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Harvey–Wiman hypermaps

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Abstract

We show that on a hypermap (α,σ) of genus $g \geqslant 2$, an automorphism ψ is either order $o(\psi) = p(1 + 2g/(p-1))$ if $(p, 1 + 2g/(p-1)) = 1$ or $o(\psi) \leqslant 2pg/(p-1)$, where p is the smallest divisor of the order of $Aut(\alpha,\sigma)$. We also give bounds on $|Aut(\alpha,\sigma)|$, namely, $|Aut(\alpha,\sigma) \leqslant 2p(g-1)/(p-3)$ if $p \geqslant 5$, $|Aut(\alpha,\sigma)| \leqslant 15(g-1)$ if $p = 3$; thus, only when $p = 2$ is the Hurwitz bound $|Aut(\alpha,\sigma)| \leqslant 84(g-1)$ effective. We define p-Harvey hypermaps as hypermaps admitting an automorphism of order $p(1 + 2g/(p-1))$ (type I) or $2pg/(p-1)$ (type II) and characterise them as p-elliptic hypermaps.

Résumé

Nous montrons que sur une hypercarte (α,σ) de genre $g \geqslant 2$, un automorphisme ψ est soit d'ordre $o(\psi) = p(1 + 2g/(p-1))$ si $(p, 1 + 2g/(p-1)) = 1$ soit $o(\psi) \leqslant 2pg/(p-1)$, où p est le plus petit diviseur de l'ordre de $Aut(\alpha,\sigma)$. Nous donnons également des bornes sur $|Aut(\alpha,\sigma)|$, à savoir $|Aut(\alpha,\sigma) \leqslant 2p(g-1)/(p-3)$ si $p \geqslant 5$, $|Aut(\alpha,\sigma)| \leqslant 15(g-1)$ si $p = 3$; ainsi, la borne d'Hurwitz n'est atteinte que pour $p = 2$ et donc $|Aut(\alpha,\sigma)| \leqslant 84(g-1)$. Nous définissons les hypercartes de p-Harvey comme celles qui admettent un automorphisme d'ordre $p(1 + 2g/(p-1))$ (type I) ou $2pg/(p-1)$ (type II) et les caracterisons comme hypercartes p-elliptiques.

1. Introduction

On a compact Riemann surface of genus $g > 1$ the maximal order for an automorphism is $4g + 2$ [12]. Harvey [8] generalized the result by arithmetic methods to the following theorem:

Theorem 1.1. *Let S be a Riemann surface of genus $g \geqslant 2$, p be the smallest divisor of $Aut(S)$ and let ψ be an automorphism of S. Then*:

(i) *either* $o(\psi) \leqslant p(1 + 2g/(p-1))$, *if* $o(\psi) = pm$ *where p and m are coprime,*

(ii) *or* $o(\psi) \leqslant 2pg/(p-1)$.

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Very sophisticated and complex computation was usually required to solve these geometrical problems and attaining a result was always a enormous work, until geometers decided to introduce an algebraic version of Riemann surfaces (see [9]). In parallel, group and graph theorists where working on embedded graphs in surfaces (see [7]). They met on the concept of *map* or its generalized version — introduced by Cori — of *hypermap.*

Their approach is combinatorial in nature; they represent a surface by a pair of permutations (α, σ) such that the group they generate is transitive: such a pair is called a hypermap. They $Aut(S)$ becomes $Aut(\alpha, \sigma)$ the centralizer of the two permutations. Machì gave a combinatorial proof of the classical result $|Aut(\alpha, \sigma)| \leqslant 84(g-1)$ for $g \geqslant 2$, where g is the genus of the hypermap (see Section 2).

In this paper, we keep this approach which allows results with elementary combinatorial proofs.

We first give a refinement of Machì's result, namely that when $Aut(\alpha, \sigma)$ is of odd order, then $|Aut(\alpha, \sigma)| \leqslant 15(g-1)$ if $p=3$ and $|Aut(\alpha, \sigma)| \leqslant 2p(g-1)/(p-3)$ if $p \geqslant 5$ where p is the smallest divisor of $|Aut(\alpha, \sigma)|$ (Theorem 4.2).

It is then possible to reprove Harvey's theorem from the hypermap point of view. We note with interest that *the combinatorial vision offers a geometrical interpretation of this theorem*! It can be seen as a generalization of results concerning a restricted type of surfaces: the so-called p-elliptic surfaces defined in [2]. These can be viewed as p-sheeted coverings of the sphere, where p is a prime. In [2] we have generalized the notion of a hyperelliptic hypermap to that of a p-elliptic hypermap: this is a hypermap admitting an automorphism of prime order p such that it is normal in $Aut(\alpha, \sigma)$ and fixes the maximum of points that an element of order p can fix, that is $2 + 2g/(p-1)$ (see below for a detailed explanation).

Now, we achieve an improvement of Harvey's theorem:

Theorem 1.2. *Let (α, σ) be a hypermap of genus $g \geqslant 2$, p be the smallest divisor of $Aut(\alpha, \sigma)$ and let ψ be an automorphism of (α, σ). Then*

(i) *either $o(\psi) = p(1 + 2g(p-1))$, where p and $1 + 2g/(p-1)$ are comprime and the quotient hypermap with respect to the subgroup of order p is planar,*

(ii) *or $o(\psi) \leqslant 2pg/(p-1)$.*

As in the case of a Riemann surface, a hypermap is *hyperelliptic* if it admits an involution ϕ fixing $2g+2$ points; this implies that ϕ is central in $Aut(\alpha, \sigma)$ (see [7, p. 459]). When p is fixed, both bounds are sharp since they are reached on p-elliptic hypermaps for infinitely many g, as we already showed in [4].

Kulkarni [10] defines a Wiman curve as Riemann surface which admits automorphisms of order $4g+2$ (type I) or $4g$ (type II). Accordingly, we define a Wiman hypermap as a hypermap admitting automorphisms of order $4g+2$ (type I) or $4g$ (type II). We then give combinatorial equivalents to Kulkarni's results:

A Wiman hypermap of type I *is hyperelliptic and its automorphism group is precisely the cyclic group C_{4g+2}.*

A Wiman hypermap of type II *is hyperelliptic and its automorphism group is precisely* C_{4g} *or* D_{4g} *(dihedral) except for* $g = 2$ *and* $g = 3$.

