Preconditioned GAOR methods for solving weighted linear least squares problems

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In this paper, we present the preconditioned generalized accelerated overrelaxation (GAOR) method for solving linear systems based on a class of weighted linear least squares problems. Two kinds of preconditioning are proposed, and each one contains three preconditioners. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the convergence rate of the preconditioned GAOR methods is indeed better than the rate of the original method, whenever the original method is convergent. Finally, a numerical example is presented in order to confirm these theoretical results.

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1. Introduction

Consider the weighted linear least squares problem

$$\min_{x \in \mathbb{R}^n} (Ax - b)^T W^{-1} (Ax - b),$$

where $W$ is the variance–covariance matrix (cf. [14]). This problem has many scientific applications. A typical source is parameter estimation in mathematical modelling.

This problem has been discussed in many books and articles. In order to solve it, man has to solve a linear system as

$$Hy = f,$$

where

$$H = \begin{pmatrix} I - B_1 & U \\ C & I - B_2 \end{pmatrix}$$

is an invertible matrix with

$$B_1 = (b_i)_{p \times p}, \quad B_2 = (b_i)_{(n-p) \times (n-p)}, \quad C = (c_i)_{(n-p) \times p}, \quad U = (u_i)_{p \times (n-p)}.$$ 

For solving general linear systems

$$Ax = b$$

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using classical iterative methods, man splits $A$ as
\[ A = D - C_i - C_u. \] (1.2)
When $D$ is a diagonal matrix, $C_i$ and $C_u$ are respectively strictly lower and strictly upper triangular matrices, there are three well known iterative methods, Jacobi, Gauss–Seidel and successive overrelaxation (SOR) methods, which were fully covered in the excellent books by Varga [10] and Young [11]. To accelerate the convergence of the SOR method, by introducing another parameter, Hadjidimos in [5] proposed accelerated overrelaxation (AOR) method. As a generalization of AOR method, in [8,9] we assume $D$ in (1.2) is nonsingular, but we do not assume that $D$ is diagonal, or that $C_i$ and $C_u$ are triangular. Then we proposed the generalized AOR (GAOR) method. When the two parameters are equal, the GAOR method reduces to GSOR method.

In order to solve the linear system (1.1) using the GAOR method, in [2,3,14] the matrix $H$ is split as
\[ H = I - \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} - \begin{pmatrix} B_1 & -U \\ 0 & B_2 \end{pmatrix}. \]
Then, for $\omega \neq 0$, a GAOR method can be defined by
\[ y^{(k+1)} = L_{\gamma,\omega} y^{(k)} + \omega g, \quad k = 0, 1, 2, \ldots, \] (1.3)
where
\[ L_{\gamma,\omega} = \begin{pmatrix} 1 & 0 \\ \gamma C & 1 \end{pmatrix}^{-1} \begin{pmatrix} (1 - \omega)I + (\omega - \gamma) \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} + \omega \begin{pmatrix} B_1 & -U \\ 0 & B_2 \end{pmatrix} \\ (1 - \omega)I + \omega B_1 \omega(\beta - 1)C - \omega \gamma CB_1 (1 - \omega)I + \omega B_2 + \omega \gamma CU \end{pmatrix} \]
is iteration matrix and
\[ g = \begin{pmatrix} 1 \\ -\gamma C \end{pmatrix} f. \]
If we take $\gamma = \omega$, then the GAOR method reduces to the GSOR method given by [12,13].

The spectral radius of the iteration matrix $L_{\gamma,\omega}$ is decisive for the convergence, and the smaller it is, the faster the method converges. In order to decrease the spectral radius of $L_{\gamma,\omega}$, an effective method is to precondition the linear system (1.1), namely,
\[ PHy = Pf, \]
where $P$ is a nonsingular.

