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Cohomology and Higher Dimensional Baer Invariants

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In this paper we work in a categorical setting which includes the most usual algebraic categories, for example, groups, rings (associative, commutative), or algebras (associative, commutative, Jordan, Lie, etc.). Such a setting is supplied by the theory of Ω-groups in the sense of Higgins [5].

Let $\mathscr{D}$ be a variety of Ω-groups and $\mathscr{B} \subseteq \mathscr{D}$ a subvariety, with associated quotient functor $U: \mathscr{D} \to \mathscr{B}$. Several authors have shown that the first Baer invariant of an object A in $\mathscr{D}$, $\mathscr{B}_1(A) = L_1 U(A)$, plays an important role in the study of the relationship between extensions of $U(A)$ in the subvariety $\mathscr{B}$ and general extensions of A in $\mathscr{D}$. In the particular case where $\mathscr{D}$ is a variety of groups, Leedham-Green and Mackay showed, in [8], the existence of a five term exact sequence of abelian groups, for any group A in $\mathscr{D}$ and any $U(A)$-module M in $\mathscr{B}$,

$$0 \to \mathscr{B}^1(U(A), M) \to \mathscr{D}^1(A, M) \to \mathrm{Hom}_A(\mathscr{B}_1(A), M) \to \mathscr{B}^2(U(A), M) \to \mathscr{D}^2(A, M). \qquad (S_1)$$

The sequence (S_1) relates the first cohomology groups in $\mathscr{D}$ with those in $\mathscr{B}$. In [11], Lue also showed the existence of a sequence in which the first four terms of S_1 appear. These sequences generalize the well known exact sequence of universal coefficients for groups cohomology

$$0 \to \mathrm{Ext}(A_{ab}, M) \to H^2(A, M) \to \mathrm{Hom}(H_2(A, \mathbb{Z}), M) \to 0,$$

which results from (S_1) when $\mathscr{D}$ is the variety of all groups and $\mathscr{B}$ the subvariety of abelian groups.

In [12] Modi defined the nth Baer invariant of an object A in $\mathscr{D}$ relative to the variety $\mathscr{B}$, using Keune's homotopical theory [7], as the value of the nth left derived functor of the quotient functor U in A, $\mathscr{B}_n(A) = L_n U(A)$, $n \geqslant 1$. In this paper we prove that $\mathscr{B}_n(A)$ is a $U(A)$-module in $\mathscr{B}$, and therefore it is an A-module in $\mathscr{D}$ via the canonical projection $A \twoheadrightarrow U(A)$. Then we show that the sequence (S_1) always exists, for any $U(A)$-module M in

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$\mathscr{B}$, and in the case where $\mathscr{B}_r(A)=0$, $1\leqslant r\leqslant n-1$, there also exists an exact sequence

$$\begin{aligned} 0 &\to \mathscr{B}^n(U(A), M) \to \mathscr{D}^n(A, M) \to \mathrm{Hom}_A(\mathscr{B}_n(A), M) \\ &\to \mathscr{B}^{n+1}(U(A), M) \to \mathscr{D}^{n+1}(A, M). \end{aligned} \qquad (\mathrm{S}_n)$$

When $\mathscr{D}$ is the variety of all groups and $\mathscr{B}$ the subvariety of abelian groups, the nth Baer invariant of a group A is just the integral homology group $H_n(A, \mathbb{Z})$, and then the sequence S_n gives the well known universal coefficients isomorphism

$$H^n(A, M) \cong \mathrm{Hom}(H_n(A, \mathbb{Z}), M)$$

for any group A with $H_r(A, \mathbb{Z})=0$, $2\leqslant r\leqslant n-1$, and any trivial A-module M. So, the existence of the sequence (S_n) can be seen as a generalization to arbitrary varieties of Ω-groups of the above classical result for cohomology of groups.