Finally, we define a p-Harvey hypermap as hypermap admitting automorphisms of order $p(1 + 2g/(p-1))$ (type I) or $2pg/(p-1)$ (type II). Of course, Wiman hypermaps are just 2-Harvey hypermaps.

Now, Propositions 6.6 and 6.7 below show that these hypermaps admit an automorphism of order p fixing $2 + 2g/(p-1)$ points. Proposition 6.1 shows that such an automorphism in the center of the p-Sylow subgroup containing it. Proposition 6.3 shows that such an automorphism generates a subgroup which is normal in the whole group $Aut(\alpha, \sigma)$, for $p \neq 3$. Thus, for $p \neq 3$:

A p-Harvey hypermap of type I *is p-elliptic and its automorphism group is precisely* $C_{p(1+2g/(p-1))}$.

A p-Harvey hypermap of type II *is p-elliptic and its automorphism group is precisely* $C_{2pg/(p-1)}$ *or* $D_{2pg/(p-1)}$.

In order to prove these results, we characterize the automorphisms of the torus.

2. Hypermaps and automorphisms

For a general introduction to the theory of hypermaps see [7]. In this section we recall a few definitions and results that will be needed in the sequel.

Definition 2.1. A *hypermap* is a pair of permutations (α, σ) on B (the set of *brins*) such that the group they generate is transitive on B. When α is a fixed point free involution, (α, σ) is a map. The cycles of α, σ and $\alpha^{-1}\sigma$ are called *edges*, *vertices* and *faces*, respectively; but if their specification in terms of edges, vertices or faces is not needed, we will refer to them as *points*.

Euler's formula gives the relationship between the numbers of cycles of these three permutations:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g,$$

where $n = card(B)$, g is a non-negative integer, called the *genus* of (α, σ), and where for any permutation $\theta, z(\theta)$ denotes the number of its cycles (cycles of length 1 are included) (see [7], p. 422]). If $g = 0$, then (α, σ) is *planar*.

Definition 2.2. An *automorphism* ϕ of a hypermap (α, σ) is a permutation commuting with both α and σ:

$$\alpha\phi = \phi\alpha \quad \text{and} \quad \sigma\phi = \phi\sigma.$$

Thus, the full automorphism group of (α, σ), denoted by $Aut(\alpha, \sigma)$ is the centralizer in $Sym(n)$ of the group generated by α and σ. A subgroup G of $Aut(\alpha, \sigma)$ is an *automorphism* *group* of (α, σ); the transivity of (α, σ) implies that $Aut(\alpha, \sigma)$ is semi-regular. Recall here that a semi-regular group is defined by the fact that all its orbits are of the same length, namely, $|G|$.

We denote by $\chi_\theta(\phi)$ the number of cycles of a permutation θ fixed by an automorphism ϕ and by $\chi(\phi)$ the total number of cycles of α, σ, and $\alpha^{-1}\sigma$ fixed by ϕ; $o(\phi)$ will be the order of ϕ. If (α, σ) is planar ($g = 0$) then $\chi(\phi) = 2$ for all non-trivial automorphisms ϕ. Moreover, $Aut(\alpha, \sigma)$ is one of C_n (cyclic), D_n (dihedral), A_4, S_4 and A_5 (see [7, p. 464]). Finally, we recall a result which is well known in the theory of Riemann surfaces: *an automorphism of prime order cannot fix only one point*. For a proof of this in the case of hypermaps see [5]. We shall need these results later.

We now define an equivalence relation R on the set B.

Definition 2.3. Let G be an automorphism group of the hypermap (α, σ). Two brins b_1 and b_2 are *equivalent*, $b_1 R b_2$, if they belong to the same orbit of G.

This leads to the following definition.

Definition 2.4. The *quotient hypermap* $(\bar{\alpha}, \bar{\sigma})$ of (α, σ) with respect to an automorphism group G, is a pair of permutations $(\bar{\alpha}, \bar{\sigma})$ acting on the set $\bar{B}$, where $\bar{B} = B/R$ and $\bar{\alpha}, \bar{\sigma}$ are the permutations induced by α and σ on $\bar{B}$.

The following Riemann–Hurwitz formula relates the genus γ of $(\bar{\alpha}, \bar{\sigma})$ to the genus g of (α, σ) (see [11]):

$$2g - 2 = card(G)(2\gamma - 2) + \sum_{\phi \in G - \{id\}} \chi(\phi). \tag{1}$$

It follows that $\gamma \leqslant g$. In case G is a cyclic group, $G = \langle \phi \rangle$, (1) becomes

$$2g - 2 = card(G)(2\gamma - 2) + \sum_{i=1}^{o(\phi)-1} \chi(\phi^i). \tag{2}$$

As mentioned above one can prove that for $g \geqslant 2$, $|Aut(\alpha, \sigma)| \leqslant 84(g-1)$.

If ϕ is an automorphism of order m, then, for all integers i, $\chi(\phi) \leqslant \chi(\phi^i)$, and when m and i are coprime $\chi(\phi) = \chi(\phi^i)$.

Let (α, σ) be a hypermap, G an automorphism group of (α, σ) and let $(\bar{\alpha}, \bar{\sigma})$ be the quotient hypermap of (α, σ) with respect to G. The proof of the following results can be found in [3]. For any element ψ in the normalizer of G in $Aut(\alpha, \sigma)$, the permutation $\bar{\psi}$, defined as $\bar{\psi} = \psi/G$, is an automorphism of $(\bar{\alpha}, \bar{\sigma})$. The two following operations on (α, σ) are equivalent:

(i) taking the quotient (α, σ) first by G and then by $\bar{\psi}$

(ii) taking the quotient (α, σ) by $\langle G, \psi \rangle$.

Definition 2.5. The permutation $\bar{\psi}$ is called the *induced automorphism* of ψ on $(\bar{\alpha}, \bar{\sigma})$. We also say that ψ induces $\bar{\psi}$ on $(\bar{\alpha}, \bar{\sigma})$.

