any graph G, $\gamma(G) + \Delta(G) \leqslant n$ and $\mathrm{ir}(G) + \Delta(G) \leqslant n$.*

For any inequality, it is interesting to discover conditions which guarantee equality. In this section, we are interested in finding those graphs which have $i(G) + \Delta(G) = n$. Examples of graphs for which $i(G) + \Delta(G) = n$ include K_n and any graph with $\Delta(G) = n - 1$ or $\Delta(G) = n - 2$.

Theorem 5. *Let G be a graph with $i(G) + \Delta(G) = n$ and let x be a vertex of degree $\Delta(G)$. Then $V - N[x]$ is an independent set.*

Proof. Suppose there is an edge in $V - N[x]$. Any independent dominating set of $V - N[x]$ will have size at most $|V - N[x]| - 1$. Thus

$$i(G) \leqslant |V - N[x]| - 1 + |\{x\}| = n - \Delta(G) - 1$$

a contradiction. □

A *subdivision* of an edge uv is obtained by introducing a new vertex w and replacing the edge uv with edges uw and vw. The converse of Theorem 5 does not hold. For example, the graph constructed by taking a vertex with 3 independent neighbours and subdividing each edge once has $\Delta(G) = 3, i = 3$ and $n = 7$.

We will now turn our attention to bipartite graphs which satisfy $i(G) + \Delta(G) = n$. A graph is *bipartite* if it has a *bipartition*, $A \cup B$, of the vertices such that every edge joins a vertex of A to a vertex of B. Thus a graph is bipartite if and only if it contains no odd cycles. A bipartite graph with all possible edges between A and B is called *complete bipartite*, and is denoted $K_{|A|,|B|}$.

Lemma 1. *If G is a connected bipartite graph with $i(G)+\Delta(G)=n$, and x is a vertex of maximum degree, then $|V-N[x]| \leqslant \Delta(G)-1$.*

Proof. Let G be a connected bipartite graph with $i(G)+\Delta(G)=n$. By Theorem 5, $V-N[x]$ is an independent set. Since G is connected, $N(x)$ must be an independent dominating set of G. So $i(G) \leqslant |N(x)| = \Delta(G)$ and $n = i(G)+\Delta(G) \leqslant 2\Delta(G)$. Thus $|V-N[x]| \leqslant \Delta(G)-1$. □

Theorem 6. *Let G be a connected bipartite graph with bipartition $A \cup B$. Then $i(G)+\Delta(G)=n$ if and only if $\Delta(G)=\max\{|A|,|B|\}$ and $i(G)=\min\{|A|,|B|\}$.*

Proof. Let G be a connected bipartite graph with bipartition $A \cup B$. Clearly, if $\Delta(G)=\max\{|A|,|B|\}$ and $i(G)=\min\{|A|,|B|\}$, then $i(G)+\Delta(G)=n$.

Conversely, suppose $i(G)+\Delta(G)=n$. Let x be a vertex of degree $\Delta(G)$. By Theorem 5, $V-N[x]$ is an independent set, and by Lemma 1, $|V-N[x]| \leqslant \Delta(G)-1$.

Without loss of generality, let $x \in A$. Then $N(x)=B$ and $V-N[x] \subset A$. Thus $|B| = \Delta(G) = |N(x)|$. Also, $A = V-N[x] \cup \{x\}$, so $|A| \leqslant \Delta(G)$. Then $\Delta(G)=\max\{|A|,|B|\}$, and since $i(G)+\Delta(G)=n$, we have $i(G)=\min\{|A|,|B|\}$. □

Suppose G is a connected bipartite graph with bipartition $A \cup B$ such that $|B| > |A| > 2$. By the above theorem, if $i(G)+\Delta(G)=n$, then there is a vertex in A which is adjacent to every vertex in B.

A connected graph is a *tree* if it contains no cycles. Thus any tree is also bipartite. For trees, the conclusions of Lemma 1 together with Theorem 5 form necessary and sufficient conditions for a tree T to have $i(T)+\Delta(T)=n$.

Theorem 7. *Let T be a tree and let x be a vertex of T with degree $\Delta(T)$. Then $i(T)+\Delta(T)=n$ if and only if $V-N[x]$ is an independent set and $|V-N[x]| \leqslant \Delta(T)-1$.*

Proof. Let T be a tree, let x be a vertex of degree $\Delta(T)$.