As an immediate consequence of the existence of the sequences (S_n), we will prove that the Baer invariants of an object A in $\mathscr{D}$ are trivial at dimensions less than n ($\mathscr{B}_r(A)=0$, $1\leqslant r\leqslant n-1$) if and only if for any $U(A)$-module M in $\mathscr{B}$, the canonical morphism $\mathscr{B}^r(U(A), M)\to\mathscr{D}^r(A, M)$ is an isomorphism for $r\leqslant n-1$ and a monomorphism for $r=n$. Let us note that there are some interesting examples in which $\mathscr{B}_1(A)$ is zero (see [6, 9, 10]). If we moreover suppose that A is a $\mathscr{B}$-perfect or a $\mathscr{B}$-splitting object (i.e., $U(A)$ is zero or a retract of a $\mathscr{B}$-free object, respectively), we obtain an isomorphism $\mathscr{D}^2(A, M)\cong\mathrm{Hom}_A(\mathscr{B}_2(A), M)$ as a direct consequence of the sequence (S_2).

1. Modules in a Variety of Ω-Groups

This section recalls some facts about modules over an object in a variety of Ω-groups. We will need these facts in the rest of the paper.

Let us suppose $\Omega=\Omega_0\cup\Omega_1\cup\Omega_2$ a set of operators (none of weight greater than 2), with only one operator of weight zero (denoted by 0), just two of weight two (denoted by $+$ and $*$, respectively), and at least one of weight one (denoted by $-$).

An Ω-group is a set A together with n-ary operations, one for each operator in Ω of weight n, which satisfy:

(i) The set A, together with the operations which correspond to the operators $+$, $-$, and 0, is a group.

(ii) The operation which corresponds to $*$ is distributive with respect to that which corresponds to + (i.e., $x * (y+z) = (x * y) + (x * z)$ and $(x+y) * z = (x * z) + (y * z)$ for all $x, y, z \in A$).

(iii) For all $\omega \in \Omega_1$, $\omega(x * y) = \omega(x) * y = x * \omega(y)$ and if $\omega \neq -$, then $\omega(x+y) = \omega(x) + \omega(y)$ for all $x, y \in A$.

The category of Ω-groups has Ω-groups as objects and Ω-group homomorphisms as morphisms, where an Ω-group homormorphism is a map which preserves all the operations.

We will suppose $\mathscr{D}$ a variety (in the sense of Birkoff) of the category of Ω-groups. Then $\mathscr{D}$ is complete, cocomplete, and has a zero object. Note also that for any surjective morphism in $\mathscr{D}$, $p: A \to B$, the set $N = \{a \in A \,/\, p(a) = 0\}$ is an object in $\mathscr{D}$, the inclusion $v: N \hookrightarrow A$ is the kernel of p, and p is the cokernel of v.

An *ideal* of an object A is a subobject which is the kernel of some morphisms with domain A. It is clear that a subobject N of A is an ideal iff it is a normal subgroup of A and for all $x \in N$ and $a \in A$ the elements $x * a$ and $a * x$ are in N.

An object M in $\mathscr{D}$ is called *singular* if it is abelian as a group and if $M * M = 0$. Given an object A, by an *A-Module* in $\mathscr{D}$ we mean a right split exact sequence in $\mathscr{D}$, $0 \to M \to E \leftrightarrows^{s} A \to 0$, with M a singular object. Such a sequence is determined, up to isomorphism, by A, M, and the *induced actions* of A on M,

$$^{a}x = s(a) + x - s(a),$$

$$a * x = s(a) * x \qquad \text{and} \qquad x * a = x * s(a), \qquad \text{for all} \quad x \in M, a \in A;$$

note that E is isomorphic to the Ω-group $M \rtimes A$ (semidirect product), whose underlying set is the cartesian product $M \times A$ and whose operations are

$$\omega(x, a) = (\omega(x), \omega(a))$$

$$(x, a) + (x', a') = (x + {}^{a}x', a + a')$$

$$(x, a) * (x', a') = (x * a' + a * x', a * a').$$

Then an A-module is just a singular object M together with certain actions, in such a way that the Ω-group $M \rtimes A$ is an object in $\mathscr{D}$.