We consider now the case in which an automorphism ϕ of prime order p is normal in $\langle \psi, \phi \rangle$, where ψ is any element of $Aut(\alpha, \sigma)$.

Proposition 2.6. *Let ψ commute with ϕ.*

(i) *If ψ is of order m where p and m are coprime, then*

$$\chi(\bar{\psi})p = \chi(\psi) + (p-1)\chi(\phi\psi).$$

(ii) *If ψ is of order pn, p and n coprime, and ϕ belongs to $\langle \psi \rangle$, then*

$$\chi(\bar{\psi})p = \chi(\psi^p) + (p-1)\chi(\psi).$$

(iii) *If ψ is of order $p^m n, m > 1, p$ and n coprime, and ϕ belongs to $\langle \psi \rangle$, then*

$$\chi(\psi) = \chi(\bar{\psi}).$$

(iv) *If ψ is of order pm, m being any integer, and ϕ does not belong to $\langle \psi \rangle$, then*

$$\chi(\bar{\psi})p = \sum_{i=0}^{p-1} \chi(\psi\phi^i)$$

and

$$\chi(\psi\phi^i) \equiv 0 \pmod p.$$

(v) *If ψ does not commute with ϕ, then*

$$\chi(\psi) = \chi(\bar{\psi}).$$

In the classical theory of Riemann surfaces, a hyperelliptic surface S is a surface admitting an involution which is central in $Aut(S)$ and fixes $2 + 2g$ points. This notion applies to hypermaps [7]. In the next definition we consider autmorphisms of prime order p to generalize the notion of hyperelliptic hypermaps.

Definition 2.7. A hypermap (α, σ) of genus $g > 1$ is said to be *p-elliptic* if it admits an automorphism ϕ of prime order p such that:

(1) the quotient hypermap $(\bar{\alpha}, \bar{\sigma})$ with respect to ϕ is planar,

(2) $\langle \phi \rangle$ is normal in $Aut(\alpha, \sigma)$.

Remark 2.8. This definition is equivalent to that given in Section 1.

Since an automorphism on the sphere fixes exactly 2 points, an automorphism ψ on a p-elliptic hypermap of genus g fixes $\chi(\psi) = 0, 1, 2, p, p+1, 2p$ or $2 + 2g/(p-1)$ points. It is a consequence of Proposition 2.6 together with the fact that a planar automorphism fixes exactly 2 points (see [4]).

Proposition 2.9. *Let (α, σ) be a hypermap and G an automorphism group; let $N(G)$ be the normalizer of G in $Aut(\alpha, \sigma)$ and $t > 0$ the number of points fixed by all non-trivial* *elements of G. Then there exists a homomorphism h from N(G) to S_t whose kernel is a cyclic group.*

We remark that when $G = \langle \phi \rangle$, then the image of h is contained in $S_{\chi(\phi)}$.

For complete proofs of these results see [3].

Theorem 2.10. *Let (α, σ) be a p-elliptic hypermap and let ψ be an automorphism of (α, σ). Then either $o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime, or $o(\psi) \leqslant 2pg/(p-1)$.*

Theorem 2.11. *Let (α, σ) be a p-elliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{pn}(cyclic) where n is a divisor of $1 + 2g/(p-1)$; C_{pn} or D_{pn} (dihedral) where n is a divisor of $2g/(p-1)$; a semi-direct product of either C_n or a lifting of D_n, by C_p where n is a divisor of $2 + 2g/(p-1)$; or is of order 12p, 24p or 60p (extensions of A_4, S_4, A_5, respectively).*

Corollary 2.12. *Let (α, σ) be a hyperelliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{2n} where n is a divisor of $2g + 1$, C_{2n} or D_{2n} where n is a divisor of 2g; $C_n \times C_2$ or an extension of D_n by C_2, where n is a divisor of $2g + 2$; or $Aut(\alpha, \sigma)$ is of order 24, 48 or 120 (extensions of A_4, S_4, A_5, respectively).*

Proposition 2.13. *Let (α, σ) be a hypermap of genus $g > 1$ such that there exists an automorphism ϕ of prime order $2g + 1$. Then, except for the case $g = 3$ and $Aut(\alpha, \sigma) = PSL_2(\mathbf{Z}_7)$ (the simple group of order 168), (α, σ) is a$(2g + 1)$-elliptic hypermap.*

Proposition 2.14. *Let (α, σ) be a hypermap of genus $g > 30$ such that there exists an automorphism ϕ of prime order $g + 1$. Then (α, σ) is a $(g + 1)$-elliptic hypermap.*

Proposition 2.15. *Let (α, σ) be a hypermap of genus $g = 2$. Then $Aut(\alpha, \sigma)$ is either trivial or C_2, or else (α, σ) is 5-elliptic, 3-elliptic or hyperelliptic.*

3. Automorphisms of the torus

Proposition 3.1. *Let (α, σ) be a hypermap of genus 1 and ψ an automorphism. Then only two cases can happen:*

(i) *either ψ fixes no point and neither does any non-trivial power of ψ.*

(ii) *or ψ fixes at least one point and then ψ is of order 2, 3, 4 or 6 with 4, 3, 2 or 1 fixed points, respectively.*

Proof. Suppose first that ϕ an automorphism of prime order p fixes points. Then, by (2), $\chi(\phi) = 2p/(p-1)$; Only two cases are possible: $o(\phi) = 2$ and $\chi(\phi) = 4$ or $o(\phi) = 3$ and $\chi(\phi) = 3$.

Suppose now that ψ is an automorphism such that one of its power ϕ of prime order fixes points. If $p = 2$, then by Theorem 2.9, there exists an homomorphism h from $\langle\psi\rangle$ to S_4, and $Ker(h) = \langle\phi\rangle$ since by (2) no other automorphism can fix these 4 points in common with $\langle\phi\rangle$. S_4 possesses elements of order 1, 2, 3, 4 thus $o(\psi) = 1, 2, 3, 4, 6, 8$. Let us show that $o(\psi) = 8$ is impossible. If we consider the induced automorphism $\bar{\psi}$ on the quotient hypermap with respect to $\langle\phi\rangle$, we know that $2 = \chi(\bar{\psi})$ since this hypermap is planar. But by Proposition 2.6, $\chi(\bar{\psi}) = \chi(\psi)$. These 2 points are fixed in common with ϕ thus $o(h(\psi)) = 2, o(\psi) = 4$ and $\chi(\psi) = 2$.