If we express $PH$ as
\[ PH = \begin{pmatrix} 1 - B_i^* & U^* \\ C^* & 1 - B_i^* \end{pmatrix}, \]
then the preconditioned GAOR method can be defined by
\[ y^{(k+1)} = L_{\gamma,\omega}^* y^{(k)} + \omega g^*, \quad k = 0, 1, 2, \ldots, \] (1.4)
where
\[ L_{\gamma,\omega}^* = \begin{pmatrix} (1 - \omega)I + \omega B_i^* & -\omega U^* \\ \omega(\beta - 1)C - \omega \gamma CB_1 (1 - \omega)I + \omega B_2 + \omega \gamma C^*U^* \end{pmatrix} \]
and
\[ g^* = \begin{pmatrix} 1 \\ -\gamma C^* \end{pmatrix} Pf. \]

In this paper, we investigate the preconditioned GAOR method defined by (1.4). In Section 2 two kinds of preconditioning are proposed and each one contains three preconditioners. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the convergence rate of the preconditioned GAOR methods is indeed better than the rate of the original method, whenever the original method is convergent. In Section 3, a numerical example is presented in order to confirm the theoretical results given in Section 2.

For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$ holds for all $i, j = 1, 2, \ldots, n$. We are calling $A$ nonnegative (positive) if $A \geq (>0)$, we say that $A - B \geq 0$ if and only if $A \geq B$. These definitions carry immediately over to vectors by identifying them with $n \times 1$ matrices. $\rho(\cdot)$ denotes the spectral radius of a matrix.

The following known results are useful in the proof of our results in the next section.

**Lemma 1.1** (Varga [10].) Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and irreducible. Then
(i) $A$ has a positive real eigenvalue equal to its spectral radius $\rho(A)$;
(ii) for $\rho(A)$, there corresponds an eigenvector $x > 0$.

**Lemma 1.2** (Berman and Plemmons [11].) Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and irreducible. If $0 \neq \alpha x \leq Ax \leq \beta x, \alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector $x$, then $\alpha < \rho(A) < \beta$ and $x$ is a positive vector.
2. Preconditioned GAOR methods and comparisons

We consider the preconditioned linear system
\[ \tilde{H}y = \tilde{f}, \]  
(2.1)
where \( \tilde{H} = (I + \tilde{S})H \) and \( \tilde{f} = (I + \tilde{S})f \) with
\[ \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \]
S is a \( p \times p \) matrix with \( 1 < p < n \).

For \( \alpha > 0 \), using the ideas of \([4,6,7]\), we take three kinds of \( S \) as follows.

\[
S_1 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & 0 & \ldots & 0 \\ \frac{\alpha}{\alpha+1} & 0 & \ldots & 0 \\ \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ b_{21} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & 0 & \ldots & 0 \\ \end{pmatrix}, \\
S_3 = \begin{pmatrix} 0 & b_{32} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} \\ \end{pmatrix}.
\]

Now, we obtain three preconditioned linear systems with coefficient matrices
\[
\tilde{H}_i = \begin{pmatrix} I - [B_1 - S_i(I - B_1)] & (I + S_i)U \\ C & I - B_2 \end{pmatrix}, \quad i = 1, 2, 3,
\]

where
\[
B_1 - S_1(I - B_1) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pp} \\ \frac{1 - b_{11}}{\alpha} b_{p1} & b_{p2} + \frac{b_{p1} b_{12}}{\alpha} & \cdots & b_{pp} + \frac{b_{p1} b_{1p}}{\alpha} \end{pmatrix},
\]
\[
B_1 - S_2(I - B_1) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} b_{11} & b_{22} + b_{21} b_{12} & \cdots & b_{2p} + b_{21} b_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} b_{11} & b_{p2} + b_{p1} b_{12} & \cdots & b_{pp} + b_{p1} b_{1p} \end{pmatrix},
\]
\[
B_1 - S_3(I - B_1) = \begin{pmatrix} b_{11} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21} b_{11} & \cdots & b_{2,p-2} + b_{21} b_{1,p-1} & b_{2p} + b_{21} b_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} + b_{p,p-1} b_{1,p-1} & \cdots & b_{p,p-2} + b_{p1} b_{p-1,p-1} & b_{pp} + b_{p1} b_{p,p-1} \end{pmatrix}.
\]

For \( i = 1, 2, 3 \), split \( \tilde{H}_i \) as
\[
\tilde{H}_i = I - \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} - \begin{pmatrix} [B_1 - S_i(I - B_1)] & -(I + S_i)U \\ 0 & B_2 \end{pmatrix}.
\]