Suppose first that $i(T)+\Delta(T)=n$. Then the result follows from Theorem 5 and Lemma 1.

Conversely, suppose that $V-N[x]$ is an independent set and $|V-N[x]| \leqslant \Delta(T)-1$. Let I be an independent dominating set of size $i(T)$. If $x \in I$, then $I=\{x\} \cup (V-N[x])$ and $i(T)+\Delta(T)=n$. So suppose $x \notin I$. Then every vertex in $N(x)$ of degree 1 is in I. Also, any vertex in $N(x)$ of degree $\geqslant 3$ is in I, in order to make $|I|$ as small as possible. Any vertex $y \in N(x)$ of degree 2 is adjacent to x and one vertex z in $V-N[x]$. Then either y or z is in I, so $|I| \geqslant \Delta(T)$. Now $N(x)$ is an independent set and all of T is dominated by the $\Delta(T)$ vertices in $N(x)$, so $|I| \leqslant \Delta(T)$. Thus $|I| = \Delta(T) = |N(x)|$, and recall that $|V-N[x]| \leqslant \Delta(T)-1$. Now $n = |x| + |N(x)| + |V-N[x]| \leqslant 1+i(T)+\Delta(T)-1$, which implies $n \leqslant i(T)+\Delta(T)$. By Theorem 4, $i(T)+\Delta(T) \leqslant n$, and so $i(T)+\Delta(T)=n$. □

A graph is a *split graph* if there is a partion $V = I \cup K$ of the vertices into an independent set I and a clique K, where a *clique* is a set of vertices whose induced subgraph is complete. It turns out that every connected split graph G has $i(G) + \Delta(G) = n$:

Theorem 8. *If G is a connected split graph, then $i(G) + \Delta(G) = n$.*

Proof. Let G be a connected split graph. Partition the vertices into $K \cup I$, where K is a clique and I is an independent set. If $|I| = 0$, then the graph is complete and clearly $i(G) + \Delta(G) = n$. So assume that $|I| \geqslant 1$. Note that $\Delta(G) \geqslant |K|$, since each vertex of K is adjacent to $|K| - 1$ vertices in the clique K and the graph is connected, so at least one vertex of K is adjacent to a vertex of I.

Using Theorem 4, it will suffice to show that $i(G) \geqslant n - \Delta(G)$. Let D be an independent dominating set with $|D| = i(G)$, and suppose that $|D| < n - \Delta(G)$. Now $|D \cap K| \leqslant 1$ since D is an independent set. We consider two cases for $|D \cap K|$.

First suppose that $|D \cap K| = 0$. Then $D = I$, and

$$n = |K| + |I| = |K| + |D| < |K| + n - \Delta(G) \leqslant |K| + n - |K| = n$$

a contradiction.

Now suppose that $|D \cap K| = 1$. Let $x = D \cap K$. Then $D = \{x\} \cup (I - N[x]) = V - N(x)$. Then

$$n - \Delta(G) > |D| = n - \deg(x) \geqslant n - \Delta(G)$$

a contradiction.

Thus, $i(G) = |D| \geqslant n - \Delta(G)$, so $i(G) + \Delta(G) = n$. □

3. Graphs which satisfy $\gamma(G) + \Delta(G) = n$

We will now consider graphs which satisfy $\gamma(G) + \Delta(G) = n$. Note that if a graph has $\gamma(G) + \Delta(G) = n$, then $i(G) + \Delta(G) = n$ also. A *star* is a graph isomorphic to $K_{1,r}$, $r \geqslant 2$. A *double star* is a graph obtained by taking two stars and joining the vertices of maximum degree with an edge. Any double star is an example of a graph with $i(G) + \Delta(G) = n$ but $\gamma(G) + \Delta(G) < n$. The complete bipartite graph $K_{m,n}$ for $2 < m \leqslant n$ is another such example. Here, $\gamma(K_{m,n}) = 2$, $\Delta(K_{m,n}) = n$ and $i(K_{m,n}) = m$. The following result follows from Theorem 5.

Theorem 9. *If G is a graph with $\gamma(G) + \Delta(G) = n$ and x is a vertex of degree $\Delta(G)$ then $V - N[x]$ is an independent set.*

Theorem 10 gives an additional necessary condition for a graph with $\gamma(G) + \Delta(G) = n$.