An A-module morphism is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \leftrightarrows & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & E' & \leftrightarrows & A & \longrightarrow & 0 \end{array}$$

or equivalently a morphism $f: M \to M'$ which is compatible with the actions. The corresponding category of A-modules is equivalent to the category of abelian group objects in the comma category $\mathscr{D}/A$, i.e., the category of A-modules in $\mathscr{D}$ in the sense of Beck. An equivalence is given by the functor which associates to any A-module M the object $M \rtimes A \to^{\mathrm{pr}} A$ in $\mathscr{D}/A$.

PROPOSITION 1. *Let M be an A-module in $\mathscr{D}$:*

(i) *Given a morphism $\psi: A' \to A$, M is an A'-module in $\mathscr{D}$ with actions*

$$\begin{aligned} {}^{a'}x &= {}^{\psi(a')}x, \\ a' * x &= \psi(a') * x, \qquad \textit{and} \\ x * a' &= x * \psi(a'), \qquad \textit{for all } x \in M \textit{ and } a' \in A'. \end{aligned}$$

We will say that M is an A'-module via ψ.

(ii) *If N is an ideal of A, and it acts trivially on M (i.e., ${}^{a}x = x$ and $a * x = 0 = x * a$, for all $x \in M$ and $a \in N$), then M is an A/N-module in $\mathscr{D}$, with the corresponding induced actions.*

Proof. (i) $M \rtimes A'$ is the pullback object in $\mathscr{D}$

$$\begin{array}{ccc} M \rtimes A' & \overset{(s,1)}{\longleftrightarrow} & A' \\ \downarrow & & \downarrow \psi \\ M \rtimes A & \overset{s}{\longleftrightarrow} & A \end{array}$$

(ii) N acts trivially on M if and only if N is an ideal of $M \rtimes A$ by the inclusion $x \mapsto (0, x)$. In this case the semidirect product $M \rtimes (A/N)$, according to the induced actions of A/N on M, is an object of $\mathscr{D}$ since it is isomorphic to $(M \rtimes A)/N$. ∎

Finally, recall that for any object $\psi: B \to A$ in $\mathscr{D}/A$, there is a natural bijection $\mathrm{Hom}_{\mathscr{D}/A}(B \to^{\psi} A,\ M \rtimes A \to A) \cong \mathrm{Der}(B, M)$, where $\mathrm{Der}(B, A)$ denotes the abelian group of *derivations* of B into the B-module, via the morphism ψ, M (i.e., maps $\eta: B \to M$ satisfying

$\eta(x + y) = \eta(x) + {}^{x}\eta(y)$,

$\eta(x * y) = (\eta(x) * y) + (x * \eta(y))$ and $\eta(\omega(x)) = \omega(\eta(x))$, for all $x, y \in B$).

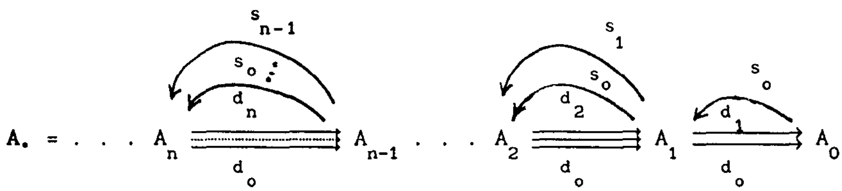
For more details see Orzech [13].

2. THE HOMOTOPY GROUPS ARE MODULES

In this Section we prove that all homotopy groups of a simplicial object $\mathbf{A}.$ in a variety of Ω-groups $\mathscr{D}$ are $\Pi_0(\mathbf{A}.)$-modules in $\mathscr{D}$. This fact will be used to see how Baer invariants are modules, since they can be calculated as the homotopy groups of certain simplicial objects.