Then the preconditioned GAOR methods for solving (2.1) are defined as follows.
\[ y^{(k+1)} = L^{(k)} y^{(k)} + \omega \tilde{g}, \quad k = 0, 1, 2, \ldots, \]
(2.2)
where for \( i = 1, 2, 3 \),
\[
L^{(k)} = \begin{pmatrix} (1 - \omega)I + \omega[B_1 - S_i(I - B_1)] & -\omega(I + S_i)U \\ (\alpha(\gamma - 1)C - \omega \gamma C[B_1 - S_i(I - B_1)]) & (1 - \omega)I + \omega B_2 + \omega \gamma C(I + S_i)U \end{pmatrix}
\]
are iteration matrices and
\[ \tilde{g} = \begin{pmatrix} 1 & 0 \\ -\gamma C & I \end{pmatrix} \tilde{f}. \]

Now, we discuss the convergence of the preconditioned GAOR methods. We give comparisons between the preconditioned GAOR methods defined by (2.2) and the corresponding GAOR method defined by (1.3).
Theorem 2.1. Let $\mathcal{L}_{Y,0}^{(1)}, \mathcal{L}_{Y,0}^{(2)}$ be the iteration matrices of the GAOR and preconditioned GAOR methods, respectively. If the matrix $H$ in (1.1) is irreducible with $C \leq 0$, $U \leq 0$, $B_1 \geq 0$, $B_2 \geq 0$, $b_{p1} > 0$, $\alpha > 0$, $\alpha > 1 - b_{11}$, and $0 < \omega \leq 1$, $0 \leq \gamma < 1$, then either

$$\rho(\mathcal{L}_{Y,0}^{(1)}) < \rho(\mathcal{L}_{Y,0}) < 1 \quad (2.3)$$

or

$$\rho(\mathcal{L}_{Y,0}^{(1)}) > \rho(\mathcal{L}_{Y,0}) > 1. \quad (2.4)$$

Proof. By direct operation we have

$$\mathcal{L}_{Y,0}^{(1)} = \begin{pmatrix} (1 - \omega)I + \omega B_1 & -\omega U \\ -\omega(1 - \gamma)C & (1 - \omega)I + \omega B_2 \end{pmatrix} + \omega \gamma \begin{pmatrix} 0 & 0 \\ -CB_1 & CU \end{pmatrix}. \quad (2.5)$$

Since $0 < \omega \leq 1$, $0 \leq \gamma < 1$, $C \leq 0$, $U \leq 0$, $B_1 \geq 0$, $B_2 \geq 0$, then

$$\begin{pmatrix} 0 & 0 \\ -CB_1 & CU \end{pmatrix} \geq 0$$

and $\mathcal{L}_{Y,0}^{(1)}$ is nonnegative. Since $H$ is irreducible, from (2.5), it is easy to see that the matrix $\mathcal{L}_{Y,0}^{(1)}$ is nonnegative and irreducible. Similarly, it can be proved that the matrix $\mathcal{L}_{Y,0}^{(1)}$ is nonnegative and irreducible since $\alpha > 1 - b_{11}$. By Lemma 1.1, there is a positive vector $x$, such that

$$\mathcal{L}_{Y,0}^{(1)}x = \lambda x, \quad (2.6)$$

where $\lambda = \rho(\mathcal{L}_{Y,0}^{(1)})$. Clearly, $\lambda = 1$ is impossible, otherwise the matrix $H$ is singular. Hence it gets either $\lambda < 1$ or $\lambda > 1$.

Now, from (2.6) and by the definitions of $\mathcal{L}_{Y,0}$ and $\mathcal{L}_{Y,0}^{(1)}$, we have

$$\mathcal{L}_{Y,0}^{(1)}x - \lambda x = (\mathcal{L}_{Y,0}^{(1)} - \mathcal{L}_{Y,0})x \quad (2.7)$$

Since $b_{p1} > 0$ and $\alpha > 0$, then $S_1 \geq 0$ and $S_1 \neq 0$. So we derive

$$\begin{pmatrix} S_1 & 0 \\ -\gamma CS_1 & 0 \end{pmatrix}x \geq 0, \quad \begin{pmatrix} S_1 & 0 \\ -\gamma CS_1 & 0 \end{pmatrix}x \neq 0.$$

If $\lambda < 1$, then

$$\mathcal{L}_{Y,0}^{(1)}x - \lambda x \leq 0, \quad \mathcal{L}_{Y,0}^{(1)}x - \lambda x \neq 0.$$

By Lemma 1.2, the inequality (2.3) is proved.

