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Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leqslant 14$

Darryn E. Bryant[a], C.A. Rodger[b,*], Erin R. Spicer[b]

[a] *Centre for Combinatorics, Department of Mathematics, The University of Queensland, Qld 4072, Australia*

[b] *Department of Discrete and Statistical Sciences, 120 Math Annex, Auburn University, Auburn, Alabama 36849-5307, USA*

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Abstract

In this paper we completely settle the embedding problem for m-cycle systems with $m \leqslant 14$. We also solve the more general problem of finding m-cycle systems of $K_v - K_u$ when $m \in \{4,6,7,8,10,12,14\}$.

1. Introduction

Throughout we will use standard graph theoretic terminology which if not defined here can be found in [2]. Let G^c denote the complement of G. If G and H are two graphs then let $G \cup H$ be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then let $G \vee H$ be the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{\{g,h\} \mid g \in V(G), h \in V(H)\}$. If H is a subgraph of G, let $G - H$ be the graph containing those edges of G which are not in H. Define $|i-j|_x = \min\{|i-j|, x-|i-j|\}$. If D_0, $D_1 \subseteq \{1,2,\ldots,\lfloor x/2 \rfloor\}$ and $S \subseteq \mathbb{Z}_x$, then define the graph $\langle D_0, S, D_1 \rangle_x$ to be the graph with the vertex set $\mathbb{Z}_x \times \mathbb{Z}_2$ and edge set $\{\{(i,0),(j,0)\} \mid |i-j|_x \in D_0\} \cup \{\{(i,0),(j,1)\} \mid i-j \pmod{x} \in S\} \cup \{\{(i,1),(j,1)\} \mid |i-j|_x \in D_1\}$.

Let $\mathbb{Z}_m = \{0,1,\ldots,m-1\}$. An *m-cycle* is a graph $(v_0, v_1, \ldots, v_{m-1})$ with vertex set $\{v_i \mid i \in \mathbb{Z}_m\}$ and edge set $\{\{v_i, v_{i+1}\} \mid i \in \mathbb{Z}_m\}$ (reducing the subscript modulo m). An *m-cycle system* of a graph G is an ordered pair (V, C) where V is the vertex set of G and C is a set of m-cycles, the edges of which partition the edges of G. An *m-cycle system* of *order n* is an m-cycle system of K_n. An m-cycle system (V, C) is said to be *embedded* in the m-cycle system (W, P) if $V \subset W$ and $C \subset P$.

* Corresponding author. E-mail: rodgecl@mail.auburn.edu. Research supported by NSF grant DMS-9225046.

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A most natural question to consider is the *embedding problem* for m-cycle systems; that is, to find the set of integers $S_m(u)$ such that any m-cycle system of order u can be embedded in an m-cycle system of order $v > u$ if and only if $v \in S_m(u)$. Doyen and Wilson solved the embedding problem for 3-cycle systems (Steiner triple systems) in 1973.

Theorem 1.1 (Doyen and Wilson [6]). $S_3(u) = \{v \mid v \equiv 1$ *or* $3 \pmod 6,\ v \geqslant 2u + 1\}$.

The embedding problem for 5-cycle systems was recently solved by Bryant and Rodger.

Theorem 1.2 (Bryant and Rodger [4]). $S_5(u) = \{v \mid v \equiv 1$ *or* $5 \pmod{10}, v \geqslant 3u/2+1\}$.

More recently, Bryant and Rodger [5] gave a general approach for embedding an m-cycle system of order u in the case m is odd and u is 1 or $m \pmod{2m}$ and a complete solution to the embedding problem for m-cycle systems was given in the cases $m = 7$ and $m = 9$.

Theorem 1.3 (Bryant and Rodger [5]). $S_7(u)=\{v \mid v\equiv 1$ *or* $7 \pmod{14},\ v \geqslant 4u/3+1\}$.

Theorem 1.4 (Bryant and Rodger [5]). $S_9(u)=\{v \mid v\equiv 1$ *or* $9 \pmod{18}, v \geqslant 5u/4+1\}$.

In this paper we completely solve the embedding problem for m-cycle systems for the remaining values of $v \leqslant 14$; that is, for $v \in \{4,6,8,10,11,12,13,14\}$.

When an m-cycle system of order u is embedded in an m-cycle system of order v, an m-cycle system of the complete graph of order v with a hole of size u, $K_v - K_u$, is obtained by deleting the m-cycle system of order u. However, the solution of the embedding problem for m-cycle systems does not completely solve the existence problem for m-cycle systems of $K_v - K_u$; it does not solve the cases where there is no m-cycle system of order u. This more general problem was solved in the case $m = 3$ by Mendelsohn and Rosa [11] and in the case $m = 5$ by Bryant et al. [3].

Theorem 1.5 (Mendelsohn and Rosa [11]). *There exists a 3-cycle system of* $K_v - K_u$ *if and only if* $v \geqslant 2u + 1$ *and*

(1) $u, v \equiv 1$ *or* $3 \pmod 6$, *or*

(2) $u \equiv v \equiv 5 \pmod 6$.

Theorem 1.6 (Bryant et al. [3]). *There exists a 5-cycle system of* $K_v - K_u$ *if and only if* $v \geqslant 3u/2 + 1$ *and*

(1) $u, v \equiv 1$ *or* $5 \pmod{10}$, *or*

(2) $u \equiv v \equiv 3 \pmod{10}$, *or*

(3) $u, v \equiv 7$ *or* $9 \pmod{10}$.

In this paper, we also settle this more general problem for the cases $m = 4, 6, 7, 8, 10, 12$ and 14. There are some obvious necessary conditions for the existence of an m-cycle system of $K_v - K_u$.

Lemma 1.1. *Let $v > u \geqslant 1$. If there exists an m-cycle system of $K_v - K_u$ then*
(1) $(v-m)(v-1) \geqslant u(u-1)$,
(2) $u \equiv v \equiv 1 \pmod 2$,
(3) $(v-u)(v+u-1) \equiv 0 \pmod{2m}$,
(4) $v \geqslant (m+1)u/(m-1)+1$ *if m is odd.*

Remark. Notice that Lemma 1.1(1) implies that if $u \neq v$ then $v \geqslant m$ and $v \geqslant u + m/2$. ($v \geqslant m$ is immediate, and $v \geqslant u + m/2$ follows since in the range $m \leqslant v \leqslant u + m/2$ the function $f(v) = (v-m)(v-1)$ is increasing, and since $f(u+m/2) < u(u-1)$.)

Proof. Since $v > u$, in order that the edges incident with a vertex of degree $v-1$ be placed in m-cycles, the number of m-cycles must be at least $(v-1)/2$. That is, $(v(v-1) - u(u-1))/2m \geqslant (v-1)/2$, so (1) follows. Each of the u vertices in the hole of $K_v - K_u$ has degree $v-u$, and each of the remaining $v-u$ vertices has degree $v-1$. Since each m-cycle includes an even number of edges incident with each vertex, (2) follows. Since $2m$ divides $\varepsilon(K_v - K_u) = (v-u)(v+u-1)$, (3) follows. Finally, (4) follows from [12]. □

We will need the following result concerning the existence of m-cycle systems of K_n.

Theorem 1.7 (Jackson [8], Kotzig [9] and Rosa [13]). *There exists an m-cycle system of K_n in each of the following cases:*
(1) *m is even and* $n \equiv 1 \pmod{2m}$;
(2) *m is odd and* $n \equiv 1$ *or* $m \pmod{2m}$.

In Section 2 we solve the existence problem for m-cycle systems of $K_v - K_u$ when m is even and $6 \leqslant m \leqslant 14$. In Section 3 we solve the embedding problem for m-cycle systems when $m \in \{11, 13\}$, supplementing Theorems 1.3 and 1.4. Finally, in Section 4 we solve the existence problem for 7-cycle systems of $K_v - K_u$.

2. Even cycles

Theorem 2.1 (Sotteau [14]). *Let m be even, and $x, y \geqslant 1$. There exists an m-cycle system of $K_{x,y}$ if and only if*
(1) $x,\ y \geqslant m/2$,
(2) $xy \equiv 0 \pmod m$, *and*
(3) $x \equiv y \equiv 0 \pmod 2$.

Proposition 2.1. *Let m be even. Suppose $u = 1$, $u > m/2$ or $v > u$. If there exists an m-cycle system of $K_v - K_u$ then there exists an m-cycle system of $K_{v+xm} - K_{u+ym}$ with $x \geqslant y \geqslant 0$ and $x \equiv y \pmod 2$.*

Proof. Let $(U \cup V, C)$ be an m-cycle system of $K_v - K_u$ where U is the set of u vertices in the hole and V is the set of the remaining $v-u$ vertices. Let U' and V' be sets of ym and $(x-y)m$ further vertices, respectively. Form an m-cycle system $(U \cup U' \cup V \cup V', C')$ of $K_{v+xm} - K_{u+ym}$ with hole $U \cup U'$ by defining $C' = C \cup C_1 \cup C_2 \cup C_3$ as follows.

Let $(U' \cup V, C_1)$ be an m-cycle system of $K_{ym,v-u}$ with bipartition U' and V. This exists by Theorem 1.2 since $v-u \geqslant m/2$ (by the remark after Lemma 1.1) or $v-u = 0$, and since $v - u \equiv 0 \pmod 2$ by Lemma 1.1(2).

Let $z \in U \cup V$. Let $((U \cup U' \cup V \cup V')\backslash\{z\}, C_2)$ be an m-cycle system of $K_{u+ym+v-u-1,(x-y)m}$ with bipartition $(U \cup U' \cup V)\backslash\{z\}$ and V'. This exists by Theorem 1.2 since $u + ym + v - u - 1 = ym + v - 1 \equiv 0 \pmod 2$ by Lemma 1.1(2), and since if $v \neq 1$ then $ym + v - 1 \geqslant m/2$ (either $v > u$ in which case $v - u \geqslant m/2$, or $v = u > m/2$ by the hypothesis of this Proposition).

Let $(\{z\} \cup V', C_3)$ be an m-cycle system of $K_{(x-y)m+1}$. This exists by Theorem 1.7 since $x \equiv y \pmod 2$. □

Proposition 2.2. *Let m be even and $u \equiv v \pmod{2m}$. Suppose $u = 1$ or $u > m/2$. There exists an m-cycle system of $K_v - K_u$.*

Proof. There exists an m-cycle system of $K_u - K_u$ since this graph has no edges. The result therefore follows from Proposition 2.1. □

Corollary 2.1. *Let $k > 1$, $m = 4$ or 8 and $v > u$. There exists an m-cycle system of $K_v - K_u$ if and only if $u \equiv v \pmod{2m}$.*

Proof. The necessity follows from Lemma 1.1(2) and (3), and the sufficiency follows from Proposition 2.2 unless $m = 8$ and $u = 3$. If $m = 8$ and $u = 3$ then by Proposition 2.1 it suffices to find an 8-cycle system of $K_{19} - K_3$. This is done in Appendix A. □

Proposition 2.3. *For $v > u$, there exists a 6-cycle system of $K_v - K_u$ if and only if $v \geqslant u + 3$ and*

(1) u, $v \equiv 1$ *or* $9 \pmod{12}$, *or*
(2) u, $v \equiv 3$ *or* $7 \pmod{12}$, *or*
(3) $u \equiv v \equiv 5 \pmod{12}$, *or*
(4) $u \equiv v \equiv 11 \pmod{12}$.