Simpl($\mathscr{D}$) will denote the category of simplicial objects in $\mathscr{D}$. The (Moore) homotopy groups of a simplicial object $\mathbf{A.}$ (shown in Scheme 1) are defined, as in the case of simplicial groups, by

$$\Pi_n(\mathbf{A.}) = \frac{\bigcap_{i=0}^{n} \operatorname{Ker}(d_i: A_n \to A_{n-1})}{d_{n+1}(\bigcap_{i=0}^{n} \operatorname{Ker}(d_i: A_{n+1} \to A_n))}, \qquad n \geqslant 0.$$



SCHEME 1

We will denote

$$N_n(\mathbf{A.}) = \bigcap_{i=0}^{n} \operatorname{Ker}(d_i: A_n \to A_{n-1})$$

and

$$I_n(\mathbf{A.}) = d_{n+1}\left(\bigcap_{i=0}^{n} \operatorname{Ker}(d_i: A_{n+1} \to A_n)\right),$$

so $\Pi_n(\mathbf{A.}) = N_n(\mathbf{A.})/I_n(\mathbf{A.})$.

LEMMA 2. *Let* $\mathbf{A.}$ *be a simplicial object in* $\mathscr{D}$. *Then, for all* $n \geqslant 0$, $I_n(\mathbf{A.})$ *is an ideal of* A_n *and therefore* $\Pi_n(\mathbf{A.})$ *is an object in* $\mathscr{D}$.

Moreover, for all $x \in A_n$ *and* $a \in N_n(\mathbf{A.})$, $n \geqslant 1$, *we have*

$$[x + a - x] = [s_0^n d_0^n(x) + a - s_0^n d_0^n(x)],$$

$$[x * a] = [s_0^n d_0^n(x) * a], \qquad \textit{and}$$

$$[a * x] = [a * s_0^n d_0^n(x)],$$

where square brackets are used to denote equivalence classes in $\Pi_n(\mathbf{A.})$.

Proof. For all $x \in A_n$ and $y \in \bigcap_{i=0}^{n} \operatorname{Ker}(d_i: A_{n+1} \to A_n)$, the elements

$$x + d_{n+1}(y) - x = d_{n+1}(s_n(x) + y - s_n(x)),$$

$$x * d_{n+1}(y) = d_{n+1}(s_n(x) * y), \text{ and}$$

$$d_{n+1}(y) * x = d_{n+1}(y * s_n(x))$$

are in $I_n(\mathbf{A.})$. Consequently $I_n(\mathbf{A.})$ is an ideal of A_n.

Now, to prove the identities in the lemma, let us first observe that

$$[x+a-x]=[a] \quad \text{and} \quad [x*a]=0=[a*x],$$

for all $a\in N_n(\mathbf{A}.)$ and $x\in A_n$, with $d_j(x)=0$ for at least one index j in $\{1, 2, ..., n-1\}$.

In fact, let us write $u+v-u-v=[\![u, v]\!]$ (the commutator); the elements

$$z=[\![s_n(x), s_n(a)]\!]-[\![s_{n-1}(x), s_{n-1}(a)]\!]+\cdots$$
$$+(-1)^{n-j-1}[\![s_{j+1}(x), s_{j+1}(a)]\!]+(-1)^{n-j}[\![s_{j+1}(x), s_j(a)]\!]\in A_{n+1}$$

and

$$z'=s_n(x)*s_n(a)-s_{n-1}(x)*s_{n-1}(a)+\cdots$$
$$+(-1)^{n-j-1}s_{j+1}(x)*s_{j+1}(a)+(-1)^{n-j}s_{j+1}(x)*s_j(a)\in A_{n+1}$$

are in $\bigcap_{i=0}^{n}\mathrm{Ker}(d_i\colon A_{n+1}\to A_n)$. Moreover, $d_{n+1}(z)=[\![x, a]\!]$ and $d_{n+1}(z')=x*a$, so $[x+a-x-a]=0=[x*a]$ in $\Pi_n(\mathbf{A}.)$; analogously $[a*x]=0$.