While, if $\lambda > 1$, then

$$\mathcal{L}_{Y,0}^{(1)}x - \lambda x \geq 0, \quad \mathcal{L}_{Y,0}^{(1)}x - \lambda x \neq 0.$$

By Lemma 1.2, the inequality (2.4) is proved. \hfill \Box

Similarly, for other two preconditioned GAOR methods, we can obtain the following convergence theorems.

Theorem 2.2. Let $\mathcal{L}_{Y,0}^{(2)}$, $\mathcal{L}_{Y,0}^{(2)}$ be the iteration matrices of the GAOR and preconditioned GAOR methods, respectively. If the matrix $H$ in (1.1) is irreducible with $C \leq 0$, $U \leq 0$, $B_1 \geq 0$, $B_2 \geq 0$, $b_{p1} > 0$, $b_{p1} > 0$ for some $i \in \{2, \ldots, p\}$, and $0 < \omega \leq 1$, $0 \leq \gamma < 1$, then either

$$\rho(\mathcal{L}_{Y,0}^{(2)}) < \rho(\mathcal{L}_{Y,0}) < 1 \quad (2.8)$$

or

$$\rho(\mathcal{L}_{Y,0}^{(2)}) > \rho(\mathcal{L}_{Y,0}) > 1. \quad (2.9)$$
In this theorem, the condition \( b_{11} > 0 \) for some \( i \in \{2, \ldots, p\} \) implies \( S_2 \neq 0 \), and the inverse is also true. The condition \( b_{11} > 0 \) ensures that the matrix \( B_1 - S_2(I - B_1) \) has the same irreducibility as \( B_1 \).

**Theorem 2.3.** Let \( L_{\gamma,a}, L_{\gamma,a}^{(3)} \) be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix \( H \) in (1.1) is irreducible with \( C \leq 0, U \leq 0, B_1 \geq 0, B_2 \geq 0, b_{i+1,1} > 0 \) for some \( i \in \{1, \ldots, p - 1\}, b_a > 0 \) whenever \( b_{i+1,1} > 0 \) for \( i \in \{1, \ldots, p - 1\} \), and \( 0 < \omega \leq 1, 0 \leq \gamma < 1 \), then either
\[
\rho(L_{\gamma,a}^{(3)}) < \rho(L_{\gamma,a}) < 1
\]
or
\[
\rho(L_{\gamma,a}^{(3)}) > \rho(L_{\gamma,a}) > 1.
\]

In the theorem the condition \( b_{i+1,1} > 0 \) for some \( i \in \{1, \ldots, p - 1\} \) implies \( S_3 \neq 0 \), and the inverse is also true. And the condition \( b_a > 0 \) whenever \( b_{i+1,1} > 0 \) ensures that the matrix \( B_1 - S_3(I - B_1) \) has the same irreducibility as \( B_1 \).

Let \( x > 0 \) be defined by (2.6). Similar to the proof of **Theorem 2.1** we can obtain
\[
L_{\gamma,a}^{(2)}x - L_{\gamma,a}^{(1)}x = (L_{\gamma,a}^{(2)}x - \lambda x) - (L_{\gamma,a}^{(1)}x - \lambda x)
\]
\[
= (\lambda - 1) \begin{pmatrix} S_2 & 0 \\ -\gamma C S_2 & 0 \end{pmatrix} x - (\lambda - 1) \begin{pmatrix} S_1 & 0 \\ -\gamma C S_1 & 0 \end{pmatrix} x
\]
\[
= (\lambda - 1) \begin{pmatrix} I & 0 \\ -\gamma C & 0 \end{pmatrix} (S_2 - S_1)x.
\]