If $p = 3$, then by Theorem 2.9, there exists an homomorphism h from $\langle\psi\rangle$ to S_3, and $Ker(h) = \langle\phi\rangle$ since by (2) no other automorphism can fix these 3 points in common with $\langle\psi\rangle$. S_3 possesses elements of order 1, 2, 3 thus $o(\psi) = 1, 3, 6, 9$. Let us show that $o(\psi) = 9$ is impossible. If we consider the induced automorphism $\bar{\psi}$ on the quotient hypermap with respect to $\langle\phi\rangle$, we know that $2 = \chi(\bar{\psi})$ since this hypermap is planar. But by Proposition 2.6, $\chi(\bar{\psi}) = \chi(\psi)$. These 2 points are fixed in common with ϕ and it is impossible to fix 2 points in common with an element that fixes 3 points. When $o(\psi) = 6$ we have $\chi(\psi) = 1$. □

4. Bounds on automorphism groups orders

By the Riemann–Hurwitz formula, we know that if a hypermap (α, σ) of genus $g > 1$ admits an automorphism group G such that the quotient hypermap with respect to it is of genus $\gamma > 1$, then $|G| \leqslant g - 1$;

We now give a bound when $\gamma = 1$.

Theorem 4.1. *Let (α, σ) be a hypermap of genus $g > 1$ and G an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 1$ then:*
$|G| \leqslant \frac{2p}{p-1}(g-1)$ where p is the smallest prime that divides the order of $Aut(\alpha, \sigma)$.

Proof. Let us consider (1) when $\gamma = 1$:

$2g - 2 = \sum_{\phi \in G - \{id\}} \chi(\phi)$; we note that $\sum_{\phi \in G - id} \chi(\phi) = \sum_{c \in (\alpha, \sigma)} (|G_c| - 1)$ where G_c is the stablizer of a point c (any cycle of α, σ and $\alpha^{-1}\sigma$). When two cycles belong to the same orbit of G then the order of the stabilizer is equal. Let $n_i = |G_{c_i}|$ where the c_i's constitute a set of representatives of G-orbits. Then, $\sum_{\phi \in G - id} \chi(\phi) = |G| \sum_{i=1}^{r} (n_i - 1)/n_i$. We now rewrite (1). $2g - 2 = |G| \sum_{i=1}^{r} (n_i - 1)/n_i$. The minimum value of the sum $\sum$ is obtained for $n_i = p$, all i's. Thus, $\sum \leqslant r(p-1)/p$ and $r|G| \leqslant 2p(g-1)/(p-1)$. □

In the next theorem we show that if $Aut(\alpha, \sigma)$ is of odd order, then in the Hurwitz bound $84(g-1)$, 84 can be replaced by 15 if $|Aut(\alpha, \sigma)|$ is divisible by 3 and $2p/(p-3)$ if its smallest divisor $p \geqslant 5$.

Theorem 4.2. *Let (α, σ) be a hypermap of genus $g > 1, G$ an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 0$ and p the smallest prime that divides the order of $Aut(\alpha, \sigma)$. Then:*

If $p \geqslant 5$, $|G| \leqslant \frac{2p}{p-3}(g-1)$;
If $p = 3$, $|G| \leqslant 15(g-1)$;
If $p = 2$, $|G| \leqslant 84(g-1)$.

Proof. Let us consider (1) with $\gamma = 0$ and as for the precedent theorem we rewrite (1): $(2g-2)/|G| = r - 2 - \sum_{i=1}^{r} 1/n_i$. We suppose $n_i \leqslant n_j$ for $i < j$.

$r \geqslant 3$ otherwise $g = 0$. The minimum value of the sum $\sum$ is obtained for $n_i = p$, all i's. Thus $(2g-2)/|G| \geqslant r - 2 - r/p = [(p-1)r - 2p]/p$. Now, $(p-1)r - 2p = 0$ implies $r = 2p/(p-1)$ which is an integer only for $p = 2$ or $p = 3$. Thus for $p \geqslant 5, |G| \leqslant \frac{2p}{(p-1)r-2p}(g-1)$ and $r \geqslant 3$ implies $|G| \leqslant \frac{2p}{p-3}(g-1)$.

If $p = 3$ and $r \geqslant 4$, $(2g-2)/|G| \geqslant r - 2 - r/3 \geqslant 4 - 2 - 4/3 = 2/3$ thus $|G| \leqslant 3(g-1)$. If $p = 3$ and $r = 3$, $(2g-2)/|G| = 1 - 1/n_1 - 1/n_2 - 1/n_3$. If $n_1 = n_2 = 3$ then $g = 0$ and $n_3 \geqslant 5$ then $(2g-2)/|G| = 2/15$; hence, $|G| \leqslant 15(g-1)$.

If $p = 2$ we have the Hurwitz bound. □

We now give a technical lemma which will be of help in the sequel:

Lemma 4.3. *Let (α, σ) be a hypermap of genus $g \geqslant 2, G$ an automorphism group such that p is the smallest divisor of its order, ϕ an automorphism of order p, normal in G and fixing two points. Then either $G = C_n$ or $G = D_n$ with $n < 2pg/(p-1)$.*