Proof. The necessity follows from Lemma 1.1(1)–(3), so we now construct 6-cycle systems of $K_v - K_u$ in each of the four cases. In view of Proposition 2.2, we may assume that either $u \not\equiv v \pmod{2m}$ or $u \equiv v \pmod{2m}$ and $u = 3$. Using Proposition

2.1, it remains to construct 6-cycle systems of $K_9 - K_1$, $K_{13} - K_9$, $K_7 - K_3$, $K_{15} - K_7$ and $K_{15} - K_3$. This is done in Appendix A. □

Proposition 2.4. *For $v > u$, there exists a* 10-*cycle system of $K_v - K_u$ if and only if $v \geqslant u + 5$ and*

(1) u, $v \equiv 1$ *or* 5 (mod 20), *or*
(2) $u \equiv v \equiv 3$ (mod 20), *or*
(3) u, $v \equiv 7$ *or* 19 (mod 20), *or*
(4) u, $v \equiv 9$ *or* 17 (mod 20), *or*
(5) u, $v \equiv 11$ *or* 15 (mod 20), *or*
(6) $u \equiv v \equiv 13$ (mod 20).

Proof. The necessity follows from Lemma 1.1(1)–(3). To prove the sufficiency, using Propositions 2.1 and 2.2 as in the case where $m = 6$, it remains to construct 10-cycle systems of $K_{25} - K_1$, $K_{21} - K_5$, $K_{17} - K_9$ and $K_{19} - K_7$ for $u \not\equiv v \pmod{2m}$, and $K_{23} - K_3$ and $K_{25} - K_5$ when $u \equiv v \pmod{2m}$. This is done in Appendix A. □

Proposition 2.5. *For $v > u$, there exists a* 12-*cycle system of $K_v - K_u$ if and only if $v \geqslant u + 6$ and*

(1) u, $v \equiv 1, 9$ (mod 24), *or*
(2) u, $v \equiv 3$, 19 (mod 24), *or*
(3) $u \equiv v \equiv 5$ (mod 24), *or*
(4) u, $v \equiv 7, 15$ (mod 24), *or*
(5) $u \equiv v \equiv 11$ (mod 24), *or*
(6) u, $v \equiv 13, 21$ (mod 24), *or*
(7) $u \equiv v \equiv 17$ (mod 24), *or*
(8) $u \equiv v \equiv 23$ (mod 24).

Proof. The necessity follows from Lemma 1.1. To prove the sufficiency, using Propositions 2.1 and 2.2 it remains to construct 12-cycle systems of $K_{25} - K_9$, $K_{19} - K_3$, $K_{15} - K_7$ and $K_{21} - K_{13}$ for $u \not\equiv v \pmod{2m}$, and $K_{27} - K_3$ and $K_{29} - K_5$ when $u \equiv v \pmod{2m}$. This is done in Appendix A. □

Proposition 2.6. *For $v > u$, there exists a* 14-*cycle system of $K_v - K_u$ if and only if $v \geqslant u + 7$ and*

(1) u, $v \equiv 1$ *or* 21 (mod 28), *or*
(2) u, $v \equiv 3$ *or* 19 (mod 28), *or*
(3) u, $v \equiv 5$ *or* 17 (mod 28), *or*
(4) u, $v \equiv 7$ *or* 15 (mod 28), *with* $(u, v) \neq (7, 15)$, *or*
(5) u, $v \equiv v \equiv 9, 13$ (mod 28), *or*
(6) $u \equiv 11$ (mod 28), *or*
(7) $u > v \equiv 23, 27$ (mod 28), *or*
(8) $u \equiv v \equiv 25$ (mod 28), *or*

Proof. The necessity follows from Lemma 1.1. To prove the sufficiency, using Propositions 2.1 and 2.2 it remains to construct 14-cycle systems of $K_{21} - K_1$, $K_{19} - K_3$, $K_{29} - K_{21}$, $K_{17} - K_5$ and $K_{43} - K_7$ when $u \not\equiv v \pmod{2m}$, and $K_{31} - K_3$, $K_{33} - K_5$ and $K_{35} - K_7$ when $u \equiv v \pmod{2m}$. This is done in Appendix A. □

3. Odd cycles

The following result is a great step towards settling the embedding problem for m-cycle systems.

Theorem 3.1 (Bryant and Rodger [5]). *Let m be odd and let u and v be 1 or m $\pmod{2m}$. Any m-cycle system of order u can be embedded in an m-cycle system of order v iff $v \geqslant (m+1)u/(m-1)+1$, except possibly for the smallest such value of v in the particular case where $u \equiv v \equiv m \pmod{2m}$ and either*

(a) *$m \equiv 1 \pmod 4$ and the remainder when u is divided by $m-1$ is odd and greater than $m/2$, or*

(b) *$m \equiv 3 \pmod 4$ and the remainder when u is divided by $(m-1)/$ is even and nonzero.*

A method for dealing with the possible exceptions in Theorem 3.1 is given by the following result.

Proposition 3.1 (Bryant and Rodger [5]). *Let $m \equiv 3 \pmod 4$, $u \equiv m \pmod{2m}$, and $u = a(m-1)/2 + \theta$ where $0 < \theta < (m-1)/2$, and let θ be even. Suppose there exists a simple graph G satisfying*

(1) *G has $2m$ vertices and is $(m-2\theta)$-regular;*

(2) *there exists an m-cycle system of $K^c_{(m-1)/2+\theta} \vee G$; and*

(3) *G^c has a 1-factorization.*

Then any m-cycle system of order u can be embedded in an m-cycle system of order v, where v is the smallest integer at least $(m+1)u/(m-1)+1$ with $v \equiv m \pmod{2m}$.

Using these two results we can get the following.

Proposition 3.2. *There exists an 11-cycle system of K_v containing an 11-cycle system of K_u iff u, $v \equiv 1$ or 11 $\pmod{22}$ and $v \geqslant 6u/5 + 1$.*

Proof. From Lemma 1.1(2) and (3), in order for there to exist an 11-cycle system of K_u, $u \equiv 1$ or 11 $\pmod{22}$, and similarly $v \equiv 1$ or 11 $\pmod{22}$, and $v \geqslant 6u/5 + 1$ by Lemma 1.1(4), so it remains to prove the sufficiency.

By Theorem 3.1, we need only consider the smallest such v in some cases, and by Proposition 3.1 these cases are settled by constructing two graphs, one for each even θ satisfying $2 \leqslant \theta \leqslant 4$. This is done in Appendix B. □

A more general result than the following is proved in [5]. This is a companion result to Proposition 3.2, dealing with the possible exceptions in Theorem 3.1.

Proposition 3.3 (Bryant and Rodger [5]). *Let* $m \equiv 1 \pmod 4$, $u \equiv m \pmod{2m}$, *and* $u = a(m-1)+\theta$, *where* θ *is odd and* $(m+1)/2 \leqslant \theta \leqslant m-2$. *Suppose there exists a simple graph* G *satisfying*

(1) G *has* $2m$ *vertices and is* $(2m-2\theta-1)$*-regular*;

(2) *there exists an* m*-cycle system of* $K_\theta^c \vee G$; *and*

(3) G^c *has a 2-factorization, each 2-factor* F *of which is either* $\langle\{d\},\emptyset,\{d\}\rangle_m$ *with* $1 \leqslant d < m/2$ *or* $\langle\emptyset,\{s,s+1\},\emptyset\rangle_m$ *with* $0 \leqslant s < m$.

Then any m*-cycle system of order* u *can be embedded in an* m*-cycle system of order* v, *where* v *is the smallest integer at least* $(m+1)u/(m-1)+1$ *with* $v \equiv m \pmod{2m}$.

We can use this to settle the embedding problem when $m = 13$.

Proposition 3.4. *There exists a* 13*-cycle system of* K_v *containing a* 13*-cycle system of* K_u *iff* $u, v \equiv 1$ *or* $13 \pmod{26}$ *and* $v \geqslant 7u/6+1$.

Proof. From Lemma 1.1(2) and (3), in order for there to exist an 13-cycle system of K_u, $u \equiv 1$ or $13 \pmod{26}$, and similarly $v \equiv 1$ or $13 \pmod{26}$, and $v \geqslant 7u/6+1$ by Lemma 1.1(4), so it remains to prove the sufficiency.

By Theorem 3.1, we need only consider the smallest such v in some cases, and by Proposition 3.3 these cases are settled by constructing three graphs, one for each odd θ satisfying $7 \leqslant \theta \leqslant 11$. This is done in Appendix B. □

4. 7-cycle systems of $K_v - K_u$

We begin with some known results that will be used later.

Theorem 4.1 (Hoffman and Rodger [7], and Laskar and Hare [10]). *There exists a* 1*-factorization of the complete multipartite graph with* $2m$ *vertices in each part.*

The following lemma is a particular case of a result of Stern and Lenz.

Lemma 4.1 (Stern and Lenz [15]). *Let* $D \subseteq \{1,2,\ldots,\lfloor x/2 \rfloor\}$ *and* $S \subseteq \mathbb{Z}_x$. *If either* $|S| \geqslant 1$ *or* $x/2 \in D$ *then* $\langle D,S,D\rangle_x$ *has a* 1*-factorization.*

Lemma 4.2. *If* 7 *divides* x *and* $D \subseteq \{x/7, 2x/7, 3x/7\}$ *then there exists a* 7*-cycle system of* $\langle D,\emptyset,D\rangle_x$.

Proof. Clearly, for each $d \in D$, $\langle\{d\},\emptyset,\{d\}\rangle_x$ is a pair of 7-cycles. □

The following three lemmas are special cases of more general results proven in [5].

Lemma 4.3. *If 7 divides x then there exists a 7-cycle system of $K_2^c \vee \langle x/7, \{0,1\}, x/7\rangle_x$.*

Lemma 4.4. *If F is a 1-factor, with at least 4 vertices then there exists a 7-cycle system of $K_3^c \vee F$.*

Lemma 4.5. *Let*
(1) $x \geqslant 5, 0 \leqslant \beta \leqslant x-5, D = \emptyset$ *and* $S = \{\beta, \beta+1, \beta+2, \beta+3, \beta+4\}$; *or*
(2) $x \geqslant 3, 0 \leqslant \beta \leqslant x-3, D = \{d\} \subseteq \{1,2,\ldots,\lfloor (x-1)/2 \rfloor\}$ *and* $S = \{\beta, \beta+1, \beta+2\}$; *or*
(3) $x \geqslant 5, 0 \leqslant \beta \leqslant x-1, D = \{d_1, d_2\} \subseteq \{1,2,\ldots,\lfloor (x-1)/2 \rfloor\}$ *and* $S = \{\beta\}$.
Then there exists a 7-cycle system of $K_1 \vee \langle D,S,D\rangle_x$.

We can now obtain the following corollary from Lemmas 4.3–4.5.

Corollary 4.1. *Let $y \geqslant 1, z \geqslant 0$ and $x \geqslant 3$. Then if*
(1) $x = (y+5z+1)/2$, $D = \{1,2,\ldots,\lfloor x/2 \rfloor\}$ *and* $S = \mathbb{Z}_x$, *or*
(2) *7 divides x*, $x = (y+5z+3)/2, D = \{1,2,\ldots,\lfloor x/2 \rfloor\} \setminus \{x/7\}$ *and* $S = \mathbb{Z}_x$, *or*
(3) *7 divides x*, $x = (y+5z+5)/2, D = \{1,2,\ldots,\lfloor x/2 \rfloor\} \setminus \{x/7, 2x/7\}$ *and* $S = \mathbb{Z}_x$, *or*
(4) *7 divides x*, $x = (y+5z+7)/2, D = \{1,2,\ldots,\lfloor x/2 \rfloor\} \setminus \{x/7, 2x/7\}$ *and* $S = \{2,3,\ldots,x-1\}$, *or*
(5) *7 divides x*, $x = (y+5z+9)/2, D = \{1,2,\ldots,\lfloor x/2 \rfloor\} \setminus \{x/7, 2x/7, 3x/7\}$ *and* $S = \{2,3,\ldots,x-1\}$,
there exists a 7-cycle system of $K_{3y+z}^c \vee \langle D,S,D\rangle_x$.