Under the conditions of **Theorems 2.1** and 2.2, if either \( \alpha > 1 \) or \( \alpha = 1 \) but \( b_{11} > 0 \) for some \( i \in \{2, \ldots, p - 1\} \), then \( S_2 \geq S_1 \geq 0 \) and \( S_2 \neq S_1 \). Hence, in this case, we have
\[
L_{\gamma,a}^{(2)}x \leq L_{\gamma,a}^{(1)}x, \quad L_{\gamma,a}^{(2)}x \neq L_{\gamma,a}^{(1)}x, \quad \text{whenever } \lambda < 1
\]
and
\[
L_{\gamma,a}^{(2)}x \geq L_{\gamma,a}^{(1)}x, \quad L_{\gamma,a}^{(2)}x \neq L_{\gamma,a}^{(1)}x, \quad \text{whenever } \lambda > 1.
\]

Let \( X = \text{diag}(x_1, \ldots, x_n) \). When \( \lambda < 1 \) it holds
\[
\|X^{-1} L_{\gamma,a}^{(2)}x\|_\infty \leq \|X^{-1} L_{\gamma,a}^{(1)}x\|_\infty.
\]

So we can expect \( \rho(L_{\gamma,a}^{(2)}) \leq \rho(L_{\gamma,a}^{(1)}) < 1 \), but we can not ensure it to be always true.

Now, we consider another class of preconditioners. Let the matrix \( S \) in (2.1) be defined by
\[
S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
with \( S = (S_b)_{(n-p) \times p} \). We also take three kinds of \( S \) as follows.

For \( n - p < p \), then
\[
S_4 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{c_{n-p,1}}{\alpha} & 0 & \cdots & 0 \end{pmatrix}, \quad S_5 = \begin{pmatrix} -c_{11} & 0 & \cdots & 0 \\ -c_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_{n-p,1} & 0 & \cdots & 0 \end{pmatrix}.
\]

If \( n - p = p \), then
\[
S_6 = \begin{pmatrix} -c_{11} & 0 & \cdots & 0 \\ -c_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -c_{n-p,p-p} \end{pmatrix}.
\]

If \( n - p = p \), then
\[
S_7 = \begin{pmatrix} -c_{11} & 0 & \cdots & 0 \\ -c_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -c_{p-p,p-p} \end{pmatrix}.
\]
If $n - p > p$, then

$$S_6 = \begin{pmatrix}
-c_{11} & 0 & \cdots & 0 \\
0 & -c_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -c_{pp}
\end{pmatrix}.$$ 

For simplicity, we assume $n - p < p$, we can express the coefficient matrix of (2.1) as

$$\tilde{H}_i = \begin{pmatrix}
I - B_1 & U \\
S_i(I - B_1) + C & I - B_2 + S_iU
\end{pmatrix}, \quad i = 4, 5, 6,$$

where

$$S_4(I - B_1) + C = \begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1p} \\
c_{21} & c_{22} & \cdots & c_{2p} \\
\vdots & \vdots & & \vdots \\
(1 - \frac{1 - b_{11}}{\alpha})c_{n-p,1} + \frac{c_{n-p,1}b_{12}}{\alpha} & \cdots & \frac{c_{n-p,p} + \frac{c_{n-p,1}b_{1p}}{\alpha}}
\end{pmatrix},$$

$$S_5(I - B_1) + C = \begin{pmatrix}
c_{11}b_{11} & c_{12} + c_{11}b_{12} & \cdots & c_{1p} + c_{11}b_{1p} \\
c_{21}b_{11} & c_{22} + c_{21}b_{12} & \cdots & c_{2p} + c_{21}b_{1p} \\
\vdots & \vdots & & \vdots \\
c_{n-p,1}b_{11} & c_{n-p,1}b_{12} & \cdots & c_{n-p,p} + c_{n-p,1}b_{1p}
\end{pmatrix},$$

$$S_6(I - B_1) + C = \begin{pmatrix}
c_{11}b_{11} & c_{12} + c_{11}b_{12} & \cdots & c_{1p} + c_{11}b_{1p} \\
c_{21} + c_{22}b_{21} & c_{22}b_{22} & \cdots & c_{2p} + c_{22}b_{2p} \\
\vdots & \vdots & & \vdots \\
c_{n-p,1} + c_{n-p,n-p}b_{n-p,1} & c_{n-p,2} + c_{n-p,n-p}b_{n-p,2} & \cdots & c_{n-p,p} + c_{n-p,n-p}b_{n-p,p}
\end{pmatrix}.$$