Theorem 10. *If G is a graph with $\gamma(G) + \Delta(G) = n$ and x is a vertex of degree $\Delta(G)$, then each vertex of $N(x)$ is adjacent to at most one vertex in $G - N[x]$.*

Proof. Suppose G is a graph with $\gamma(G)+\Delta(G)=n$ and let x be a vertex of degree $\Delta(G)$. Let $y\in N(x)$ and suppose y is adjacent to r vertices in $V-N[x]$, with $r>1$. Then we can dominate the whole graph using y, x and the $n-\Delta(G)-1-r$ vertices of $V-N[x]$ that are not in $N(y)$. Then we have $\gamma(G)\leqslant n-\Delta(G)-1-r+2= n-\Delta(G)-r+1<n-\Delta(G)$, a contradiction. □

Theorem 10 is not a necessary condition for a graph G which satisfies $i(G)+\Delta(G)=n$. For example, any double star has $i(G)+\Delta(G)=n$ and a vertex of $N(x)$ adjacent to more than one vertex in $V-N[x]$. The converse of Theorem 10 does not hold. For example, the star $K_{1,r}$ with every edge subdivided has $\gamma(G)=r$, $\Delta(G)=r$ and $n=2r+1$.

Theorem 11. *Let G be a graph with $\Delta(G)=1$. Then $\gamma(G)+\Delta(G)=n$ if and only if $G=K_2\cup(n-2)K_1$.*

Proof. Certainly, if $G=K_2\cup(n-2)K_1$, then $\Delta(G)=1$ and $\gamma(G)+\Delta(G)=n$. So suppose $\Delta(G)=1$ and $\gamma(G)+\Delta(G)=n$. Then each component of G is a K_1 or K_2, with at least one K_2 component, and $\gamma(G)=n-1$. If there is more than one K_2 component, then G can be dominated in fewer than $n-1$ vertices. Thus G has one K_2 component and every other component is a K_1. □

Theorem 12. *Let G be a graph with $\Delta(G)=2$. Then $\gamma(G)+\Delta(G)=n$ if and only if either*
(1) *$G=H\cup(n-3)K_1$ and $H=P_3$ or C_3, or*
(2) *$G=H\cup(n-4)K_1$ and $H=P_4$ or C_4.*

Proof. Certainly, any of the graphs listed have $\Delta(G)=2$ and $\gamma(G)=n-2$. So suppose that G is a graph with $\Delta(G)=2$ and $\gamma(G)+\Delta(G)=n$. Since $\Delta(G)=2$, each component is C_k with $k\geqslant 3$, P_k with $k\geqslant 2$ or K_1. Also, there must be at least one component that is C_k with $k\geqslant 3$ or P_k with $k\geqslant 3$. Let H be a largest component with k vertices. Then $\gamma(H)=\lceil k/3\rceil$, and $\gamma(G-H)\leqslant n-k$, so that $\gamma(G)\leqslant\lceil k/3\rceil+n-k$. So if $\gamma(G)=n-2$ then $k-\lceil k/3\rceil\leqslant 2$, and $k\geqslant 3$. Therefore $k=3$ or 4. If H is P_4 or C_4, then the rest of the graph is $(n-4)K_1$, since $\gamma(G)=n-2$. If H is P_3 or C_3 then the rest of the graph is $(n-3)K_1$. □

It is straightforward to see that any graph G with $\Delta(G)=n-1$ or $n-2$ also satisfies $\gamma(G)+\Delta(G)=n$. We will now turn our attention to bipartite graphs.

Theorem 13. *Let G be a connected bipartite graph with bipartition $A\cup B$, and let $x\in A$ be a vertex of degree $\Delta(G)$. Then $\gamma(G)+\Delta(G)=n$ if and only if*
(1) *$\Delta(G)=|B|$,*
(2) *each vertex in B has degree at most 2, and*
(3) *B contains a vertex of degree 1 or every vertex in A has degree at least 2.*

Proof. Suppose G is a connected bipartite graph which satisfies $\gamma(G)+\Delta(G)=n$. Let x be a vertex of degree $\Delta(G)$, and without loss of generality suppose that $x\in A$. Then $N(x)\subseteq B$, so that $|N(x)|=\Delta(G)\leqslant|B|$. Now $\gamma(G)+\Delta(G)=n$ implies $i(G)+\Delta(G)=n$. By Theorem 6, $|N(x)|=\Delta(G)=\max\{|A|,|B|\}\geqslant|B|$. Thus $\Delta(G)=|B|$ and $N(x)=B$. By Theorem 10, each vertex of B is adjacent to at most one vertex in $V-N[x]$, and since G is bipartite, each vertex in B has degree at most 2.