Proof. If x is odd then let $g = |D|$ and let $D' = D = \{d_1, d_2, \ldots, d_g\}$, if x is even then let $g = |D| - 1$ and let $D' = D \setminus \{x/2\} = \{d_1, d_2, \ldots, d_g\}$, and in either case let $1 \leqslant d_1 < d_2 < \cdots < d_g$. Also let $t_i = 1$ for $1 \leqslant i \leqslant z$ and $t_i = 3$ for $z+1 \leqslant i \leqslant z+y$. We write $\langle D,S,D\rangle_x = \bigcup_{i=1}^{z+y} \langle D_i, S_i, D_i\rangle_x$ where for $1 \leqslant i \leqslant z+y$, D_i and S_i are defined recursively as follows.

Let $\alpha = 0$, let β_1 be such that $S = \{\beta_1+1, \beta_1+2, \ldots, x-1\}$ and let $\gamma_1 = \min\{3-t_1, g\}$. For $2 \leqslant i \leqslant z+y-1$ define $\alpha_i = \alpha_{i-1} + \gamma_{i-1}$, $\beta_i = \beta_{i-1} + (7 - 2t_{i-1} - 2\gamma_{i-1})$, and $\gamma_i = \min\{3 - t_i, g - \alpha_i\}$. Finally, define $D_i = \{d_{\alpha_i+1}, d_{\alpha_i+2}, \ldots, d_{\alpha_i+\gamma_i}\}$ and $S_i = \{\beta_i+1, \beta_i+2, \ldots, \beta_i + (7 - 2t_i - 2\gamma_i)\}$.

Now, for $1 \leqslant i \leqslant z$ there is a 7-cycle system of $K_1 \vee \langle D_i, S_i, D_i\rangle_x$ by Lemma 4.5 and for $z+1 \leqslant i \leqslant z+y-1$ there is a 7-cycle system of $K_3^c \vee \langle D_i, S_i, D_i\rangle_x$ by Lemma 4.4.

It is straightforward to check that $D_i \subseteq D'$ and $S_i \subseteq S$ for $1 \leqslant i \leqslant z+y-1$ and that for $1 \leqslant i < j \leqslant z+y-1$, $D_i \cap D_j = \emptyset$ and $S_i \cap S_j = \emptyset$.

We now show that in each of the cases (1)–(5), either $D \setminus \bigcup_{i=1}^{z+y-1} D_i = \emptyset$ and $S \setminus \bigcup_{i=1}^{z+y-1} S_i = \{x-1\}$ or $D \setminus \bigcup_{i=1}^{z+y-1} D_i = \{x/2\}$ and $S \setminus \bigcup_{i=1}^{z+y-1} S_i = \emptyset$, so that $\langle D_{z+y}, S_{z+y}, D_{z+y}\rangle_x = \langle D,S,D\rangle_x \setminus (\bigcup_{i=1}^{z+y-1} \langle D_i, S_i, D_i\rangle_x)$ is a 1-factor.

In each case $|E(\bigcup_{i=1}^{z+y-1} \langle D_i, S_i, D_i\rangle_x)| = x \sum_{i=1}^{z+y-1} (7 - 2t_i) = x(\sum_{i=1}^{z} 5 + \sum_{i=z+1}^{z+y-1} 1) = x(5z+y-1)$. Hence, $|E(\langle D,S,D\rangle_x)| - |E(\bigcup_{i=1}^{z+y-1} \langle D_i, S_i, D_i\rangle_x)|$ is
(1) $x(2x-1) - x(5z+y-1) = x$ in case (1);

(2) $x(2x-1)-2x-x(5z+y-1)=x(2x-1)-2x-x(2x-4)=x$ in case (2);
(3) $x(2x-1)-4x-x(5z+y-1)=x(2x-1)-4x-x(2x-6)=x$ in case (3);
(4) $x(2x-1)-6x-x(5z+y-1)=x(2x-1)-6x-x(2x-8)=x$ in case (4);
and is
(5) $x(2x-1)-8x-x(5z+y-1)=x(2x-1)-8x-x(2x-10)=x$ in case (5).

So in any case either $D\setminus\bigcup_{i=1}^{z+y-1}D_i=\emptyset$ and $S\setminus\bigcup_{i=1}^{z+y-1}S_i=\{x-1\}$ or $D\setminus\bigcup_{i=1}^{z+y-1}D_i=\{x/2\}$ and $S\setminus\bigcup_{i=1}^{z+y-1}S_i=\emptyset$, so $\langle D_{z+y},S_{z+y},D_{z+y}\rangle_x=\langle D,S,D\rangle_x-(\bigcup_{i=1}^{z+y-1}\langle D_i,S_i,D_i\rangle_x)$ is a 1-factor. Hence, by 4.4 there is a 7-cycle system of $K_{t_{z+y}}^c\vee\langle D_{z+y},S_{z+y},D_{z+y}\rangle_x$, and the union of the 7-cycle systems of $K_{t_i}^c\vee\langle D_i,S_i,D_i\rangle_x$ for $1\leqslant i\leqslant z+y$ forms the required 7-cycle system. □

Lemma 4.6. *For any odd integer $T\geqslant 3$, and for all odd integers B with $T-2\lfloor T/3\rfloor\leqslant B\leqslant T-2$ there exist integers z and y with $z\geqslant 0$ and $y\geqslant 1$ such that $3y+z=T$ and $z+y=B$.*

Proof. If $B=T-2\lfloor T/3\rfloor$ then let $y=\lfloor T/3\rfloor$ and $z=T-3y$. To complete the proof we proceed inductively, so suppose that z' and y' satisfy the conditions of the lemma and that $B<T-2$. Then it is clear that $y'\geqslant 2$, and so if we let $y=y'-1$ and $z=z'+3$ we see that $y\geqslant 1, z\geqslant 0, 3y+z=T$ and $z+y=B+2$. □

Lemma 4.7. *In each of the cases:*
(1) $u\equiv 3 \pmod{14}$ *and* $v\equiv 5 \pmod{14}$,
(2) $u\equiv 5 \pmod{14}$ *and* $v\equiv 3 \pmod{14}$,
(3) $u\equiv 9 \pmod{14}$ *and* $v\equiv 13 \pmod{14}$,
(4) $u\equiv 13 \pmod{14}$ *and* $v\equiv 9 \pmod{14}$,
(5) $u,v\equiv 11 \pmod{14}$,
if $4u/3+1\leqslant v\leqslant 6u-13$ then there exists a 7-cycle system of K_v-K_u.

Proof. By Lemma 4.6, with $T=u$, we can choose integers $z\geqslant 0$ and $y\geqslant 1$ such that $3y+z=u$ and $z+y=B$ for any odd integer B with $u-2\lfloor u/3\rfloor\leqslant B\leqslant u-2$. Now if we apply Corollary 4.1(1) with $x=(v-u)/2$ we obtain a 7-cycle system of K_v-K_u with $v=u+5z+y+1=u+5(3B-u)/2+(u-B)/2+1=7B-u+1$ (substituting $y=(u-B)/2$ and $z=(3B-u)/2$) for each odd integer B in the range $u-2\lfloor u/3\rfloor\leqslant B\leqslant u-2$. It remains to show that this covers all v in the given range.

For given u, the smallest v for which we obtain a 7-cycle system of K_v-K_u is $v_{\min}=7(u-2\lfloor u/3\rfloor)-u+1=6u-14\lfloor u/3\rfloor+1$. Hence, $v_{\min}\leqslant 6u-14(u-2)/3+1=(4u+31)/3$ and so, $v_{\min}-(4u/3+1)\leqslant 28/3<14$. The largest v for which we obtain a 7-cycle system of K_v-K_u is $v_{\max}=7(u-2)-u+1=6u-13$. Clearly, if we increase B by 2 we increase v by 14 and so we obtain a 7-cycle system of K_v-K_u for all v satisfying the conditions of the theorem. □

Lemma 4.8. *If* $u \equiv 3 \pmod{14}$ *and* $v \equiv 3 \pmod{14}$ *and if* $4u/3+1 \leqslant v \leqslant 6u-15$ *then there exists a 7-cycle system of* $K_v - K_u$, *except possibly when* $u \equiv 1 \pmod 3$ *and* $4u/3+1 \leqslant v \leqslant 4u/3+15$.

Proof. By Lemma 4.6, with $T = u-2$, we can choose integers $z \geqslant 0$ and $y \geqslant 1$ such that $3y+z = u-2$ and $z+y = B$ for any odd integer B with $u-2-2\lfloor (u-2)/3 \rfloor \leqslant B \leqslant u-4$. Now if we apply Corollary 4.1(5) with $x = (v-u)/2$, Lemmas 4.2 and 4.3, we obtain a 7-cycle system of $K_v - K_u$ with $v = u+5z+y+9 = u+5(3B-u+2)/2+(u-B-2)/2+9 = 7B-u+13$ (substituting $y = (u-B-2)/2$ and $z = (3B-u+2)/2$) for each odd integer B in the range $u-2-2\lfloor (u-2)/3 \rfloor \leqslant B \leqslant u-4$. It remains to show that this covers all v in the given range.

For given u, the smallest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\min} = 7(u-2-2\lfloor (u-2)/3 \rfloor)-u+13 = 6u-14\lfloor (u-2)/3 \rfloor -1$. Now, if $u \equiv 0$ or $2 \pmod 3$ then $\lfloor (u-2)/3 \rfloor \leqslant (u-3)/3$ and so $v_{\min}-(4u/3+1) \leqslant 6u-14(u-3)/3-1-(4u/3+1) = 12 < 14$. Also, if $u \equiv 1 \pmod 3$ then $\lfloor (u-2)/3 \rfloor = (u-4)/3$ and so $v_{\min}-(4u/3+15) = 6u-14(u-4)/3-1-(4u/3+15) = 8/3 < 14$. The largest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\max} = 7(u-4)-u+13 = 6u-15$. Clearly, if we increase B by 2 we increase v by 14 and, so we obtain a 7-cycle system of $K_v - K_u$ for all v satisfying the conditions of the theorem. □

Lemma 4.9. *If* $u \equiv 5 \pmod{14}$ *and* $v \equiv 5 \pmod{14}$ *and if* $4u/3+1 \leqslant v \leqslant 6u-11$ *then there exists a 7-cycle system of* $K_v - K_u$.

Proof. By Lemma 4.6, with $T = u$, we can choose integers $z \geqslant 0$ and $y \geqslant 1$ such that $3y+z = u$ and $z+y = B$ for any odd integer B with $u-2\lfloor u/3 \rfloor \leqslant B \leqslant u-2$. Now if we apply Corollary 4.1(2) with $x = (v-u)/2$ and Lemma 4.2, we obtain a 7-cycle system of $K_v - K_u$ with $v = u+5z+y+3 = u+5(3B-u)/2+(u-B)/2+3 = 7B-u+3$ (substituting $y = (u-B)/2$ and $z = (3B-u)/2$) for each odd integer B in the range $u-2\lfloor u/3 \rfloor \leqslant B \leqslant u-2$. It remains to show that this covers all v in the given range.

For given u, the smallest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\min} = 7(u-2\lfloor u/3 \rfloor)-u+3 = 6u-14\lfloor u/3 \rfloor +3$. Hence, $v_{\min} \leqslant 6u-14(u-2)/3+3 = (4u+37)/3$ and, so, $v_{\min}-(4u/3+1) \leqslant 34/3 < 14$. The largest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\max} = 7(u-2)-u+3 = 6u-11$. Clearly, if we increase B by 2 we increase v by 14 and, so we obtain a 7-cycle system of $K_v - K_u$ for all v satisfying the conditions of the theorem. □

Lemma 4.10. *If* $u \equiv 9 \pmod{14}$ *and* $v \equiv 9 \pmod{14}$ *and if* $4u/3+1 \leqslant v \leqslant 6u-17$ *then there exists a 7-cycle system of* $K_v - K_u$, *except possibly when* $u \equiv 1 \pmod 3$ *and* $4u/3+1 \leqslant v \leqslant 4u/3+15$.

Proof. By Lemma 4.6, with $T = u-2$, we can choose integers $z \geqslant 0$ and $y \geqslant 1$ such that $3y+z = u-2$ and $z+y = B$ for any odd integer B with $u-2-2\lfloor (u-2)/3 \rfloor \leqslant B \leqslant u-4$.