For $i = 4, 5, 6$, split $\tilde{H}_i$ as

$$\tilde{H}_i = I - \begin{pmatrix}
0 & 0 \\
-S_i(I - B_1) - C & 0
\end{pmatrix} - \begin{pmatrix}
B_1 & -U \\
0 & B_2 - S_iU
\end{pmatrix}.$$

Applying the GAOR method to the preconditioned linear systems (2.1), respectively, we have the corresponding preconditioned GAOR methods.

$$y^{(k+1)} = L^{(i)}_{Y;\omega} y^{(k)} + \omega\tilde{g}^{(i)}, \quad k = 0, 1, 2, \ldots, \quad i = 4, 5, 6,$$

where

$$L^{(i)}_{Y;\omega} = (1 - \omega)I + \omega f^{(i)} + \omega g^{(i)}, \quad f^{(i)} = \begin{pmatrix}
B_1 & -U \\
0 & B_2 - S_iU
\end{pmatrix},$$

$$g^{(i)} = \begin{pmatrix}
0 & 0 \\
S_i(I - B_1) + C & I
\end{pmatrix}.$$

Similar to Theorems 2.1–2.3 we can prove the following convergence theorems.

**Theorem 2.4.** Let $L^{(i)}_{Y;\omega}$ and $L^{(4)}_{Y;\omega}$ be the iteration matrices of the GAOR and preconditioned GAOR methods, respectively. If the matrix $H$ in (1.1) is an irreducible with $C \leq 0, U \leq 0, B_1 \geq 0, B_2 \geq 0, c_{n-p,1} < 0, \alpha > 0, \beta > 1 - b_{11},$ and $0 < \omega \leq 1, 0 < \gamma < 1$, then either

$$\rho(L^{(i)}_{Y;\omega}) < \rho(L^{(4)}_{Y;\omega}) < 1$$

or

$$\rho(L^{(i)}_{Y;\omega}) > \rho(L^{(4)}_{Y;\omega}) > 1.$$
Theorem 2.5. Let \( \mathcal{L}_{\gamma, \omega} \) and \( \mathcal{L}^{(5)}_{\gamma, \omega} \) be the iteration matrices of the GAOR and preconditioned GAOR methods, respectively. If the matrix \( H \) in (1.1) is an irreducible with \( C \leq 0, U \leq 0, B_1 \geq 0, B_2 \geq 0, b_{11} > 0, c_{11} < 0 \) for some \( i \in \{1, \ldots, n-p\} \), and \( 0 < \omega \leq 1, 0 \leq \gamma < 1 \), then either

\[
\rho(\mathcal{L}^{(5)}_{\gamma, \omega}) < \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) < 1
\]
or

\[
\rho(\mathcal{L}^{(5)}_{\gamma, \omega}) > \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) > 1.
\]

Theorem 2.6. Let \( \mathcal{L}_{\gamma, \omega} \) and \( \mathcal{L}^{(6)}_{\gamma, \omega} \) be the iteration matrices of the GAOR and preconditioned GAOR methods, respectively. If the matrix \( H \) in (1.1) is an irreducible with \( C \leq 0, U \leq 0, B_1 \geq 0, B_2 \geq 0, c_{ii} < 0 \) for some \( i \in \{1, \ldots, n-p\} \), and \( 0 < \omega \leq 1, 0 \leq \gamma < 1 \), then either

\[
\rho(\mathcal{L}^{(6)}_{\gamma, \omega}) < \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) < 1
\]
or

\[
\rho(\mathcal{L}^{(6)}_{\gamma, \omega}) > \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) > 1.
\]

For \( \rho(\mathcal{L}^{(4)}_{\gamma, \omega}) \) and \( \rho(\mathcal{L}^{(5)}_{\gamma, \omega}) \) we can give a similar comparison analysis as \( \rho(\mathcal{L}^{(1)}_{\gamma, \omega}) \) and \( \rho(\mathcal{L}^{(2)}_{\gamma, \omega}) \) above. We can change \( S_i \) into \( S^*_i, i = 1, \ldots, 6, \) in the preconditioners. The convergence results are similar.

3. A numerical example

Now let us consider an example to illustrate the theoretical results above.