Consequently, if $x, y\in A_n$ are elements such that $d_j(x)=d_j(y)$ for some $0\leqslant j\leqslant n-1$, then $[x+a-x]=[y+a-y]$, $[x*a]=[y*a]$, and $[a*x]=[a*y]$ for all $a\in N_n(\mathbf{A}.)$. So

$$\begin{aligned}[x+a-x]&=[s_{n-1}d_n(x)+a-s_{n-1}d_n(x)]\\&=[s^2_{n-2}\,d^2_{n-2}(x)-s^2_{n-2}\,d^2_{n-2}(x)]\\&=[s^3_{n-3}d^3_{n-3}(x)+a-s^3_{n-3}d^3_{n-3}(x)]\\&=\cdots=[s^n_0d^n_0(x)+a-s^n_0d^n_0(x)]\end{aligned}$$

and analogously

$$[x*a]=[s^n_0d^n_0(x)*a] \quad \text{and} \quad [a*x]=[a*s^n_0d^n_0(x)]. \ \blacksquare$$

The following proposition is already known in particular contexts, for example, groups or associative algebras.

Proposition 3. *Let* $\mathbf{A}.$ *be a simplicial object in* $\mathscr{D}$. *Then* $\Pi_n(\mathbf{A}.)$ *is a* $\Pi_0(\mathbf{A}.)$*-module in* $\mathscr{D}$, *for all* $n\geqslant 1$, *with actions*

$$^{[y]}[x]=[s^n_0(y)+x-s^n_0(y)],$$
$$[y]*[x]=[s^n_0(y)*x] \quad \textit{and} \quad [x]*[y]=[x*s^n_0(y)]$$

for $x\in N_n(\mathbf{A}.)$ *and* $y\in N_0(\mathbf{A}.)=A_0$.

Proof. The identities in Lemma 2 warrant that $\Pi_n(\mathbf{A}.)$ is a singular object in $\mathscr{D}$.

To prove that $\Pi_n(\mathbf{A}.)$ is a $\Pi_0(\mathbf{A}.)$-module in $\mathscr{D}$, let us first observe that it is an A_n-module.

$N_n(\mathbf{A}.)$ is an ideal of A_n, then we have a short exact sequence $N_n(\mathbf{A}.) \hookrightarrow A_n \twoheadrightarrow^{p} A_n/N_n(\mathbf{A}.)$ and by pulling back a long p, we obtain a right split short exact sequence

$$N_n(\mathbf{A}.) \xrightarrow{\ i\ } A_n \times_{A_n/N_n(\mathbf{A}.)} A_n \underset{q}{\overset{\Delta}{\rightleftarrows}} A_n,$$

where $i(a) = (a, 0)$, $q(x, z) = z$ and $\Delta(x) = (x, x)$. Since $I_n(\mathbf{A}.)$ is an ideal of A_n, it is also an ideal of $A_n \times_{A_n/N_n(\mathbf{A}.)} A_n$ via the inclusion $I_n(\mathbf{A}.) \hookrightarrow N_n(\mathbf{A}.) \hookrightarrow^{i} A_n \times_{A_n/N_n(\mathbf{A}.)} A_n$. So we have a diagram of short exact sequences in $\mathscr{D}$

$$\begin{array}{ccccc}
I_n(\mathbf{A}.) & \overline{\overline{\qquad\qquad}} & I_n(\mathbf{A}.) & & \\
\downarrow & & \downarrow\cap & & \\
N_n(\mathbf{A}.) & \xrightarrow{\ i\ } & A_n \times_{A_n/N_n(\mathbf{A}.)} A_n & \underset{q}{\overset{\Delta}{\rightleftarrows}} & A_n \\
\downarrow & & \downarrow & & \| \\
\Pi_n(\mathbf{A}.) & \hookrightarrow & \dfrac{A_n \times_{A_n/N_n(\mathbf{A}.)} A_n}{I_n(\mathbf{A}.)} & \underset{\bar{q}}{\overset{\bar{\Delta}}{\rightleftarrows}} & A_n
\end{array}$$

in which the bottom row is an A_n-module in $\mathscr{D}$. The actions of A_n on $\Pi_n(\mathbf{A}.)$ are

$${}^{x}[a] = [x + a - x], \qquad x * [a] = [x * a], \qquad \text{and} \qquad [a] * x = [a * x].$$