Proof. Since ϕ is normal in G there exists a homomorphism h from G to S_2; thus, either $G = C_n$ or $G = D_n$ and $p = 2$. Let $\theta \in C_n$ be an automorphism of prime order q fixing the maximal number of points and γ the genus of the quotient hypermap with respect to θ. If $p = q$, then by (1), $g = \gamma |ker(h)|$ thus $|G| \leqslant 2g < 2pg/(p-1)$. We will now suppose that $p \neq q$. If $\gamma = 0, \chi(\theta) = 2 + 2g/(q-1)$ then the results on p-elliptic hypermaps apply and $n \leqslant 2qg/(q-1) < 2pg/(p-1)$. If $\gamma = 1, \chi(\theta) = 2(g-1)/(q-1)$, then $n = 2q, 3q, 4q, 6q$ i.e. $p = 2$ and $\chi(\phi) \geqslant 4$ or $p = 3$ and $\chi(\phi) \geqslant 3$, by Proposition 3.1 and results on induced automorphisms. This is a contradiction. If $\gamma \geqslant 2$, we proceed by induction on the genus. $n/q < 2p\gamma/(p-1)$ and by (1) $2g - 2 \geqslant q(2\gamma - 2) + 2(q-1)$ i.e. $q\gamma \leqslant 2q$; hence, $n < 2pg/(p-1)$. □

Theorem 4.4. *Let (α, σ) be a hypermap of genus $g \geqslant 2$, ψ be an automorphism of (α, σ) and p the smallest divisor of $o(\psi)$. Then either*

(i) *$o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime and the quotient hypermap with respect to the subgroup of order p is planar, or*

(ii) $o(\psi) \leqslant 2pg/(p-1)$.

Proof. Let ψ be an automorphism of (α, σ) such that the quotient hypermap with respect to it is of genus $\gamma_0 = 0$ (otherwise Theorem 4.1 applies). Let ϕ be a power of ψ of prime order p where p is the smallest divisor of $o(\psi)$ and γ the genus of $(\bar{\alpha}, \bar{\sigma})$ the quotient hypermap with respect to $\langle\phi\rangle$. We denote induced automorphisms on $(\bar{\alpha}, \bar{\sigma})$ by a bar. If $\chi(\phi) = 0$, $2g - 2 + 2o(\psi) = \sum_{i=1}^{o(\psi)-1} \chi(\psi^i) = \sum_{i=1}^{o(\psi)/p-1} \chi(\psi^{pi}) = 2g - 2 + 2o(\psi)/p$, which is impossible. Thus, $\chi(\phi) \geqslant 2$. By Lemma 4.3, we know that $\chi(\phi) = 2$ implies $o(\psi) \leqslant 2g$. Thus assume that $\chi(\phi) \geqslant 3$.

If $\gamma = 0$, then the results on p-elliptic hypermaps apply and $o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime, or $o(\psi) \leqslant 2pg/(p-1)$.

If $\gamma = 1$, $\chi(\phi) = 2(g-1)/(p-1)$ and by Proposition 3.1 $o(\psi) = 6$, 4, 3 or 2 since $\gamma_0 = 0$. Since p is the smallest divisor of $o(\psi)$, either $p = 2$ and $o(\psi) = 12$, 8, 6 or 4, or $p = 3$ and $o(\psi) = 9$. When $p = 2$ the bound is $4g$ and $3g$ for $p = 3$, since it is easy to verify that there is no automorphism of order 9 nor 12 on a hypermap of genus 2 (considering induced automorphisms).

If $\gamma > 1$, we proceed by induction on the genus. Let q be the smallest prime that divides the order of the induced automorphism group.

By induction, $o(\bar{\psi}) \leqslant 2q\gamma/(q-1)$ and by (1) $2g \geqslant 2p\gamma$ thus $o(\psi) = p\,o(\bar{\psi}) \leqslant 2pq\gamma/(q-1) \leqslant 2qg/(q-1) \leqslant 2pg/(p-1)$. Or $o(\bar{\psi}) = q(1 + 2\gamma/(q-1))$, where q and $1 + 2\gamma/(q-1)$ are coprime and the quotient hypermap with respect to the subgroup of order q is planar. Let $\bar{\theta}$ be a generator of it. Thus, $o(\bar{\theta}) = q$ and $\chi(\bar{\theta}) = 2 + 2\gamma/(q-1)$, so that $o(\psi) = pq(1 + 2\gamma/(q-1))$. If $\chi(\phi) \geqslant p + 2$, then $2g - 2 \geqslant p(2\gamma - 2) + (p-1)(p+2)$ i.e. $2g \geqslant 2p\gamma + p(p-1)$ that is $2g/(p-1) \geqslant p(1 + 2\gamma/(p-1))$. Thus, $o(\psi) \leqslant p^2(1 + 2\gamma/(p-1) \leqslant 2pg/(p-1)$. If $\chi(\phi) < p + 2$, then there exists a homomorphism h from $\langle\psi\rangle$ to $S_{\chi(\phi)}$ and $\langle\psi\rangle/ker(h) = C_p$ or is trivial since p is its smallest divisor. If $\langle\psi\rangle/ker(h) = id$ then by (1) $|ker(h)| \leqslant 2g + 1 \leqslant 2pg/(p-1)$ thus $o(\psi) \leqslant 2pg/(p-1)$. If $\langle\psi\rangle/ker(h) = C_p$ then $\chi(\phi) = p$ or $\chi(\phi) = p + 1$ and $q = p$. Since $\chi(\theta) = \chi(\bar{\theta}) = 2 + 2\gamma/(p-1)$ and $o(\theta) = p^2, \theta$ fixes $2 + 2\gamma/(p-1)$ points in common with ϕ but then $h(\theta) = id$ since $\chi(\phi) = p$ or $\chi(\phi) = p + 1$; and since p does not divide $1 + 2\gamma/(p-1)$ by hypothesis it means that $\langle\psi\rangle/ker(h) = C_p$ does not happen. Thus the result follows. □

The following improvement of Harvey's theorem can now be obtained:

Theorem 4.5. *Let (α, σ) be a hypermap of genus $g \geqslant 2$, p be the smallest divisor of $Aut(\alpha, \sigma)$ and let ψ be an automorphism of (α, σ). Then either*

(i) $o(\psi) = p(1 + 2g/(p-1))$, *where p and $1 + 2g/(p-1)$ are coprimes and the quotient hypermap with respect to the subgroup of order p is planar,*

(ii) *or* $o(\psi) \leqslant 2pg/(p-1)$.