Now if we apply Corollary 4.1(4) with $x = (v-u)/2$, Lemmas 4.2 and 4.3, we obtain a 7-cycle system of $K_v - K_u$ with $v = u+5z+y+7 = u+5(3B-u+2)/2+(u-B-2)/2+7 = 7B - u + 11$ (substituting $y = (u - B - 2)/2$ and $z = (3B - u + 2)/2$) for each odd integer B in the range $u - 2 - 2\lfloor (u-2)/3 \rfloor \leqslant B \leqslant u - 4$. It remains to show that this covers all v in the given range.

For given u, the smallest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\min} = 7(u-2-2\lfloor (u-2)/3 \rfloor)-u+11 = 6u-14\lfloor (u-2)/3 \rfloor -3$. Now, if $u \equiv 0$ or $2 \pmod 3$ then $\lfloor (u-2)/3 \rfloor \leqslant (u-3)/3$ and so $v_{\min}-(4u/3+1) \leqslant 6u-14(u-3)/3-3-(4u/3+1) = 10 < 14$. Also, if $u \equiv 1 \pmod 3$ then $\lfloor (u-2)/3 \rfloor = (u-4)/3$ and so $v_{\min} - (4u/3 + 15) = 6u - 14(u-4)/3 - 3 - (4u/3 + 15) = 2/3 < 14$. The largest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\max} = 7(u-4) - u + 11 = 6u - 17$. Clearly, if we increase B by 2 we increase v by 14 and so we obtain a 7-cycle system of $K_v - K_u$ for all v satisfying the conditions of the theorem. □

Lemma 4.11. *If* $u \equiv 13 \pmod{14}$ *and* $v \equiv 13 \pmod{14}$ *and if* $4u/3 + 1 \leqslant v \leqslant 6u - 9$ *then there exists a 7-cycle system of* $K_v - K_u$.

Proof. By Lemma 4.6, with $T = u$, we can choose integers $z \geqslant 0$ and $y \geqslant 1$ such that $3y+z = u$ and $z+y = B$ for any odd integer B with $u-2\lfloor u/3 \rfloor \leqslant B \leqslant u-2$. Now if we apply Corollary 4.1(3) with $x = (v-u)/2$ and Lemma 4.2 we obtain a 7-cycle system of $K_v - K_u$ with $v = u + 5z + y + 5 = u + 5(3B - u)/2 + (u - B)/2 + 5 = 7B - u + 5$ (substituting $y = (u - B)/2$ and $z = (3B - u)/2$) for each odd integer B in the range $u - 2\lfloor u/3 \rfloor \leqslant B \leqslant u - 2$. It remains to show that this covers all v in the given range.

For given u, the smallest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\min} = 7(u - 2\lfloor u/3 \rfloor) - u + 5 = 6u - 14\lfloor u/3 \rfloor + 5$. Hence, $v_{\min} \leqslant 6u - 14(u-2)/3 + 5 = (4u+43)/3$ and so, $v_{\min} - (4u/3+1) \leqslant 40/3 < 14$. The largest v for which we obtain a 7-cycle system of $K_v - K_u$ is $v_{\max} = 7(u-2) - u + 5 = 6u - 9$. Clearly, if we increase B by 2 we increase v by 14 and so we obtain a 7-cycle system of $K_v - K_u$ for all v satisfying the conditions of the theorem. □

Notice that Theorem 1.3 and Lemmas 4.7–4.11 yield, for given u, a 7-cycle system of $K_v - K_u$ for all the possible values (as allowed by Lemma 1.1) of $v \leqslant 6u - 17$, except in the cases

(a) $u \equiv 31 \pmod{42}$ and $v = 4u/3 + 11/3$ and

(b) $u \equiv 37 \pmod{42}$ and $v = 4u/3 + 5/3$.

We now proceed to remove these exceptions.

Lemma 4.12. *There exists a 4-regular graph* G *on 14 vertices such that there is a 7-cycle system of* $K_4^c \vee G$ *and such that there is a 1-factorization of* G^c.

Proof. Let $G = \langle \emptyset, \{0,1,2,3\}, \emptyset \rangle_x$ (so G^c is 4-regular and has a 1-factorization by Lemma 4.1) and let the vertices in K_4^c be $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. The set C given below forms a 7-cycle system of $K_4^c \vee G$:

$$C = \{ ((0,0),(0,1),\infty_4,(1,0),\infty_3,(2,0),(2,1)),((0,0),(1,1),\infty_4,(2,0),\infty_2,(1,0),(3,1)),((0,0),\infty_1,(6,1),(3,0),\infty_4,(4,0),\infty_2),((0,0),\infty_3,(6,1),(4,0),\infty_1,(5,0),\infty_4),((1,0),(1,1),\infty_3,(3,0),\infty_2,(6,0),(2,1)),((1,0),(4,1),\infty_4,(6,0),\infty_3,(0,1),\infty_1),((2,0),(3,1),\infty_4,(2,1),\infty_3,(4,0),(4,1)),((2,0),(5,1),\infty_4,(6,1),\infty_2,(1,1),\infty_1),((3,0),(3,1),\infty_3,(5,0),(5,1),\infty_1,(4,1)),((3,0),(5,1),\infty_3,(4,1),\infty_2,(2,1),\infty_1),((4,0),(0,1),(6,0),(1,1),(5,0)\infty_2,(5,1)),((5,0),(0,1),\infty_2,(3,1),\infty_1(6,0),(6,1))\}. \quad \square$$

Lemma 4.13. *There exists a 2-regular graph G on* 14 *vertices such that there is a 7-cycle system of $K_4^c \vee G$ and such that there is a 1-factorization of G^c.*

Proof. Let $G = \langle \emptyset, \{0,1\}, \emptyset \rangle_x$ (so G^c is 2-regular and has a 1-factorization by Lemma 4.1) and let the vertices in K_4^c be $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. The set C given below forms a 7-cycle system of $K_4^c \vee G$:

$$C = \{((0,0),(0,1),\infty_4,(1,0),\infty_3,(2,0),\infty_1),((0,0),(1,1),\infty_4,(2,0),\infty_2,(3,0),\infty_3),((0,0),\infty_2,(6,1),(5,0),\infty_3,(4,0),\infty_4),((1,0),(1,1),\infty_3,(6,0),\infty_4,(3,0),\infty_1),((1,0),(2,1),\infty_4,(5,0),\infty_1,(0,1),\infty_2),((2,0),(2,1),\infty_3,(0,1),(6,0),\infty_1,(3,1)),((3,0),(3,1),\infty_4,(5,1),\infty_2,(4,0),(4,1)),((4,0),(5,1),\infty_3,(3,1),\infty_2,(1,1),\infty_1),((5,0),(5,1),\infty_1,(6,1),\infty_4,(4,1),\infty_2),((6,0),(6,1),\infty_3,(4,1),\infty_1,(2,1),\infty_2)\}. \quad \square$$

Lemma 4.14. *If $u \equiv 31 \pmod{42}$ and $v = 4u/3 + 11/3$ then there is a 7-cycle system of $K_v - K_u$.*

Proof. Let $u = 42k + 31$ so that $v = 56k + 45$ and let $G = G_1 \vee G_2 \vee \cdots \vee G_{14k+9} \vee G'$ where $G_i \cong K_3$ for $1 \leqslant i \leqslant 14k + 9$ and $G' \cong K_4$ so $G \cong K_u$. Also let $H = H_1 \vee H_2 \vee \cdots \vee H_{k+1}$ where $H_i \cong K_{14}$ for $1 \leqslant i \leqslant k + 1$ so $G^c \vee H \cong K_v - K_u$. By Lemma 4.12, there is a 4-regular subgraph H_i' of H_i such that there exists a 7-cycle system of $G'^c \vee H_i'$ and such that $H_i - H_i'$ has a 1-factorization. Hence, by Theorem 4.1 there is a 1-factorization of $H - (\bigcup_{i=1}^{k+1} H_i')$. Now, since there are 28 edges in each H_i', there are $7(k + 1)(14k + 9)$ edges in $H - (\bigcup_{i=1}^{k+1} H_i')$ and hence $14k + 9$ 1-factors. Let these 1-factors be $F_1, F_2, \ldots, F_{14k+9}$. By Lemma 4.4, there is a 7-cycle system of $G_i \vee F_i$

for $1 \leqslant i \leqslant 14k+9$. The union of these 7-cycle systems with the 7-cycle systems of $G' \vee H_i'$, $(1 \leqslant i \leqslant k+1)$ yield a 7-cycle system of $G^c \vee H$. □

Lemma 4.15. *If* $u \equiv 37 \pmod{42}$ *and* $v = 4u/3 + 5/3$ *then there is a 7-cycle system of* $K_v - K_u$.

Proof. Let $u = 42k+37$ so that $v = 56k+51$ and let $G = G_1 \vee G_2 \vee \cdots \vee G_{14k+11} \vee G'$ where $G_i \cong K_3$ for $1 \leqslant i \leqslant 14k+11$ and $G' \cong K_4$ so $G \cong K_u$. Also let $H = H_1 \vee H_2 \vee \cdots \vee H_{k+1}$ where $H_i \cong K_{14}$ for $1 \leqslant i \leqslant k+1$ so $G^c \vee H \cong K_v - K_u$. By Lemma 4.13, there is a 2-regular subgraph H_i' of H_i such that there exists a 7-cycle system of $G'^c \vee H_i'$ and such that $H_i - H_i'$ has a 1-factorization. Hence, by Theorem 4.1 there is a 1-factorization of $H - (\bigcup_{i=1}^{k+1} H_i')$. Now, since there are 14 edges in each H_i', there are $7(k+1)(14k+11)$ edges in $H - (\bigcup_{i=1}^{k+1} H_i')$ and hence $14k+11$ 1-factors. Let these 1-factors be $F_1, F_2, \ldots, F_{14k+11}$. By Lemma 4.4, there is a 7-cycle system of $G_i \vee F_i$ for $1 \leqslant i \leqslant 14k+11$. The union of these 7-cycle systems with the 7-cycle systems of $G' \vee H_i'$, $(1 \leqslant i \leqslant k+1)$ yield a 7-cycle system of $G^c \vee H$. □

We now have the following corollary to Theorem 1.3 and Lemmas 4.7–4.14 and 4.15.

Corollary 4.2. *For all* $v \leqslant 6u - 17$ *and satisfying the conditions of Lemma* 1.1, *there exists a 7-cycle system of* $K_v - K_u$.

We need one more lemma before we prove the main theorem for 7-cycle systems of $K_v - K_u$.

Lemma 4.16. *There is a 7-cycle system of* $K_{17} - K_3$ *and of* $K_{19} - K_3$.