Suppose that B has no vertex of degree one. Then every vertex in B has degree exactly 2. If $V-N[x]$ has a vertex a of degree one, then a has neighbour $b\in B$. Then the set $(V-N[x])-\{a\}\cup\{b\}$ is a dominating set for G, and so $\gamma(G)\leqslant n-\Delta(G)-1$, a contradiction. Thus either B has a vertex of degree one or every vertex in $V-N[x]$ has degree at least 2.

Conversely, suppose that G is a connected bipartite graph, with a vertex $x\in A$ of degree $\Delta(G)=|B|$ and suppose every vertex of B has degree at most 2.

Suppose first that B has a vertex b of degree one. We will show that $\gamma(G)+\Delta(G)=n$. Of all dominating sets of G of size $\gamma(G)$, let D be one containing the fewest members of B. Since either b or x is in D and $|B\cap D|$ is minimal, we may assume $x\in D$. Suppose there exists $d\in B\cap D$. Then d cannot be its own private neighbour, since $x\in D$. So d has a private neighbour $a\in A$. Now d has exactly one neighbour in $V-N[x]$ since $\deg(d)\leqslant 2$, so the dominating set $D-\{d\}\cup\{a\}$ has size $\gamma(G)$ and has fewer members of B than D, a contradiction. Thus $|B\cap D|=0$ and $\gamma(G)+\Delta(G)=n$.

Now suppose that every vertex in B has degree 2, and every vertex in $V-N[x]$ has degree at least 2. Of all the dominating sets of size $\gamma(G)$, let D be one with fewest members of B.

Suppose $x\notin D$. Then x is dominated by a vertex $b_1\in B$, where b_1 is adjacent to $a\in V-N[x]$. Note that a has at least 2 neighbours in B. If $a\in D$, we get the dominating set $D-\{b_1\}\cup\{x\}$, contradicting the choice of D. So $a\notin D$, and then all of $N(a)$ are in D, since each vertex of $N(a)$ is adjacent to exactly one vertex in $V-N[x]$. But then the dominating set $D-N(a)\cup\{x,a\}$ contradicts the choice of D. Thus we must have $x\in D$.

Suppose $b_2\in B\cap D$. Then b_2 has a neighbour $a\in V-N[x]$. The dominating set $D-\{b_2\}\cup\{a\}$ contradicts the choice of D.

Thus $|D\cap B|=0$, so $\gamma(G)=|D|=|A|=n-\Delta(G)$. □

Trees form a subclass of bipartite graphs. A tree is a *wounded spider* if the tree is $K_{1,r}$, $r\geqslant 0$, with at most $r-1$ of the edges subdivided. Thus, a star is a wounded spider. It is easy to see that every wounded spider T satisfies $\gamma(T)+\Delta(T)=n$.

Corollary 2. *Let T be a tree. Then $\gamma(T)+\Delta(T)=n$ if and only if T is a wounded spider.*

Proof. Clearly by Theorem 13 every wounded spider has $\gamma(T)+\Delta(T)=n$. So suppose that T is a tree with $\gamma(T)+\Delta(T)=n$, and suppose that $\deg(x)=\Delta(T)$. Let $A\cup B$ be a bipartition of V such that $x\in A$. By Theorem 13, $N(x)=B$. Since T contains no cycles, there cannot be a vertex of degree 2 in $A - \{x\}$. Thus, either there are no vertices in $A - \{x\}$, in which case T is a star, or there is at least one vertex in $A - \{x\}$, which has degree one. By Theorem 13, there must be a vertex of degree one in B. Hence T is a wounded spider. $\square$

Theorems 9 and 10 yield conditions which are necessary for an arbitrary graph G to achieve $\gamma(G) + \Delta(G) = n$. We finish this section by showing these conditions are sufficient for a graph G to satisfy $\gamma(G) + \Delta(G) \geqslant n - 1$.