Now, we consider $\Pi_n(\mathbf{A}.)$ as A_0-module in $\mathscr{D}$ via the morphism $s_0^n \colon A_0 \to A_n$. Then, using Proposition 1, since $\Pi_0(\mathbf{A}.)$ is the quotient of A_0 by the ideal $d_1(\mathrm{Ker}(d_0 \colon A_1 \to A_0))$, to see that $\Pi_n(\mathbf{A}.)$ is a $\Pi_0(\mathbf{A}.)$-module with the announced actions, we only have to observe that $d_1(\mathrm{Ker}(d_0 \colon A_1 \to A_0))$ acts trivially on $\Pi_n(\mathbf{A}.)$.

Let $a \in N_n(\mathbf{A}.)$ and $y \in \mathrm{Ker}(d_0 \colon A_1 \to A_0)$. The elements of A_{n+1}

$$z = s_0^n(y) + s_n(a) - s_0^n(y) - s_n(s_0^{n-1}(y) + a - s_0^{n-1}(y))$$

and

$$z' = s_0^n(y) * s_n(a) - s_n(s_0^{n-1}(y) * a)$$

are in $\bigcap_{i=0}^{n} \mathrm{Ker}(d_i \colon A_{n+1} \to A_n)$. Moreover

$$d_{n+1}(z) = (s_0^n d_1(y) + a - s_0^n d_1(y)) - (s_0^{n-1}(y) + a - s_0^{n-1}(y))$$

and

$$d_{n+1}(z') = s_0^n d_1(y) * a - s_0^{n-1}(y) * a.$$

Then

$${}^{d_1(y)}[a] = [s_0^{n-1}(y) + a - s_0^{n-1}(y)] \quad \text{and} \quad d_1(y) * [a] = [s_0^{n-1}(y) * a],$$

but $d_0(y) = 0$, so using Lemma 2 we have

$$[s_0^{n-1}(y) + a - s_0^{n-1}(y)] = [s_0^n d_0(y) + a - s_0^n d_0(y)] = [a]$$

and

$$[s_0^{n-1}(y) * a] = [s_0^n d_0(y) * a] = 0.$$

Therefore ${}^{d_1(y)}[a] = [a]$, $d_1(y) * [a] = 0$, and, analogously, $[a] * d_1(y) = 0$. ∎

3. An Exact Sequence in Cohomology of Simplicial Ω-Groups

The object of this section is to establish an exact sequence of abelian groups

$$0 \longrightarrow H^n(\mathrm{Cosk}^n(\mathbf{A.}), M) \xrightarrow{\mathbf{d}^*} H^n(\mathbf{A.}, M) \xrightarrow{\chi} \mathrm{Hom}_{\pi_0(\mathbf{A.})}(\Pi_n(\mathbf{A.}), M) \swarrow$$
$$H^{n+1}(\mathrm{Cosk}^n(\mathbf{A.}), M) \xrightarrow{\mathbf{d}^*} H^{n+1}(\mathbf{A.}, M),$$

$n \geqslant 1$, for any simplicial object $\mathbf{A.}$ and any $\Pi_0(\mathbf{A.})$-module M in a variety of Ω-groups $\mathscr{D}$. This sequence will be used to deduce the exact sequence (S_n) (which was announced in the Introduction). We will also identify the corresponding sequence for $n = 1$ with that obtained by Leedham–Green and Mackay in [8].

We will start by recalling some aspects of simplicial objects and their cohomology.

Let $\mathbf{A.}$ be a simplicial object in $\mathscr{D}$, $A = \Pi_0(\mathbf{A.})$, and M an A-module in $\mathscr{D}$; the nth *cohomology group* of $\mathbf{A.}$ with coefficients in M. $H^n(\mathbf{A.}, M)$, is defined as the nth cohomology of the cochain complex of abelian groups

$$\mathrm{Der}(\mathbf{A.}, M) = \cdots \mathrm{Der}(A_{n-1}, M) \xrightarrow{\delta^{n-1}} \mathrm{Der}(A_n, M)$$
$$\xrightarrow{\delta^n} \mathrm{Der}(A_{n+1}, M) \to \cdots,$$

where $\delta^n = \sum_{i=0}^{n} (-1)^i d_i^*$ and M is an A_n-module via the canonical morphism $A_n \to A$.