Proof. Let the vertex set of $K_{17} - K_3$ be $\mathbb{Z}_{17}$ with the hole elements being 14,15 and 16. Then the set C given below forms a 7-cycle system of $K_{17} - K_3$:

$$\begin{aligned} C = \{&(0,1,16,2,15,3,4),(0,2,14,1,15,4,5),(0,3,16,4,14,5,6),\\ &(0,7,16,5,15,6,8),(0,9,16,6,14,3,10),(0,11,16,8,15,7,12),\\ &(0,13,16,10,15,9,14),(0,15,13,1,11,12,16),(1,2,13,3,12,4,6),\\ &(1,3,11,2,12,5,7),(1,4,13,5,11,6,9),(1,5,10,2,9,3,8),\\ &(1,10,14,7,13,6,12),(2,3,7,4,11,8,5),(2,4,10,6,7,9,8),\\ &(2,6,3,5,9,10,7),(4,8,14,11,13,12,9),\\ &(7,8,13,10,12,15,11),(8,10,11,9,13,14,12)\}.\end{aligned}$$

Let the vertex set of $K_{19} - K_3$ be $\mathbb{Z}_{19}$ with the hole elements being 16,17 and 18. Then the set C given below forms a 7-cycle system of $K_{19} - K_3$:

$$C = \{(0,1,18,2,17,3,4),(0,2,16,1,17,4,5),(0,3,18,4,16,5,6),$$

$$(0,7,18,5,17,6,8),(0,9,18,6,16,3,10),(0,11,18,8,17,7,12),$$

$$(0,13,18,10,17,9,14),(0,15,18,12,17,11,16),(0,17,15,1,13,14,18),$$

$$(1,2,15,3,14,4,6),(1,3,13,2,14,5,7),(1,4,15,5,13,6,9),$$

$$(1,5,12,2,11,3,8),(1,10,16,7,15,6,11),(1,12,16,8,15,10,14),$$

$$(2,3,12,4,13,7,6),(2,4,11,5,10,7,8),(2,5,9,3,7,4,10),$$

$$(2,7,14,6,12,8,9),(3,5,8,4,9,10,6),(7,9,16,13,17,14,11),$$

$$(8,10,13,9,15,12,11),(8,13,12,9,11,15,14),$$

$$(10,11,13,15,16,14,12)\}. \quad \square$$

Theorem 4.2. *There exists a 7-cycle system of $K_v - K_u$ if and only if*
(a) $v \geqslant 7$, *and*
(b) $v \geqslant 4u/3 + 1$, *and*
(c) $u \equiv v \equiv 11 \pmod{14}$, *or* $u, v \equiv 1$ *or* $7 \pmod{14}$, *or* $u, v \equiv 3$ *or* $5 \pmod{14}$, *or* $u, v \equiv 9$ *or* $13 \pmod{14}$.

Proof. Necessity follows from Lemma 1.1. The proof of sufficiency is by induction on v. The result was proved for $u, v \equiv 1$ or $7 \pmod{14}$ in [5]. Also, by Corollary 4.2 and Lemma 4.16, the result is true for $v \leqslant 6u - 17$ and for $v < 31$. Let u be given and suppose that for all $s < v$, with $K_s - K_u$ satisfying the conditions of the theorem, there is a 7-cycle system of $K_s - K_u$. We show that we can always find an s such that there exists a 7-cycle system of $K_s - K_u$ and of $K_v - K_s$ and hence of $K_v - K_u$.

Let s be the largest integer $s \leqslant 3(v-1)/4$ for which there is a 7-cycle system of $K_s - K_u$. First we need to show that such an s exists. By assumption, if $3(v-1)/4 - (4u/3+1) \geqslant 14$ then such an s exists. Since $v \geqslant 31$, $3(v-1)/4-(4u/3+1) \geqslant (129-8u)/6$ which is at least 14 for $u \leqslant 5$ and so we can assume $u \geqslant 9$. It is straightforward to show, since we can assume $v > 6u-17$, that $3(v-1)/4-(4u/3+1) > (19u-87)/6$, which is at least 14 for $u \geqslant 9$. Hence, we can always find an s with $3(v-1)/4-14 \leqslant s \leqslant 3(v-1)/4$ such that there is a 7-cycle system of $K_s - K_u$.

Since $s \leqslant 3(v-1)/4$, we have $v \geqslant 4s/3 + 1$ and since $s \geqslant 3(v-1)/4 - 14$, we have $6s-17 \geqslant (9v-211)/2$ which is at least v for $v \geqslant 31$. Hence, we have $4s/3+1 \leqslant v \leqslant 6s-17$ and so there exists a 7-cycle system of $K_v - K_s$ by Corollary 4.2. $\square$

Appendix A

For each of the cases $m = 6, 10, 12$ and 14, the following names a graph G and then proves the existence of an m-cycle system of G.

$m = 6$:
$K_9 - K_1$ $(\mathbb{Z}_3 \times \mathbb{Z}_3, \{((0,0),(0,1),(0,2),(1,2),(2,1),(1,0)) + (i,j), ((0,0),(1,2),(1,0),(0,1),(1,1),(2,2)) + (i,j) \mid i,j \in \mathbb{Z}_3\})$.
$K_7 - K_3$ $(\mathbb{Z}_7, \{(0,3,4,5,2,6),(0,4,1,3,6,5),(1,5,3,2,4,6)\})$ with the hole having vertex set $\mathbb{Z}_3$.
$K_{13} - K_9$ This follows from Proposition 2.1 using $K_7 - K_3$.
$K_{15} - K_7$ This follows from Proposition 2.1 using $K_9 - K_1$.
$K_{15} - K_3$ This follows using $K_7 - K_3$ and $K_{15} - K_7$.

$m = 8$:
$K_{19} - K_3(\{\infty_0,\infty_1,\infty_2\} \cup (\mathbb{Z}_9 \times \mathbb{Z}_2), \{((0,0),(2,0),(0,1),(1,1),(4,1),(3,1),(1,0),(5,0)) + (i,0), ((\infty_0,(0,0),\infty_1,(0,1),\infty_2,(1,0),(4,0),(7,1)) + (i,0) \mid i \in \mathbb{Z}_8\} \cup \{((0,0),(0,1),(4,1),(4,0),(5,1),(6,0),(2,0),(1,1)) + (i,0) \mid i \in \mathbb{Z}_4\} \cup \{((0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0))\})$.

$m = 10$:
$K_{25} - K_1$ This is a 10-cycle system of K_{25} so is known to exist (see [1]).
$K_{21} - K_5$ $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_2), \{((0,0),(0,1),(3,1),(6,0),(5,1),(4,0),(4,1),(7,1)(2,0),(1,1)) + (i,0), ((0,0),(2,1),(7,0),(5,1),(7,1),(0,1),(4,0),(3,0),\infty_0,(4,1)) + (i,0), ((4,0),(0,0),(7,0),\infty_0,(0,1),(4,1),(3,1),(1,1),(3,0),(6,1)) + (i,0) \mid i \in \mathbb{Z}_4\} \cup \{(\infty_1,(0,0),(2,0),(5,0),\infty_2,(0,1),\infty_3,(1,0),\infty_4,(1,1)) + (i,0) \mid i \in \mathbb{Z}_8\})$.
$K_{19} - K_7$ $(\{\infty_i \mid i \in \mathbb{Z}_7\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_2), \{\{(0,0),(2,1),(1,0),(0,1),(1,1),(3,0),(5,1),(4,0),(3,1),(4,1)) + (i,0), ((0,0),(3,1),\infty_0,(1,0),\infty_1,(1,1),\infty_2,(3,0),(0,1),\infty_3) + (i,0), ((3,0),(0,0),\infty_2,(4,1),\infty_1,(4,0),\infty_0,(0,1),(3,1),\infty_3) + (i,0) \mid i \in \mathbb{Z}_3\} \cup \{((0,0),(1,0),\infty_4,(0,1),(2,1),\infty_5,(3,0),(3,1),\infty_6,(2,0)) + (i,0) \mid i \in \mathbb{Z}_6\}$.
$K_{17} - K_9$ $(\{\infty_i | i \in \mathbb{Z}_9\} \cup \mathbb{Z}_8, \{(\infty_1,5,7,6,2,1,4,0,\infty_2,3),(\infty_1,1,5,3,\infty_3,6,\infty_4,7,\infty_5,2),(\infty_1,4,\infty_4,3,7,\infty_2,6,5,\infty_5,0),(\infty_1,7,1,\infty_2,2,0,\infty_3,4,\infty_5,6),(\infty_2,5,0,\infty_4.2,\infty_3,1,\infty_5,3,4),(\infty_3,7,0,6,4,2,3,1,\infty_4,5),(\infty_6,1,0,3,\infty_7,6,\infty_8,2,\infty_0,5),(\infty_6,2,5,4,\infty_7,7,\infty_8,3,\infty_0,6),(\infty_6,3,6,1,\infty_7,0,\infty_8,4,\infty_0,7),(\infty_6,4,7,2,\infty_7,5,\infty_8,1,\infty_0,0)\}$.
$K_{23} - K_3$ $(\{\infty_i \mid i \in \mathbb{Z}_3\} \cup (\mathbb{Z}_{10} \times \mathbb{Z}_2), \{((0,0),(1,0),(4,0),(4,1),(3,1),(8,1),(9,1),(9,0),(6,0),(5,0)) + (i,0) \mid i \in \mathbb{Z}_5\} \cup \{((7,0),(9,0),\infty_1,(9,1),\infty_2,(8,0),(7,1),(3,1),(0,1),(8,1)) + (i,0) \mid i \in \mathbb{Z}_{10}\} \cup \{((0,0),(4,0),(9,1),\infty_3,(6,0),(0,1),(2,0),(8,1),(5,0),(2,1)) + (i,0) \mid i \in \mathbb{Z}_{10}\}$.
$K_{25} - K_5$ $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup (\mathbb{Z}_{10} \times \mathbb{Z}_2), \{(0,0),(5,0),(1,0),(3,0),(0,1),(4,1),(9,1),(5,1),(8,0),(6,0)) + (i,0) \mid i \in \mathbb{Z}_5\} \cup \{((0,j),(i,j),(2i,j),(3i,j),\ldots,(9i,j)) \mid i \in \{1,3\}, j \in \mathbb{Z}_2\} \cup \{(\infty_0,(0,0),(5,1),(1,0),(4,1),(2,0),(3,1),(3,0),\infty_1,(2,1)) + (i,0), ((\infty_2,(0,1),(4,0),(2,1),(4,1),(5,0),\infty_3,(5,1),\infty_4,(6,0)) + (i,0) \mid i \in \mathbb{Z}_{10}\})$.

$m = 12$:
$K_{25} - K_9$ $(\{\infty_i \mid i \in \mathbb{Z}_9\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_2)$,

$\{((0,0),(0,1),(1,0),(2,0),(2,1),(3,0),(4,0),(4,1),(5,0),(6,0),(6,1),(7,0)) + (i,0) \mid i \in \mathbb{Z}_2\} \cup \{(0,0),(3,1),(4,1),(7,0),(5,0),(1,1),(5,1),(1,0),(3,0),(0,1),(7,1),(4,0)) +(i,0) \mid i \in \mathbb{Z}_4\} \cup \{((0,0),(1,1),(3,1),(5,0),\infty_3,(5,1),\infty_2,(4,0),\infty_1,(4,1),\infty_0,(3,0)) +(i,0),((0,0),(2,1),(5,1),\infty_4,(5,0),\infty_8,(7,1),\infty_7,(6,0),\infty_6,(4,1),\infty_5) + (i,0) \mid i \in \mathbb{Z}_8\})$.

$K_{19} - K_3$ $(\{\infty_i \mid i \in \mathbb{Z}_3\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_2)$
$\{((0,0),(0,1),(1,0),(2,0),(2,1),(3,0),(4,0),(4,1),(5,0),(6,0),(6,1),(7,0)) + (i,0) \mid i \in \mathbb{Z}_2\} \cup \{((0,0),(3,1),(4,1),(7,0),(5,0),(1,1),(5,1),(1,0),(3,0),(0,1),(7,1),(4,0)) +(i,0) \mid i \in \mathbb{Z}_4\} \cup \{((0,0),(3,0),(1,1),(4,1),(2,0),(3,1),(5,1),\infty_2,(5,0),\infty_1,(6,1),\infty_0) +(i,0) \mid i \in \mathbb{Z}_8\})$.

$K_{15} - K_7$ $(\{\infty_i \mid i \in \mathbb{Z}_7\} \cup (\mathbb{Z}_4 \times \mathbb{Z}_2)$,
$\{((0,0),(0,1),\infty_5,(1,0),(2,0),\infty_2,(1,1),(2,1),\infty_3,(3,0),\infty_4,(3,1)),((1,0),(1,1),\infty_1, (2,0),(0,0),\infty_3,(0,1),(3,1),\infty_5,(3,0),\infty_6,(2,1)),((2,0),(3,1),\infty_7,(1,0),(3,0),\infty_2, (0,1),(1,1),\infty_5,(0,0),\infty_4,(2,1))((3,0),(3,1),\infty_6,(2,0),(1,1),\infty_3,(1,0),(0,1),\infty_7, (0,0),\infty_1,(2,1)),((0,1),\infty_4,(1,0),\infty_2,(2,1),(0,0),(1,1),(3,1),\infty_3,(2,0),(3,0),\infty_1,), ((0,0),\infty_2,(3,1),(1,0),\infty_6,(1,1),\infty_7,(2,0),\infty_5,(2,1),(0,1),(3,0)),(\infty_6,(0,0),(1,0), \infty_1,(3,1),(2,1),\infty_7,(3,0),(1,1),\infty_4,(2,0),(0,1))\})$.