Proof. Let $o(\psi) \leqslant 2qg/(q-1)$ where q is the smallest divisor of $o(\psi)$. Since the function $f(q) = 2qg/(q-1)$ is decreasing, Result (ii) is true for p the smallest prime that divides the order of $Aut(\alpha, \sigma)$.

Suppose now that $o(\psi) = q(1 + 2g/(q-1))$, with $o(\psi) > 2pg/(p-1)$. Since q is the smallest divisor of $o(\psi)$, $|Aut(\alpha, \sigma)| \geqslant pq(1 + 2g/(q-1)) > 2p^2g/(p-1)$.

For $p \geqslant 5$, we know by Theorem 4.2 that $|Aut(\alpha,\sigma)| \leqslant 2p(g-1)/(p-3)$; thus, $2p^2g/(p-1) < 2p(g-1)/(p-3)$, i.e. $p(p-3)g < (p-1)(g-1)$ which is impossible.

If $p = 3$, $q(1 + 2g/(q-1)) > 2pg/(p-1) = 3g$ implies $q(1 + 2g/(q-1)) = 3g + 1$ or $q(1 + 2g/(q-1)) = 3g + 2$. Thus, $g = (q-1)^2/(q-3)$ or $g = (q-2)(q-1)/(q-3)$. Hence, $q - 3$ divides 4 or 2 that is $q = 5$ and $g = 8$ or $q = 7$ and $g = 9$ in the first case and $q = 5$ and $g = 6$ in the second case. But then 2 or 3 divides $o(\psi)$ in the first case and 3 divides $o(\psi)$ in the second case.

If $p = 2, q(1 + 2g/(q-1)) > 2pg/(p-1) = 4g$ implies $q(1 + 2g/(q-1)) = 4g + 1$, i.e. $2g = (q-1)^2/(q-2)$ but then g is not an integer, a contradiction. □

5. Wiman hypermaps

Definition 5.1. A *Wiman hypermap*, is a hypermap of genus $g \geqslant 2$ admitting an automorphism of order $4g + 2$ (type I) or an automorphism of order $4g$ (type II).

Theorem 5.2. *Let (α, σ) be a Wiman hypermap of type* I; *then (α, σ) is hyperelliptic and $Aut(\alpha, \sigma) = C_{4g+2}$.*

Proof. By Theorem 4.4 we know that the square of the automorphism of order $4g + 2$ fixes $2g + 2$ points. Thus (α, σ) *is hyperelliptic and the theorems* on hyperelliptic hypermaps allow to conclude that $Aut(\alpha, \sigma) = C_{4g+2}$. □

We give now a technical lemma which is used in the proof of the next theorem.

Lemma 5.3. *Let (α, σ) a hypermap of genus* 3, *admitting an automorphism of order* 12 *such that its power of order* 2 *fixes* 4 *points then (α, σ) is not hyperelliptic and $Aut(\alpha, \sigma) = C_{12}$ or is of order* 48.

Proof. Let ψ be the automorphism of order 12 and ϕ its power of order 2. ψ^4 is of order 3 and fixes 2 or 5 points by (1). If ψ^4 fixes 2 points then for all i $\chi(\psi^i) \leqslant 2$ which contradicts (1) applied to $\langle\psi\rangle$. If ψ^4 fixes 5 points then it fixes one point in common with ϕ; thus, $\chi(\psi) = 1$. Hence, $\langle\psi\rangle$ is its own normalizer in $Aut(\alpha, \sigma)$. If (α, σ) is 3-elliptic, then $Aut(\alpha, \sigma) = C_{12}$. If (α, σ) is not 3-elliptic, then $|Aut(\alpha, \sigma)|$ is 48, 84, 120 or 144 by the Sylow theorems and the fact that $|Aut(\alpha, \sigma)| \leqslant 84(g-1)$. Now, there is no automorphism of order 5 on a hypermap of genus 3 (since it would fix only one point) and a hypermap of genus 3 admitting an automorphism of order $7 = 2g + 1$ $Aut(\alpha, \sigma)$ is C_7, C_{14} or is of order 168.

These hypermaps are not hyperelliptic for otherwise the hyperelliptic involution would normalize $\langle\psi\rangle$ in $Aut(\alpha, \sigma)$ and thus would be equal to ϕ itself. However ϕ fixes only 4 points, hence the contradiction. □

Theorem 5.4. *Let (α, σ) be a Wiman hypermap of type II. Then, two cases may occur:*
(i) *(α, σ) is hyperelliptic and $Aut(\alpha, \sigma) = C_{4g}, D_{4g}$, or $|Aut(\alpha, \sigma)| = 48$, an extension of S_4 by C_2 and $g = 2$.*
(ii) *(α, σ) is not hyperelliptic then $g = 3$ and $Aut(\alpha, \sigma) = C_{12}$ or $|Aut(\alpha, \sigma)| = 48$.*

Proof. Let ψ be the automorphism of order $4g$ where ϕ is its power of order 2. Note that an automorphism of order 2 fixes an even number of points. We know that the quotient hypermap with respect to $\langle\psi\rangle$ is of genus $\gamma_0 = 0$, since $|\langle\psi\rangle| \geqslant 4g - 4$. Thus, by (2), $\sum_{i=1}^{4g-1} \chi(\psi^i) = 10g - 2$.

If $\chi(\phi) = 0$, then $10g - 2 = \sum_{i=1}^{4g-1} \chi(\psi^i) = \sum_{i=1}^{2g-1} \chi(\psi^{2i}) \leqslant 6g - 2$, which is impossible.

If $\chi(\phi) = 2$, then Proposition 4.3 applies and there is no automorphism of order $4g$; a contradiction.

If $\chi(\phi) = 4$, there exists a homomorphism h from $\langle\psi\rangle$ to S_4; then $ker(h)$ contains $\langle\psi^4\rangle$ or $ker(h) = \langle\psi^3\rangle$. If $ker(h)$ contains $\langle\psi^4\rangle$, then $\sum_{i=1}^{g-1} \chi((\psi^4)^i) \geqslant 4g - 4$ by (1) we have that $\sum_{i=1}^{g-1} \chi((\psi^4)^i) \leqslant 4g - 2$ and we have an equality. The non-trivial elements of the group $\langle\psi^4\rangle$ must all fix 4 points except two of them which fix 5 points and are therefore of order 3; but it is impossible for an element to fix 4 points in common with an element that fixes 5 points. If $ker(h) = \langle\psi^3\rangle$, then 3 divides g and by (1) we have $2g - 2 \geqslant (-2)4g/3 + 4(4g/3 - 1)$ i.e. $2g + 2 \geqslant 8g/3$. Thus $g = 3$ and $o(\psi) = 12$. By Lemma 5.3, we have the result.