Theorem 14. *Let G be a connected graph, and let x be a vertex of degree $\Delta(G)$. If $V - N[x]$ is an independent set and every vertex in $N(x)$ is adjacent to at most one vertex in $V - N[x]$, then either $\gamma(G) + \Delta(G) = n$ or $\gamma(G) + \Delta(G) = n - 1$.*

Proof. Suppose G is a connected graph and x is a vertex of degree $\Delta(G)$. Suppose also that $V - N[x]$ is an independent set and every vertex in $N(x)$ is adjacent to at most one vertex in $V - N[x]$. Any minimal dominating set of G will contain at least $|V - N[x]|$ vertices in order to dominate all the vertices in $V - N[x]$. Also, we can dominate G by using x and all of $V - N[x]$. So we have $|V - N[x]| \leqslant \gamma(G) \leqslant |V - N[x]| + |\{x\}|$. Thus $n - \Delta(G) - 1 \leqslant \gamma(G) \leqslant n - \Delta(G)$. $\square$

By strengthening the hypothesis, we find a sufficient condition to guarantee that a graph G satisfies $\gamma(G) + \Delta(G) = n$.

Theorem 15. *Let G be a connected graph and let x be a vertex of degree $\Delta(G)$. If $V - N[x]$ is an independent set, every vertex in $N(x)$ is adjacent to at most one vertex of $V - N[x]$ and $N(x)$ contains a vertex of degree one, then $\gamma(G) + \Delta(G) = n$.*

Proof. Let G be a connected graph and let $x \in V$ be a vertex of degree $\Delta(G)$. Suppose further that $V - N[x]$ is an independent set, every vertex in $N(x)$ is adjacent to at most one vertex of $V - N[x]$ and $y \in N(x)$ has degree one.

Let S be a minimal dominating set of size $\gamma(G)$. By Theorem 14 either $|S| = n - \Delta(G)$ or $|S| = n - \Delta(G) - 1$. Suppose $|S| = n - \Delta(G) - 1$. Since y has degree one, S must contain either x or y. So there are at most $n - \Delta(G) - 2$ vertices in S available to dominate $V - N[x]$. But at least $|V - N[x]| = n - \Delta(G) - 1$ vertices are necessary to dominate $V - N[x]$. This contradicts our assumption that $|S| = n - \Delta(G) - 1$. Therefore $|S| = \gamma(G) = n - \Delta(G)$. $\square$

This added condition is not, however, a necessary one. The graph C_4 satisfies $\gamma(C_4) + \Delta(C_4) = 4$ but C_4 does not satisfy the hypothesis of Theorem 15.

4. The irredundance number, ir(G)

We now turn our attention to the lower irredundance number, ir(G). By Corollary 1, $\mathrm{ir}(G) + \Delta(G) \leqslant n$ for any graph G. Since $\mathrm{ir}(G) \leqslant \gamma(G)$, we know any graph G

satisfying $\mathrm{ir}(G) + \Delta(G) = n$ will also satisfy $\gamma(G) + \Delta(G) = n$. Thus any graph with $\mathrm{ir}(G)+\Delta(G)=n$ and a vertex x of degree $\Delta(G)$ will satisfy the hypothesis for Theorems 9 and 10. So $V - N[x]$ will be an independent set and any vertex in $N(x)$ will be adjacent to at most one vertex of $V - N[x]$. It is straightforward to check that for the bipartite graphs listed in Theorem 13, $\mathrm{ir}(G) + \Delta(G) = n$. Thus, the bipartite graphs with $\gamma(G) + \Delta(G) = n$ also will have $\mathrm{ir}(G) + \Delta(G) = n$. Furthermore, for any tree T, $\mathrm{ir}(T) + \Delta(T) = n$ if and only if T is a wounded spider.

It is possible to find examples of graphs for which $\gamma(G) + \Delta(G) = n$ and $\mathrm{ir}(G) + \Delta(G) < n$. For example, consider the graph G constructed as follows: Let k be a positive integer. For $1 \leqslant i \leqslant k$, form the graph G_i by taking vertices $u_{i1}, u_{i2}, u_{i3}, u_{i4}, v_{i1}, v_{i2}, v_{i3}$. Add edges $u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}$ and $v_{i1}u_{i1}, v_{i2}u_{i2}, v_{i2}u_{i3}, v_{i3}u_{i4}$. Take a vertex x and join it to u_{ij} for $1 \leqslant j \leqslant 4$ and $1 \leqslant i \leqslant k$. Add a vertex y and edge xy. The graph G has $|N(x)| = \Delta(G) = 4k + 1$. Since $V - N[x]$ is an independent set, every vertex in $N(x)$ is adjacent to at most one vertex of $V - N[x]$ and x has a neighbour of degree one, by Theorem 15, $\gamma(G) + \Delta(G) = n$. An irredundant set S can be found by taking y together with u_{i2}, u_{i3} for $1 \leqslant i \leqslant k$. This set S is a maximal irredundant set for G. Therefore $\mathrm{ir}(G) \leqslant 2k + 1$ and this is an example of a graph with $\gamma(G)+\Delta(G)=n$ and $\mathrm{ir}(G) + \Delta(G) \leqslant n - k$.