$K_{21} - K_{13}$ $(\{\infty_1 \mid i \in \mathbb{Z}_{13}\} \cup (\mathbb{Z}_4 \times \mathbb{Z}_2)$,
$\{((0,0),(1,0),\infty_3,(2,1),(2,0),\infty_2,(0,1),(3,1),\infty_4,(3,0),(1,1),\infty_1),((1,0),(2,0),\infty_3, (3,1),(3,0),\infty_2,(1,1),(0,1),\infty_4,(0,0),(2,1),\infty_1),((2,0),(3,0),\infty_3,(0,1),(0,0),\infty_2, (2,1),(1,1),\infty_4,(1,0),(3,1),\infty_1),((3,0),(0,0),\infty_3,(1,1),(1,0),\infty_2,(3,1),(2,1),\infty_4, (2,0),(0,1),\infty_1),((0,0),\infty_5,(0,1),\infty_6,(1,0),\infty_7,(1,1),\infty_8,(2,0),\infty_9,(2,1),\infty_{10}), ((2,0),\infty_{13},(2,1),\infty_{12},(3,0),\infty_{11},(3,1),\infty_{10},(1,0),\infty_9,(1,1),\infty_6),((0,0),\infty_6,(3,1), \infty_8,(1,0),(3,0),(2,1),\infty_{11},(2,0),\infty_{10},(1,1),\infty_{12}),((\infty_7,(2,0),\infty_{12},(3,1),\infty_9,(0,0), \infty_8,(0,1),\infty_{13},(3,0),(0,1),(2,1)),((0,1),\infty_{10},(3,0),\infty_7,(3,1),\infty_5,(1,0),\infty_{13},(1,1), (2,0),(0,0),\infty_{11}),((0,1),\infty_7,(0,0),\infty_{13},(3,1),(1,1),\infty_5,(3,0),\infty_8,(2,1), (1,0),\infty_{12}),((0,1),\infty_9,(3,0),\infty_6,(2,1),\infty_5,(2,0),(3,1),(0,0),(1,1),\infty_{11},(1,0))\})$.

$K_{27} - K_3$ This follows using $K_{19} - K_3$ and $K_{27} - K_{19}$(from $K_{15} - K_7$).

$K_{29} - K_5$ $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup (\mathbb{Z}_{12} \times \mathbb{Z}_2),\{((0,0),(0,1),(1,0),(11,1),(2,0),(10,1),(4,1), (8,0),(5,1),(7,0),(6,1),(6,0)) + (i,0) \mid i \in \mathbb{Z}_6\} \cup \{((0,0),(1,1),(11,0),(2,1), \infty_2,(5,0),\infty_3,(0,1),\infty_1,(3,0),(1,0),(4,0)) + (i,0) \mid i \in \mathbb{Z}_{12}\} \cup \{((0,0),(4,1),(0,1), (3,1),(1,1),(2,1),\infty_4,(1,0),\infty_0,(6,1),(11,0),(5,1)) + (i,0) \mid i \in \mathbb{Z}_{12}\} \cup \{(0,j),(i,j),(2i,j),(3i,j),\ldots,(11i,j) \mid i = 5 \text{ and } j \in \mathbb{Z}_2, \text{ or } i = 1 \text{ and } j = 0\})$.

$m = 14$:

$K_{21} - K_1$ This is a 14-cycle system of K_{21} so is known to exist [1].

$K_{19} - K_3$ $(\{\infty_i \mid i \in \mathbb{Z}_3\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_2)$,
$\{((0,0),(1,1),(4,1),(6,0),(2,1),(3,1),(7,1),(6,1),(2,0),(0,1),(5,1),(4,0),(7,0), (3,0)) + (i,0) \mid i \in \mathbb{Z}_4\} \cup \{((0,0),(2,1),(2,0),(3,0),(5,0),(4,1),(7,0),\infty_0,(6,1),\infty_1, (6,0),\infty_2,(5,1),(3,1)) + (i,0) \mid i \in \mathbb{Z}_8\})$.

$K_{17} - K_5$ $(\{\infty_i \mid i \in \mathbb{Z}_5\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_2),\{((0,0),(2,1),(0,1),(5,1),\infty_5,(5,0),(4,0),(1,0), (4,1),(1,1),(3,0),(3,1),\infty_2,(2,0)),((1,0),(3,1),(1,1),(0,1),\infty_4,(0,0),(5,0),(2,0), (5,1),(2,1),(4,0),(4,1),\infty_2,(3,0)),((2,0),(4,1),(2,1),(1,1),\infty_4,(1,0),(0,0),(3,0),$

$(0,1),(3,1),(5,0),(5,1),\infty_1,(4,0)),((3,0),(5,1),(3,1),(2,1),\infty_4,(2,0),(1,0),\infty_3,$
$(1,1),\infty_5,(0,0),(0,1),\infty_1,(5,0)),((4,0),(0,1),(4,1),(3,1),\infty_4,(3,0),(2,0),\infty_3,$
$(2,1),\infty_5,(1,0),(1,1),\infty_2,(0,0)),((5,0),(1,1),(5,1),(4,1),\infty_4,(4,0),(3,0),\infty_3,(3,1),$
$\infty_1,(2,0),(2,1),\infty_2,(1,0)),(\infty_1,(1,1),(0,0),(3,1),(4,0),\infty_2,(5,1),(1,0),(0,1),$
$(5,0),\infty_3,(4,1),\infty_5,(3,0)),((0,0),(5,1),\infty_3,(4,0),(1,1),(2,0),$
$\infty_5,(0,1),\infty_2,(5,0),(4,1),(3,0),(2,1),\infty_1),((0,0),\infty_3,(0,1),$
$(2,0),(3,1),\infty_5,(4,0),(5,1),\infty_4,(5,0),(2,1),(1,0),\infty_1,(4,1))\}).$

$K_{29}-K_{21}$ $(\{\infty_i \mid i\in\mathbb{Z}_{21}\}\cup(\mathbb{Z}_4\times\mathbb{Z}_2),$
$\{((0,0),(0,1),\infty_5,(1,0),\infty_4,(1,1),\infty_3,(2,0),\infty_2,(2,1),\infty_1,(3,0),\infty_6,(3,1)),((1,0),$
$(1,1),\infty_5,(2,0),\infty_4,(2,1),\infty_3,(3,0),\infty_2,(3,1),\infty_1,(0,0),\infty_6,(0,1)),((2,0),(2,1),$
$\infty_5,(3,0),\infty_4,(3,1),\infty_3,(0,0),\infty_2,(0,1),\infty_1,(1,0),\infty_6,(1,1)),((3,0),(3,1),\infty_5,$
$(0,0),\infty_4,(0,1),\infty_3,(1,0),\infty_2,(1,1),\infty_1,(2,0),\infty_6,(2,1)),((0,0),(1,1),\infty_7,(1,0),$
$\infty_8,(3,1),\infty_9,(2,0),\infty_{10},(0,1),\infty_{11},(3,0),\infty_{12},(2,1)),((1,0),(2,1),\infty_7,(2,0),$
$\infty_8,(0,1),\infty_9,(3,0),\infty_{10},(1,1),\infty_{11},(0,0),\infty_{12},(3,1)),((2,0),(3,1),\infty_7,(3,0),$
$\infty_8,(1,1),\infty_9,(0,0),\infty_{10},(2,1),\infty_{11},(1,0),\infty_{12},(0,1)),((3,0),(0,1),\infty_7,(0,0),$
$\infty_8,(2,1),\infty_9,(1,0),\infty_{10},(3,1),\infty_{11},(2,0),\infty_{12},(1,1)),((0,0),(1,0),\infty_{13},(0,1),$
$\infty_{14},(2,0),\infty_{15},(2,1),\infty_{16},(3,1),\infty_{17},(1,1),\infty_{18},(3,0)),((0,1),(1,1),\infty_{13},(2,0),$
$\infty_{16},(3,0),\infty_{14},(3,1),\infty_{18},(0,0),\infty_{15},(1,0),\infty_{17},(2,1)),((0,0),(2,0),\infty_{18},(0,1),$
$\infty_{19},(1,0),\infty_{20},(1,1),\infty_{21},(3,0),\infty_{13},(3,1),(2,1),y_{14}),(\infty_{13},(0,0),\infty_{16},(0,1),$
$\infty_{20},(3,0),(2,0),\infty_{19},(1,1),(3,1),\infty_{21},(1,0),\infty_{18},(2,1)),(\infty_{17},(0,0),\infty_{20},$
$(2,0),\infty_{21},(2,1),\infty_{19},(3,1),(0,1),\infty_{15},(1,1),\infty_{16},(1,0),(3,0)),(\infty_{17},(0,1),\infty_{21},$
$(0,0),\infty_{19},(3,0),\infty_{15},(3,1),\infty_{20},(2,1),(1,1),\infty_{14},(1,0),(2,0))\}).$

$K_{43}-K_7$ $(\{\infty_i \mid i\in\mathbb{Z}_7\}\cup(\mathbb{Z}_{18}\times\mathbb{Z}_2),\{((0,0),(1,0),(6,0),(15,1),(10,1),(9,1),(0,1),$
$(1,1),(6,1),(15,0),(10,0),(9,0),(11,0),(2,0))+(i,0)\mid i\in\mathbb{Z}_9\}\cup\{((0,0),(3,0),(4,1),$
$(8,1),(14,1),\infty_1,(17,0),\infty_2,(12,1),(10,1),(8,0),(4,0),(3,1),(0,1))+(i,0)\mid i\in\mathbb{Z}_{18}\}$
$\cup\{((0,0),(12,0),(7,1),(3,0),(0,1),(2,0),(5,1),(9,0),(14,1),(3,1),\infty_4,(17,0),\infty_3,$
$(6,1))+(i,0)\mid i\in\mathbb{Z}_{18}\}\cup\{((2,0),(9,0),(17,0),\infty_5,(16,1),\infty_6,(16,0),\infty_7,(5,1),(15,1),$
$(7,0),(0,1),(6,0),(12,1))+(i,0)\mid i\in\mathbb{Z}_{18}\}).$

$K_{41}-K_9$ $(\{\infty_i \mid i\in\mathbb{Z}_9\}\cup(\mathbb{Z}_{16}\times\mathbb{Z}_2),\{((1,0),(2,0),(10,1),(9,1),(12,0),(4,0),(1,1),$
$(2,1),(10,0),(9,0),(12,1),(14,1),(6,1),(4,1))+(i,0)\mid i\in\mathbb{Z}_8\}\cup\{((0,0),(7,0),(12,0),$
$(8,1),(2,0),\infty_1,(15,1),\infty_2,(15,0),\infty_3,(2,1),(8,0),(12,1),(7,1))+(i,0)\mid i\in\mathbb{Z}_{16}\}$
$\cup\{((0,0),(6,0),(10,0),(15,1),(5,1),(9,1),\infty_6,(14,0),\infty_5,(14,1),$
$\infty_4,(12,0),(7,1),(0,1))+(i,0)\mid i\in\mathbb{Z}_{16}\}\cup\{((4,0),(7,0),(9,0),(2,1),$
$(3,0),(5,1),(8,1),(10,0),(11,1),\infty_9,(12,0),\infty_8,(13,1),\infty_7)+(i,0)\mid i\in\mathbb{Z}_{16}\}).$