If $\chi(\phi) \geqslant 6$, then $2g - 2 \geqslant 2(2\gamma - 2) + 6$, that is, after computation, $2g \geqslant 4\gamma + 4$ which is impossible for $\gamma \geqslant 2$ (by Theorem 4.4).

Let $\gamma = 1$. Since there exist powers of ψ not in $\langle\phi\rangle$ that fix points, the same holds for $\bar{\psi}$. Now, $o(\bar{\psi}) \neq 3$ since $2g$ is even and $o(\bar{\psi}) = 2$ means $g = 1$, a contradiction. $o(\bar{\psi}) = 4$ or $o(\bar{\psi}) = 6$ mean $g = 2$ or $g = 3$ but $\chi(\phi) \geqslant 6$ implies that $g > 3$.

Hence, $\gamma = 0$ that is $\chi(\phi) = 2g + 2$; the hypermap is hyperelliptic and $Aut(\alpha, \sigma) = C_{4g}$, $Aut(\alpha, \sigma) = D_{4g}$ or $g = 2$ and $|Aut(\alpha, \sigma)| = 48$ a central extension of S_4 by C_2. □

6. Harvey hypermaps

We recall that a normal subgroup of order p in a p-group is contained in the center of the p-group.

Proposition 6.1. *Let (α, σ) be a hypermap of genus $g \geqslant 2$, p a prime dividing the order of $Aut(\alpha, \sigma)$, and $\mathscr{P}$ a p-group. Let $\phi \in \mathscr{P}$ be an automorphism of prime order p such that $\chi(\phi) = 2 + 2g/(p - 1)$. Then ϕ is in the center of $\mathscr{P}$.*

Proof. Let ψ of order p be a central element in $\mathscr{P}$ and γ the genus of the quotient hypermap $(\bar{\alpha}, \bar{\sigma})$ with respect to $\langle\psi\rangle$; on this quotient hypermap, all automorphisms are induced. Since $\psi\phi = \phi\psi$ we can also consider the quotient hypermap with respect to $\langle\phi\rangle$; on this quotient hypermap ψ is induced and since this hypermap is of genus 0, $\chi(\psi) = 0, p$ or $2p$. We suppose for convenience that p is an odd prime since if $p = 2$ the map is hyperelliptic and ϕ is in the center of $Aut(\alpha, \sigma)$.

Let $\gamma \geqslant 2$. By induction on the genus, we suppose that $\forall\bar{\theta}\in\mathscr{P}/\langle\psi\rangle$, $\overline{\psi\theta} = \overline{\theta\phi}$. If $\bar{\theta}$ is identity then θ is a power of ψ therefore commutes with ϕ. If $\bar{\theta}$ is not identity then $\theta\phi\theta^{-1} = \psi^i\phi$ for some $i \neq 0 \bmod p$. Thus, $\theta^j\phi\theta^{-j} = \psi^{ij}$ for all $j \neq 0 \bmod p$. Since $\langle\psi^j\rangle = \langle\psi\rangle$, Now, by (1), $p(2 + 2\gamma/(p-1)) = p\chi(\bar{\phi}) = \sum_{j=1}^{p}\chi(\psi^{ij}\phi) = p(2 + 2g/(p-1)$; hence, $g \leqslant 1$, a contradiction.

Let $\gamma = 1$. Then $\chi(\psi) = (2g-2)/(p-1)$ and $\chi(\phi) = 2 + 2g/(p-1)$. Thus $p - 1$ divides 2, i.e. $p = 3$. Then $g = 4$ or $g = 7$ since $\chi(\psi) = p$ or $2p$. By (1) $\chi(\bar{\phi}) = 3$. If $g = 4$, since $\chi(\phi) = 6$, $\chi(\bar{\phi}) = 2$, a contradiction. If $g = 7, \chi(\phi) = 9$ and $\chi(\psi) = 6$. Thus, there exists a homomorphism from $\mathscr{P}$ to S_6. Hence, $|\mathscr{P}| \leqslant 27$. Thus, $\forall\bar{\theta}\in\mathscr{P}/\langle\psi\rangle$, $\overline{\phi\theta} = \overline{\theta\phi}$ and we use the same argument then for the case $\gamma \geqslant 2$. Let $\gamma = 0$, the induced p-Sylow is C_{p^n} because of planarity. Thus, $\forall\bar{\theta}\in\mathscr{P}/\langle\psi\rangle$, $\overline{\phi\theta} = \overline{\theta\phi}$ and we use the same argument as that for the case $\gamma \geqslant 2$. □

Corollary 6.2. *Let (α, σ) be a hypermap of genus $g \geqslant 2$ and G a nilpotent automorphism group. Let $\phi \in G$ be an automorphism of prime order p such that $\chi(\phi) = 2 + 2g/(p-1)$. Then ϕ is in the center of G.*

Proof. By Proposition 6.1, ϕ is central in the p-Sylow group and since G is nilpotent it is a direct product of its Sylow so that ϕ is in the center of G. □

Theorem 6.3. *Let (α, σ) be a hypermap of genus $g \geqslant 2, p \neq 3$ the smallest prime dividing the order of $Aut(\alpha, \sigma)$. Let ϕ be an automorphism of order p such that $\chi(\phi) = 2 + 2g/(p-1)$. Then (α, σ) is p-elliptic for the automorphism ϕ.*

Proof. Let m be the number of conjugates of $\langle\phi\rangle$. By (1), we know that $2g - 2 \geqslant -2|Aut(\alpha, \sigma)| + m(p-1)[2 + 2g/(p-1)]$ i.e. $|Aut(\alpha, \sigma)| \geqslant -g + 1 + m(p - 1 + g)$. Thus $|Aut(\alpha, \sigma)| \geqslant (m-1)(g-1) + mp$.