Example 3.1. The coefficient matrix \( H \) in (1.1) is given by

\[
H = \begin{pmatrix}
1 - B_1 & U \\
C & 1 - B_2
\end{pmatrix},
\]

where \( B_1 = \left((b^{(1)}_i)_{p\times p}, B_2 = (b^{(2)}_i)_{(p-n)\times (p-n)}, C = (c_i)_{(p-n)\times p}, U = (u_i)_{p\times(p-n)}\right) \) with

\[
b^{(1)}_i = \frac{1}{10 \times (i + 1)}, \quad i = 1, \ldots, p,
\]

\[
b^{(2)}_i = \frac{1}{30} - \frac{1}{30 \times j + i}, \quad i < j, i = 1, \ldots, p - 1, j = 2, \ldots, p,
\]

\[
b^{(1)}_i = \frac{1}{30} - \frac{1}{30 \times (i - j + 1) + i}, \quad i > j, i = 2, \ldots, p, j = 1, \ldots, p - 1,
\]

\[
b^{(2)}_i = \frac{1}{10 \times (p + i + 1)}, \quad i = 1, \ldots, n - p,
\]

\[
b^{(2)}_i = \frac{1}{30} - \frac{1}{30 \times (p + j) + p + i}, \quad i < j, i = 1, \ldots, n - p + 1, j = 2, \ldots, n - p,
\]

\[
b^{(2)}_i = \frac{1}{30} - \frac{1}{30 \times (i - j + 1) + p + i}, \quad i > j, i = 2, \ldots, n - p, j = 1, \ldots, n - p - 1,
\]

\[
c_i = \frac{1}{30 \times (p + i - j + 1) + p + i} - \frac{1}{30}, \quad i = 1, \ldots, n - p, j = 1, \ldots, p,
\]

\[
u_i = \frac{1}{30 \times (p + j) + i} - \frac{1}{30}, \quad i = 1, \ldots, p, j = 1, \ldots, n - p.
\]

Table 1 displays the spectral radii of the corresponding iteration matrices with some random chosen parameters \( \omega, \gamma, p \) and \( \alpha \), where \( \rho_i = \rho(\mathcal{L}^{(i)}_{\gamma, \omega}), i = 1, \ldots, 6 \). The methods have been implemented in Matlab and the output has been produced with the help of Matlab 6.51.

From Table 1, in accordance to the theory for the example we see that \( \rho(\mathcal{L}^{(1)}_{\gamma, \omega}) < \rho(\mathcal{L}^{(p)}_{\gamma, \omega}), i = 1, \ldots, 6, \rho(\mathcal{L}^{(2)}_{\gamma, \omega}) < \rho(\mathcal{L}^{(1)}_{\gamma, \omega}) \) and \( \rho(\mathcal{L}^{(5)}_{\gamma, \omega}) < \rho(\mathcal{L}^{(4)}_{\gamma, \omega}) \) when \( \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) < 1 \). While \( \rho(\mathcal{L}^{(1)}_{\gamma, \omega}) > \rho(\mathcal{L}^{(p)}_{\gamma, \omega}), i = 1, \ldots, 6, \) when \( \rho(\mathcal{L}^{(p)}_{\gamma, \omega}) > 1 \). These are in concord with Theorems 2.1–2.6.

In addition, the preconditioned GAOR methods need fewer iteration numbers than the original GAOR method when all iterations are started from the same vector and terminated rule. So, the preconditioned GAOR methods are superior to the original GAOR method.
Table 1
The spectral radii of the GAOR and preconditioned GAOR iteration matrices

<table>
<thead>
<tr>
<th>n</th>
<th>ω</th>
<th>γ</th>
<th>p</th>
<th>α</th>
<th>ρ(2γ,ω)</th>
<th>ρ1</th>
<th>ρ2</th>
<th>ρ3</th>
<th>ρ4</th>
<th>ρ5</th>
<th>ρ6</th>
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<td>3</td>
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<td>0.1397</td>
<td>0.1348</td>
<td>0.1384</td>
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<td>5</td>
<td>3</td>
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<td>5</td>
<td>3</td>
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<td>0.3830</td>
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<td>0.3768</td>
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<td>0.6297</td>
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<td>0.6347</td>
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<td>0.6308</td>
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<td>16</td>
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<td>0.9144</td>
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<tr>
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References