$K_{37}-K_{13}$ $(\{\infty_i \mid i\in\mathbb{Z}_{13}\}\cup(\mathbb{Z}_{12}\times\mathbb{Z}_2),\{((0,0),(1,0),(3,0),(9,1),(8,0),(7,1),(6,1),$
$(0,1),(1,1),(2,0),(3,1),(9,0),(7,0),(6,0))+(i,0)\mid i\in\mathbb{Z}_6\}\cup\{((5,0),(8,1),(6,0),$
$(9,0),\infty_4,(10,1),\infty_3,(10,0),\infty_2,(11,1),\infty_1,(11,0),(9,1),(5,1))+(i,0)\mid i\in\mathbb{Z}_{12}\}$
$\cup\{((2,0),(7,0),(11,0),\infty_5,(11,1),\infty_6,(10,0),\infty_7,(10,1),$
$\infty_8,(9,0),(5,1),(3,1),(6,1))+(i,0)\mid i\in\mathbb{Z}_{12}\}\cup\{((0,1),(3,0),(8,1),$
$\infty_{13},(8,0),(3,1),\infty_9,(10,0),\infty_{10},(11,1),\infty_{11},$
$(11,0),\infty_{12},(5,1))+(i,0)\mid i\in\mathbb{Z}_{12}\}).$

$K_{43}-K_7$ $(\{\infty_i \mid i\in\mathbb{Z}_7\}\cup(\mathbb{Z}_{18}\times\mathbb{Z}_2),\{((0,0),(1,0),(6,0),$
$(15,1),(10,1),(9,1),(0,1),(1,1),(6,1),(15,0),(10,0),(9,0),$

$(11,0),(2,0)) + (i,0) \mid i \in \mathbb{Z}_9\} \cup \{((0,0),(3,0),(4,1),(8,1),(14,1),$
$\infty_1,(17,0),\infty_2,(12,1),(10,1),(8,0),(4,0),(3,1),(0,1)) + (i,0) \mid i \in \mathbb{Z}_{18}\}$
$\cup\{((0,0),(12,0),(7,1),(3,0),(0,1),(2,0),(5,1),(9,0),(14,1),(3,1),\infty_4,$
$(17,0),\infty_3,(6,1)) + (i,0) \mid i \in \mathbb{Z}_{18}\} \cup \{((2,0),(9,0),(17,0),\infty_5,$
$(16,1),\infty_6,(16,0),\infty_7,(5,1),(15,1),(7,0),(0,1),(6,0),(12,1)) + (i,0) \mid i \in \mathbb{Z}_{18}\})$.
$K_{31} - K_3$ This follows using $K_{19} - K_3$ and $K_{31} - K_{19}$ (from $K_{17} - K_5$).
$K_{33} - K_5$ This follows using $K_{17} - K_5$ and $K_{33} - K_{17}$ (from $K_{19} - K_3$).
$K_{35} - K_7$ $(\{\infty_i \mid i \in \mathbb{Z}_7\}, \{((0,0),(7,0),(1,0),(6,1),(12,0),(12,1),(8,1),(0,1),$
$(7,1),(1,1),(5,1),(5,0),(13,1),(8,0)) + (i,0) \mid i \in \mathbb{Z}_7\} \cup \{((0,j),(i,j),(2i,j),\ldots,$
$(13i,j)) \mid i \in \{1,3,5\}, j \in \mathbb{Z}_2\} \cup \{((0,0),(6,1),(2,0),(5,1),(3,0),(1,1),(3,1),(8,0),$
$(4,1),(7,0),\infty_0,(8,1),\infty_1,(12,0)) + (i,0),((0,0),(4,0),(5,1),(6,0),(13,1),$
$\infty_2,(7,0),\infty_3,(7,1),\infty_4,(8,0),\infty_5,(8,1),\infty_6) + (i,0) \mid i \in \mathbb{Z}_{14}\})$.

Appendix B

For each of the cases $m = 11$ and 13 and for each $\theta \in \{2,4\}$ and $\theta \in \{7,9,11\}$ respectively, the following lists a graph G on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_2$ and a set C of m-cycles which form an m-cycle system: of $K^c_{(m-1/2)+\theta} \vee G$ when $m = 11$, the vertices in $K^c_{(m-1)/2+\theta}$ being $\infty_1, \infty_2, \ldots, \infty_{5+\theta}$; and of $K^c_\theta \vee G$ when $m = 13$, the vertices of K^c_θ being $\infty_1, \infty_2, \ldots, \infty_\theta$.

$m = 11$:
$\theta = 2$. Let $G = \langle\{1,2,3\}, \{0\}, \{1,2,3\}\rangle_{11}$, and $C = \{((0,0),(3,0),(6,0),(9,0),(1,0),$
$(4,0),(7,0),(10,0),(2,0),(5,0),(8,0)),\ ((0,1),(3,1),(6,1),(9,1),(1,1),(4,1),(7,1),$
$(10,1),(2,1),(5,1),(8,1)),((1,0),(0,0),(0,1),(1,1),\infty_1,(2,1),\infty_2,(3,1),\infty_3,(4,1),\infty_4),$
$((6,0),(8,0),(10,0),(1,0),\infty_1,(2,0),\infty_2,(3,0),\infty_3,(4,0),\infty_4),$
$((1,0),(3,0),(5,0),(7,0),\infty_1,(8,0),\infty_2,(9,0),\infty_3,(0,0),\infty_5),$
$((6,0),(5,0),(5,1),(6,1),\infty_1,(7,1),\infty_2,(8,1),\infty_3,(9,1),\infty_5),$
$((0,0),(2,0),(4,0),(6,0),\infty_6,(8,0),\infty_3,(5,0),\infty_5,(9,0),\infty_4),$
$((7,0),(6,0),(6,1),(7,1),\infty_3,(1,1),\infty_5,(2,1),\infty_7,(3,1),\infty_6),$
$((7,0),(9,0),(0,0),(10,0),\infty_7,(2,0),\infty_6,(3,0),\infty_5,(4,0),\infty_2),$
$((10,0),(9,0),(9,1),(10,1),\infty_1,(0,0),\infty_7,(3,0),\infty_4,(5,0),\infty_2),$
$((10,0),(10,1),(8,1),(6,1),\infty_2,(4,1),\infty_5,(7,1),\infty_4,(9,1),\infty_6),$
$((3,0),(2,0),(2,1),(3,1),\infty_4,(10,0),\infty_5,(7,0),\infty_3,(6,0),\infty_1),$
$((6,1),(4,1),(2,1),(0,1),\infty_7,(1,0),\infty_6,(1,1),\infty_4,(2,0),\infty_3),$
$((0,1),(9,1),(7,1),(5,1),\infty_5,(8,0),\infty_4,(7,0),\infty_7,(5,0),\infty_1),$
$((5,0),(4,0),(4,1),(5,1),\infty_2,(1,0),\infty_3,(10,0),\infty_1,(8,1),\infty_6),$
$((8,0),(7,0),(7,1),(8,1),\infty_4,(5,1),\infty_3,(10,1),\infty_6,(4,1),\infty_7),$
$((4,0),(3,0),(3,1),(4,1),\infty_1,(9,0),\infty_6,(2,1),\infty_4,(6,1),\infty_7),$
$((2,0),(1,0),(1,1),(2,1),\infty_3,(0,1),\infty_2,(9,1),\infty_7,(10,1),\infty_5),$
$((9,0),(8,0),(8,1),(9,1),\infty_1,(5,1),\infty_6,(0,0),\infty_2,(1,1),\infty_7),$
$((5,1),(3,1),(1,1),(10,1),\infty_4,(0,1),\infty_6,(6,1),\infty_5,(8,1),\infty_7),$

$((0,1),(10,1),\infty_2,(6,0),\infty_7,(7,1),\infty_6,(4,0),\infty_1,(3,1),\infty_5)\}$.
$\theta = 4$. Let $G = \langle\{1\},\{0\},\{1\}\rangle_{11}$ and
$C = \{((0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0)),$
$((0,0),(0,1),(1,1),(1,0),\infty_1,(2,0),\infty_2,(3,0),\infty_3,(4,0),\infty_4),$
$((1,1),(2,1),\infty_1,(0,1),\infty_2,(3,1),\infty_3,(4,1),\infty_4,(5,1),\infty_5),$
$((2,0),(2,1),\infty_2,(0,0),\infty_1,(3,0),\infty_4,(1,0),\infty_5,(4,0),\infty_6),$
$((2,1),(3,1),\infty_1,(1,1),\infty_2,(4,1),\infty_5,(0,1),\infty_3,(5,1),\infty_6),$
$((3,0),(3,1),\infty_4,(2,0),\infty_3,(0,0),\infty_5,(2,1),\infty_7,(1,0),\infty_8),$
$((3,1),(4,1),\infty_1,(4,0),\infty_2,(1,0),\infty_3,(1,1),\infty_4,(2,1),\infty_8),$
$((4,0),(4,1),\infty_6,(0,1),\infty_4,(6,1),\infty_1,(5,0),\infty_2,(5,1),\infty_7),$
$((4,1),(5,1),\infty_1,(6,0),\infty_3,(5,0),\infty_5,(3,1),\infty_6,(0,0),\infty_7),$
$((5,0),(5,1),\infty_8,(0,0),\infty_9,(1,0),\infty_6,(3,0),\infty_5,(6,0),\infty_4),$
$((5,1),(6,1),\infty_2,(6,0),\infty_6,(1,1),\infty_7,(0,1),\infty_8,(4,1),\infty_9),$
$((6,0),(6,1),\infty_3,(2,1),\infty_9,(1,1),\infty_8,(5,0),\infty_6,(7,0),\infty_7),$
$((6,1),(7,1),\infty_1,(7,0),\infty_2,(8,0),\infty_3,(9,0),\infty_4,(10,0),\infty_5),$
$((7,0),(7,1),\infty_2,(9.0),\infty_1,(10,0),\infty_9,(8,1),\infty_4,(9,1),\infty_5),$
$((10,1),(0,1),\infty_9,(8,0),\infty_1,(8,1),\infty_7,(9,1),\infty_3,(7,0),\infty_4),$
$((7,1),(8,1),\infty_5,(10,1),\infty_8,(4,0),\infty_9,(9,0),\infty_6,(10,0),\infty_7),$
$((8,0),(8,1),\infty_6,(10,1),\infty_9,(3,0),\infty_7,(6,1),\infty_8,(7,1),\infty_4),$
$((8,1),(9,1),\infty_1,(10,1),\infty_7,(3,1),\infty_9,(2,0),\infty_5,(9,0),\infty_8),$
$((9,1),(9,0),\infty_7,(2,0),\infty_8,(6,0),\infty_9,(7,1),\infty_5,(8,0),\infty_6),$
$((9,1),(10,1),\infty_2,(10,0),\infty_3,(7,1),\infty_6,(6,1),\infty_9,(7,0),\infty_8),$
$((10,0),(10,1),\infty_3,(8,1),\infty_2,(9,1),\infty_9,(5,0),\infty_7,(8,0),\infty_8)\}$.