If $p \geqslant 5$, we know by Theorem 4.2 that $|Aut(\alpha, \sigma)| \leqslant 2p(g-1)/(p-3)$. Thus, $(m-1)(g-1) + mp \leqslant 2p(g-1)/(p-3)$ and $(m-1) < 2p/(p-3) = 2 + 6/(p-3)$. Hence, if $p > 5$ then $m \leqslant 5$ and any element which conjugates ϕ would be of order smaller than or equal to 5, a contradiction.

Note now that if there are exactly p conjugates of ϕ, there exists an element θ of order $o(\theta) = p$ which conjugates ϕ; then θ cyclically permutes all p conjugates of ϕ. But we know by Proposition 6.1 that θ must commute with at least the one which belongs to its p-Sylow, which is impossible. Thus, $p = 2$ and $m \leqslant 6$ is a contradiction.

If $p = 3$ and $|Aut(\alpha, \sigma)| \leqslant 4(g-1)$ then $m \leqslant 4$ the observation we made for $p = 5$ applies here and we get a contradiction.

If $p = 3$ and $4(g-1) > |Aut(\alpha, \sigma)| \leqslant 15(g-1)$, then, with the notation of Theorem 4.2, $r = 3$ and $\frac{1}{2} \leqslant 1/n_1 + 1/n_2 + 1/n_3 < 1$ i.e. $\frac{1}{6} \leqslant 1/n_2 + 1/n_3 < \frac{2}{3}$.

Such a situation is reached for instance when $n_2 = n_3 = 5$ and then $|Aut(\alpha, \sigma)| = 15(g-1)$. □

Definition 6.4. A *p-Harvey hypermap*, is a hypermap of genus $g \geqslant 2$ admitting an automorphism of order $p(1 + 2g/(p-1))$ (type I) or an automorphism of order $2pg/(p-1)$ (type II) where p is the smallest prime dividing the order of $Aut(\alpha, \sigma)$.

Remark 6.5. A Wiman hypermap is a 2-Harvey hypermap.

Theorem 6.6. *Let (α, σ) be a p-Harvey hypermap of type* I *where $p \neq 3$. Then (α, σ) is p-elliptic and $Aut(\alpha, \sigma) = C_{p(1+2g/(p-1))}$.*

Proof. By Theorem 4.4 we know that the power of order p of the automorphism of order $p(1 + 2g/(p-1))$ fixes $2 + 2g/(p-1)$ points. Since $p \neq 3$, Proposition 6.3 applies, so that (α, σ) is p-elliptic and the theorems on p-elliptic hypermaps allows us to conclude that $Aut(\alpha, \sigma) = C_{p(1+2g/(p-1))}$. □

Proposition 6.7. *A p-Harvey hypermap of type* II *admits an automorphism of order p fixing $2 + 2g/(p-1)$ points.*

Proof. Let ψ be the automorphism of order $2pg/(p-1)$ and ϕ is one of its power of order p. Because of Theorem 5.4 we can assume p to be odd. We know that the quotient hypermap with respect to $\langle\psi\rangle$ is of genus $\gamma_0 = 0$, since $|\langle\psi\rangle| \geqslant 2p(g-1)/(p-1)$. Thus, by (2), $\sum_{i=1}^{2g/(p-1)} \chi(\psi^i) = 2g - 2 + 4pg/(p-1)$.

If $\chi(\phi) = 0$, then $2g - 2 + 4pg/(p-1) = \sum_{i=1}^{2pg/(p-1)-1} \chi(\psi^i)$ i.e. $2g - 2 + 4pg/(p-1) = \sum_{i=1}^{2g/(p-1)-1} \chi(\psi^{pi}) = 2g - 2 + 4g/(p-1)$, which is impossible.

If $\chi(\phi) = 2$, then Lemma 4.3 applies and there is no automorphism of order $2pg/(p-1)$, a contradiction.

If $3 \leqslant \chi(\phi) < 2p$, there exists a homomorphism h from $\langle\psi\rangle$ to $S_{\chi(\phi)}$. If $\langle\psi\rangle/ker(h) = id$, then by (1), $o(\psi) = ker(h)| \leqslant 2g + 1$. Thus $p = 2g + 1$ and $\psi = \phi$ which fixes $3 = 2 + 2g/(p-1)$ points. If $\langle\psi\rangle/ker(h) \neq id$, then $o(h(\psi)) = q$ a prime such that $p \leqslant q < 2p$ otherwise p would not be the smallest prime. Thus, $\chi(\psi^q) \geqslant q$ and by (1) and if $\gamma > 0$, $2g - 2 \geqslant q(2pg/q(p-1) - 1$ i.e. $q - 2 \geqslant 2g/(p-1)$ which is impossible since q divides $2g/(p-1)$. Thus $\gamma = 0$ and $\chi(\phi) = 2 + g/(p-1)$ points.

If $\chi(\phi) \geqslant 2p$, then $2g - 2 \geqslant p(2\gamma - 2) + 2p(p-1)$; that is after computation $2g/(p-1) \geqslant p - 2 + p(1 + 2\gamma/(p-1))$ which is impossible for $\gamma \geqslant 2$.

Let $\gamma = 1$. Since there exist powers of ψ not in $\langle\phi\rangle$ that fix points, the same happens for $\bar{\psi}$. Now, $o(\bar{\psi}) \neq 2, 4, 6$ since p is odd and the smallest. If $o(\bar{\psi}) = 3 = g$ thus $\chi(\phi) = 2$ which is a case already considered.

Hence, $\gamma = 0$ that is $\chi(\phi) = 2 + 2g/(p-1)$. □

Theorem 6.8. *A p-Harvey hypermap of type* II *where* $p \neq 3$ *is p-elliptic and* $Aut(\alpha, \sigma) = C_{2pg/(p-1)}$ *or* $D_{2pg/(p-1)}$.

Proof. Immediate consequence of Propositions 6.7 and 6.3. □

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