5. The upper parameters and minimum degree

The upper domination parameters, $\beta(G), \Gamma(G)$ and $\mathrm{IR}(G)$ will be combined with minimum degree for Gallai-type results.

Theorem 16. *For any graph G, $\mathrm{IR}(G) + \delta(G) \leqslant n$.*

Proof. Let S be a maximal irredundant set of size $\mathrm{IR}(G)$ and let $x \in S$. Since S is irredundant, there is a vertex y such that $y \in N[x] - N[S - \{x\}]$. We consider two cases.

Case 1: $y = x$. Then x is not adjacent to any vertex in S, and must have at least $\delta(G)$ neighbours in $V - S$. Thus $n - \mathrm{IR}(G) = |V - S| \geqslant \delta(G)$ and $\mathrm{IR}(G)+\delta(G) \leqslant n$.

Case 2: $y \neq x$. By the choice of y, $y \notin S$ and $N(y) \cap S = \{x\}$. Then $N[y] - \{x\} \subset V - S$, so that $n - \mathrm{IR}(G) = |V - S| \geqslant |N[y] - \{x\}| \geqslant \delta(G)$ and so $\mathrm{IR}(G) + \delta(G) \leqslant n$. □

Using Theorems 16 and 1, we get the following corollary:

Corollary 3. *For any graph G, $\Gamma(G) + \delta(G) \leqslant n$ and $\beta(G) + \delta(G) \leqslant n$.*

We will first consider graphs for which $\beta(G) + \delta(G) = n$.

Theorem 17. *Let G be a connected graph and let I be a maximal independent set of G such that $|I| = \beta(G)$. Then $\beta(G) + \delta(G) = n$ if and only if for each $x \in I$, we have $\deg(x) = \delta(G)$ and $V - N(x)$ is an independent set.*

Proof. First suppose that $\beta(G) + \delta(G) = n$. Let $x \in I$, then $I \subseteq V - N(x)$, so $\beta(G) = |I| \leqslant n - |N(x)| \leqslant n - \delta(G) = \beta(G)$. Thus $|N(x)| = \delta(G)$, and $|V - N(x)| = |I|$ so $V - N(x)$ is an independent set.

Now suppose that for each $x \in I$, $\deg(x) = \delta(G)$ and $V - N(x)$ is an independent set. Then $\beta(G) \geqslant |V - N(x)| = n - \delta(G)$, and by Corollary 3 we must have $\beta(G) = n - \delta(G)$. □

It follows from Theorem 17 that the vertices of any graph which satisfies $\beta(G) + \delta(G) = n$ can be partitioned into two sets, $I \cup J$, where I is an independent set of size $\beta(G)$, J is a set of size $\delta(G)$ and each vertex in I is adjacent to every vertex in J. The above result also implies that if T is a tree, then $\delta(G) + \beta(G) = n$ if and only if $T = K_{1,n-1}$. In fact, $K_{1,n-1}$ is the only graph with $\beta(G) + \delta(G) = n$ and $\delta(G) = 1$. Furthermore, if G is bipartite, we have $\beta(G) + \delta(G) = n$ if and only if $G = K_{\delta(G),\beta(G)}$. If G is a split graph, then $\delta(G) + \beta(G) = n$ if and only if the vertices of G can be partitioned into $K \cup I$, where $|K| = \delta(G)$, $|I| = \beta(G)$, I is an independent set and K is a clique.

We now turn our attention to graphs which satisfy $\Gamma(G) + \delta(G) = n$. The *Cartesian product* of two graphs G and H is denoted $G \times H$. The vertices of $G \times H$ are the ordered pairs (g,h) where $g \in V(G)$ and $h \in V(H)$. The edge set is given as follows: (g_1,h_1) is adjacent to (g_2,h_2) in $G \times H$ if and only if either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$. For any graph, if $\beta(G) + \delta(G) = n$ then $\Gamma(G) + \delta(G) = n$. The converse does not hold, for example the graph $K_2 \times K_r$, the generalised prism, has $\beta(G) = 2$, $\Gamma(G) = r$, $\delta(G) = r$ and $n = 2r$. It is known that for chordal graphs [5] and bipartite graphs [1] we have $\beta(G) = \Gamma(G) = \mathrm{IR}(G)$. Thus the bipartite and split graphs with $\Gamma(G) + \delta(G) = n$ also will have $\beta(G) + \delta(G) = n$.