$m = 13$:
$\theta = 7$. Let $G = \langle\{1,2,3,4\},\ \{0,1,2\},\ \{1,2,3,4\}\rangle_{13}$ and $C = \{((0,0),(i,0),(2i,0),\dots,$
$(12i,0)) \mid i \in \{1,2,3,4\}\} \cup \{((0,1),(i,1),(2i,1),\dots,(12i,1)) \mid i \in \{2,3,4\}\} \cup$
$\{(2,0),(3,1),\infty_1,(4,1),\infty_2,(5,1),\infty_3,(6,1),\infty_4,(7,1),\infty_5,(8,1),\infty_6),$
$((0,0),(0,1),(1,1),(1,0),\infty_1,(2,0),\infty_3,(3,0),\infty_5,(4,0),\infty_7,(5,0),\infty_2),$
$((2,0),(2,1),(3,1),(3,0),\infty_7,(9,1),\infty_2,(10,1),\infty_4,(11,1),\infty_6,(12,1),\infty_5),$
$((4,0),(4,1),(5,1),(5,0),\infty_1,(6,0),\infty_2,(8,0),\infty_3,(9,0),\infty_4,(10,0),\infty_6),$
$((6,0),(6,1),(7,1),(7,0),\infty_7,(0,1),\infty_5,(1,1),\infty_3,(2,1),\infty_4,(3,1),\infty_6),$
$((8,1),(8,0),(9,1),(9,0),\infty_5,(11,0),\infty_6,(12,0),\infty_7,(0,0),\infty_1,(3,0),\infty_2),$
$((10,0),(10,1),(11,1),(11,0),\infty_4,(8,1),\infty_7,(12,1),\infty_1,(0,1),\infty_2,(2,1),\infty_5),$
$((1,1),(0,0),(2,1),(1,0),\infty_2,(4,0),\infty_3,(6,0),\infty_4,(8,0),\infty_5,(3,1),\infty_7),$
$((2,1),(1,1),(12,0),(12,1),\infty_4,(0,0),\infty_3,(10,0),\infty_2,(11,0),\infty_1,(5,1),\infty_6),$
$((12,0),(0,1),(11,0),(12,1),\infty_3,(4,1),\infty_4,(5,1),\infty_5,(6,1),\infty_6,(7,1),\infty_2),$
$((0,1),(12,1),(10,0),(11,1),\infty_7,(10,1),\infty_5,(0,0),\infty_6,(1,0),\infty_4,(7,0),\infty_3),$
$((12,1),(11,1),(9,0),(10,1),\infty_1,(8,1),\infty_3,(9,1),\infty_4,(5,0),\infty_6,(1,1),\infty_2),$
$((8,0),(10,1),(9,1),(7,0),\infty_5,(6,0),\infty_7,(9,0),\infty_2,(3,1),\infty_3,(11,1),\infty_1),$
$((7,0),(8,1),(7,1),(5,0),\infty_3,(11,0),\infty_7,(2,1),\infty_1,(4,0),\infty_4,(0,1),\infty_6),$
$((1,0),(3,1),(4,1),(2,0),\infty_2,(7,0),\infty_1,(9,0),\infty_6,(10,1),\infty_3,(7,1),\infty_7),$
$((4,1),(3,0),(5,1),(6,1),\infty_1,(10,0),\infty_7,(2,0),\infty_4,(12,0),\infty_3,(1,0),\infty_5),$

$((5,1),(4,0),(6,1),(5,0),\infty_5,(9,1),\infty_1,(1,1),\infty_4,(3,0),\infty_6,(8,0),\infty_7),$
$((9,1),(8,1),(6,0),(7,1),\infty_1,(12,0),\infty_5,(11,1),\infty_2,(6,1),\infty_7,(4,1),\infty_6)\}.$
$\theta = 9$. Let $G = \langle\{1,2\}, \{0,1,2\}, \{1,2\}\rangle_{13}$ and $C = \{((0,0),(2,0),(4,0),(6,0),(8,0),$
$(10,0),(12,0),(1,0),(3,0),(5,0),(7,0),(9,0),(11,0)),$
$((0,1),(2,1),(4,1),(6,1),(8,1),(10,1),(12,1),(1,1),(3,1),(5,1),(7,1),(9,1),(11,1)),$
$((1,0),(0,0),(0,1),(1,1),\infty_1,(2,1),\infty_2,(3,1),\infty_3,(4,1),\infty_4,(5,1),\infty_5),$
$((2,0),(1,0),(1,1),(2,1),\infty_3,(3,0),\infty_5,(4,0),\infty_7,(5,0),\infty_9,(6,0),\infty_2),$
$((3,0),(2,0),(2,1),(3,1),\infty_4,(6,1),\infty_5,(7,1),\infty_6,(8,1),\infty_7,(9,1),\infty_8),$
$((4,0),(3,0),(3,1),(4,1),\infty_5,(0,1),\infty_7,(8,0),\infty_9,(9,0),\infty_2,(10,0),\infty_4),$
$((5,0),(4,0),(4,1),(5,1),\infty_6,(10,1),\infty_7,(11,1),\infty_8,(12,1),\infty_9,(0,1),\infty_1),$
$((6,0),(5,0),(5,1),(6,1),\infty_6,(11,0),\infty_8,(12,0),\infty_1,(0,0),\infty_3,(1,0),\infty_4),$
$((7,0),(6,0),(6,1),(7,1),\infty_7,(1,1),\infty_8,(2,1),\infty_9,(3,1),\infty_1,(4,1),\infty_6),$
$((8,0),(7,0),(7,1),(8,1),\infty_8,(2,0),\infty_1,(6,0),\infty_3,(9,0),\infty_5,(11,0),\infty_4),$
$((9,0),(8,0),(8,1),(9,1),\infty_9,(5,1),\infty_1,(6,1),\infty_2,(11,1),\infty_3,(12,1),\infty_4),$
$((10,0),(9,0),(9,1),(10,1),\infty_8,(0,0),\infty_2,(3,0),\infty_4,(12,0),\infty_6,(5,0),\infty_5),$
$((11,0),(10,0),(10,1),(11,1),\infty_9,(6,1),\infty_8,(5,0),\infty_4,(2,1),\infty_7,(3,0),\infty_1),$
$((12,0),(11,0),(11,1),(12,1),\infty_1,(8,1),\infty_2,(9,1),\infty_3,(8,0),\infty_5,(6,0),\infty_7),$
$((0,0),(12,0),(12,1),(0,1),\infty_2,(5,0),\infty_3,(11,0),\infty_7,(3,1),\infty_5,(8,1),\infty_9),$
$((1,1),(0,0),(2,1),(1,0),\infty_6,(3,0),\infty_9,(11,0),\infty_2,(5,1),\infty_8,(7,1),\infty_3),$
$((1,0),(3,1),(2,0),(4,1),\infty_7,(5,1),\infty_3,(8,1),\infty_4,(9,1),\infty_5,(10,1),\infty_1),$
$((4,1),(3,0),(5,1),(4,0),\infty_8,(1,0),\infty_9,(1,1),\infty_6,(0,0),\infty_4,(7,1),\infty_2),$
$((4,0),(6,1),(5,0),(7,1),\infty_1,(8,0),\infty_2,(1,0),\infty_7,(12,1),\infty_5,(11,1),\infty_6),$
$((7,1),(6,0),(8,1),(7,0),\infty_1,(4,0),\infty_2,(1,1),\infty_4,(10,1),\infty_3,(2,0),\infty_9),$
$((12,0),(1,1),\infty_5,(0,0),\infty_7,(2,0),\infty_6,(6,0),\infty_8,(9,0),\infty_1,(10,0),\infty_3),$
$((0,1),(12,0),\infty_2,(7,0),\infty_9,(10,0),\infty_8,(8,0),\infty_6,(2,1),\infty_5,(2,0),\infty_4),$
$((7,0),(9,1),(8,0),(10,1),\infty_9,(4,0),\infty_3,(0,1),\infty_6,(9,1),\infty_1,(11,1),\infty_4),$
$((10,1),(9,0),(11,1),(10,0),\infty_7,(6,1),\infty_3,(7,0),\infty_8,(3,1),\infty_6,(12,1),\infty_2),$
$((10,0),(12,1),(11,0),(0,1),\infty_8,(4,1),\infty_9,(12,0),\infty_5,(7,0),\infty_7,(9,0),\infty_6)\}.$
$\theta = 11$. $G = \langle\{1\},\{0\},\{1\}\rangle_{13}$ and $C = \{((0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),$
$(7,0),(8,0),(9,0),(10,0),(11,0),(12,0)),$
$((0,0),(0,1),(1,1),(1,0),\infty_1,(2,0),\infty_2,(3,0),\infty_3,(4,0),\infty_4,(5,0),\infty_5),$
$((1,1),(2,1),\infty_{11},(3,1),\infty_{10},(4,1),\infty_9,(5,1),\infty_8,(6,1),\infty_7,(7,1),\infty_6),$
$((2,0),(2,1),\infty_{10},(6,0),\infty_9,(7,0),\infty_8,(8,0),\infty_7,(9,0),\infty_6,(10,0),\infty_5),$
$((2,1),(3,1),\infty_7,(8,1),\infty_2,(9,1),\infty_3,(10,1),\infty_4,(11,1),\infty_5,(7,0),\infty_6),$
$((3,0),(3,1),\infty_2,(11,0),\infty_3,(12,0),\infty_4,(0,0),\infty_6,(1,0),\infty_7,(2,0),\infty_8),$
$((3,1),(4,1),\infty_{11},(0,1),\infty_{10},(1,1),\infty_9,(2,1),\infty_8,(7,1),\infty_5,(8,1),\infty_6),$
$((4,0),(4,1),\infty_1,(5,0),\infty_2,(6,0),\infty_3,(7,0),\infty_4,(8,0),\infty_5,(9,0),\infty_8),$
$((4,1),(5,1),\infty_{11},(6,1),\infty_{10},(7,1),\infty_9,(8,1),\infty_8,(9,1),\infty_7,(10,1),\infty_6),$
$((5,0),(5,1),\infty_1,(10,0),\infty_2,(12,0),\infty_5,(1,0),\infty_8,(6,0),\infty_{11},(7,0),\infty_7),$
$((5,1),(6,1),\infty_1,(9,1),\infty_4,(12,1),\infty_7,(0,1),\infty_8,(1,1),\infty_{11},(10,1),\infty_2),$
$((6,0),(6,1),\infty_2,(0,0),\infty_3,(1,0),\infty_4,(2,0),\infty_6,(3,0),\infty_7,(4,0),\infty_5),$
$((6,1),(7,1),\infty_1,(11,1),\infty_7,(1,1),\infty_5,(2,1),\infty_4,(3,1),\infty_3,(12,1),\infty_9),$
$((7,0),(7,1),\infty_{11},(0,0),\infty_{10},(1,0),\infty_9,(2,0),\infty_3,(5,0),\infty_8,(11,0),\infty_1),$

$((7,1),(8,1),\infty_{11},(9,1),\infty_{10},(10,1),\infty_1,(12,1),\infty_2,(0,1),\infty_3,(1,1),\infty_4)$,
$((8,0),(8,1),\infty_3,(11,1),\infty_2,(9,0),\infty_1,(12,0),\infty_6,(11,0),\infty_4,(5,1),\infty_{10})$,
$((8,1),(9,1),\infty_5(0,1),\infty_1,(1,1),\infty_2,(2,1),\infty_7,(4,1),\infty_4,(3,0),\infty_{10})$,
$((9,0),(9,1),\infty_9,(10,0),\infty_{11},(4,0),\infty_{10},(12,1),\infty_8,(0,0),\infty_7,(5,1),\infty_3)$,
$((9,1),(10,1),\infty_5,(3,0),\infty_{11},(2,0),\infty_{10},(7,0),\infty_2,(4,0),\infty_9,(0,1),\infty_6)$,
$((10,0),(10,1),\infty_8,(3,1),\infty_9,(5,0),\infty_{11},(1,0),\infty_2,(8,0),\infty_1,(6,0),\infty_7)$,
$((0,1),(12,1),\infty_{11},(8,0),\infty_6,(11,1),\infty_9,(11,0),\infty_5,(6,1),\infty_3,(10,0),\infty_4)$,
$((10,1),(11,1),\infty_8,(12,0),\infty_{10},(9,0),\infty_4,(6,0),\infty_6,(4,0),\infty_1,(0,0),\infty_9)$,
$((11,0),(11,1),\infty_{10},(10,0),\infty_8,(4,1),\infty_2,(7,1),\infty_3,(8,0),\infty_9,(9,0),\infty_{11})$,
$((11,1),(12,1),\infty_6,(5,1),\infty_5,(4,1),\infty_3,(2,1),\infty_1,(3,0),\infty_9,(12,0),\infty_{11})$,
$((12,0),(12,1),\infty_5,(3,1),\infty_1,(8,1),\infty_4,(6,1),\infty_6,(5,0),\infty_{10},(11,0),\infty_7)\}$.

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