Theorem 18. *Let G be a graph with $\Gamma(G) + \delta(G) = n$ and $\beta(G) + \delta(G) < n$. Then G contains $H = K_2 \times K_\Gamma$ as an induced subgraph, and every vertex in $G - H$ is adjacent to every vertex in H.*

Proof. Let G be a graph with $\Gamma(G) + \delta(G) = n$ and $\beta(G) + \delta(G) < n$. Let S be a minimal dominating set of size $\Gamma(G)$.

Claim. *Every vertex in S has a neighbour in S.*

Proof. Suppose to the contrary that there is a vertex $x \in S$ such that x has no neighbours in S. There are at least $\delta(G)$ vertices in $N(x)$, none of which is in S. Since $\Gamma(G) + \delta(G) = n$, we must have $\deg(x) = \delta(G)$ and $V - N(x) = S$. Since every vertex in S must have a private neighbour, S must be an independent set, so that $\Gamma(G) \leqslant \beta(G)$, a contradiction. □

Since every vertex in S has a neighbour in S, each vertex in S must have a private neighbour in $V - S$.

Claim. *Every vertex in S has exactly one private neighbour.*

Proof. Suppose that $x \in S$ has at least two private neighbours. Let $y \in S$, where $y \neq x$. Then y is not adjacent to itself, and is not adjacent to any private neighbour of $S - y$. But the number of private neighbours of $S - y$ is at least $\Gamma(G)$, so we have found $\Gamma(G) + 1$ vertices not adjacent to y. But y has at least $\delta(G) = n - \Gamma(G)$ neighbours, a contradiction. $\square$

Proof of Theorem 18 (*conclusion*). From this, we can see that the number of private neighbours of S is exactly $\Gamma(G)$, one for each vertex in S. Let P denote the set of private neighbours of S. Now each vertex in P is adjacent to exactly one vertex in S, so has at least $\delta(G) - 1$ neighbours in $V - S$. Thus every vertex in P is adjacent to every vertex not in S, in particular, the private neighbours form a clique of size $\Gamma(G)$. Each vertex in S has one private neighbour, so is not adjacent to itself and $\Gamma(G) - 1$ of the private neighbours. Thus each vertex in S has degree at least $\delta(G)$ so is adjacent to its one private neighbour, to every other vertex in S and to every vertex not in $S \cup P$. Hence, S forms a clique of size $\Gamma(G)$. Thus the graph induced by $S \cup P$ is the graph $H = K_2 \times K_{\Gamma(G)}$, and every vertex not in $S \cup P$ is adjacent to every vertex in H. $\square$

Finally, we turn our attention to graphs for which $\mathrm{IR}(G) + \delta(G) = n$, and show that they are precisely the ones with $\Gamma(G) + \delta(G) = n$.

Theorem 19. *For any graph G, $\mathrm{IR}(G) + \delta(G) = n$ if and only if $\Gamma(G) + \delta(G) = n$.*

Proof. First, let G be a graph with $\Gamma(G) + \delta(G) = n$. Then it follows immediately from the fact that $\Gamma(G) \leqslant \mathrm{IR}(G)$ and Theorem 16 that $\mathrm{IR}(G) + \delta(G) = n$.

Now suppose that G is a graph with $\mathrm{IR}(G) + \delta(G) = n$ and let S be a maximal irredundant set for G with $|S| = \mathrm{IR}(G)$. We will show that S is dominating, and since S is irredundant it will be a minimal dominating set. So suppose that S is not dominating. Then there is a $w \in V - S$ such that w is not adjacent to any vertex of S. We must then have $N[w] \subseteq V - S$. But $|N[w]| \geqslant \delta(G) + 1$, which implies $\delta(G) + 1 \leqslant |N[w]| \leqslant |V - S| = n - \mathrm{IR}(G)$, and so $\mathrm{IR}(G) + \delta(G) \leqslant n - 1$, a contradiction. Thus S is a dominating set. Since $\Gamma(G) \geqslant |S| = \mathrm{IR}(G) \geqslant \Gamma(G)$, we must have $\mathrm{IR}(G) = \Gamma(G)$. $\square$

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