The elements of the group $Z^n(\mathbf{A}., M) = \text{Ker}(\delta^n)$ are called *n-cocycles* and those in $B^n(\mathbf{A}., M) = \text{Im}(\delta^{n-1})$ *n-coboundaries*. Thus $H^n(\mathbf{A}., M) = Z^n(\mathbf{A}., M)/B^n(\mathbf{A}., M)$, $n \geqslant 0$. Clearly $H^n(\mathbf{A}., -)$ is a functor from the category of $\Pi_0(\mathbf{A}.)$-modules in $\mathscr{D}$ into the category of abelian groups; moreover, a simplicial morphism $\mathbf{f}.: \mathbf{A}. \to \mathbf{B}.$ induces a natural group homomorphisms $H^n(\mathbf{f}., M): H^n(\mathbf{B}., M) \to H^n(\mathbf{A}., M)$, for any $\Pi_0(\mathbf{B}.)$-module M, where M is also considered as a $\Pi_0(\mathbf{A}.)$-module via the morphism $\Pi_0(\mathbf{f}.) : \Pi_0(\mathbf{A}.) \to \Pi_0(\mathbf{B}.)$, in such a way that

$$H^n(\mathbf{f}., -): H^n(\mathbf{B}., -) \to H^n(\mathbf{A}., -)$$

is a natural transformation between functors from the category of $\Pi_0(\mathbf{B}.)$-modules to abelian groups. Let us note also that if $\mathbf{f}.$ and $\mathbf{g}.: \mathbf{A}. \to \mathbf{B}.$ are homotopic simplicial morphisms then the homomorphisms $H^n(\mathbf{f}., M)$ and $H^n(\mathbf{g}., M)$ are the same, and in particular if $\mathbf{A}.$ and $\mathbf{B}.$ are homotopically equivalents then $H^n(\mathbf{A}., M) \cong H^n(\mathbf{B}., M)$.

Let us now recall that an n-truncated simplicial object consists of objects $A_0, A_1, ..., A_n$ and the usual face and degeneracy morphisms between them. If we designate by $\text{Tr}^n\,\text{Simpl}(\mathscr{D})$ the category of n-truncated simplicial objects of $\mathscr{D}$, the functor "truncation at level n"

$$\text{tr}^n: \text{Simpl}(\mathscr{D}) \to \text{Tr}^n\,\text{Simpl}(\mathscr{D}),$$

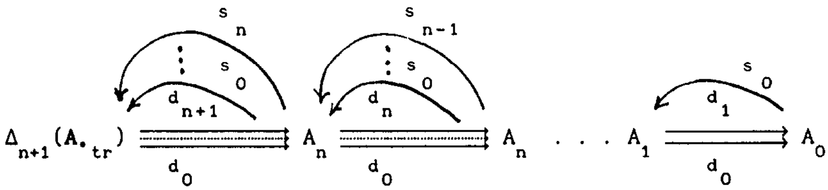
admits a right adjoint cosk^n, called the n-coskeleton functor. A construction of the n-coskeleton of a truncated complex can be done by using simplicial kernels, as follows:

Given an n-truncated simplicial object $\mathbf{A}._{\text{tr}}$, $n \geqslant 1$, the $n+1$-*simplicial kernel* of $\mathbf{A}._{\text{tr}}$ is an object, denoted by $\Delta_{n+1}(\mathbf{A}._{\text{tr}})$, together with morphisms $d_i: \Delta_{n+1}(\mathbf{A}._{\text{tr}}) \to A_n$, $0 \leqslant i \leqslant n+1$, which is universal with respect to the property $d_i d_j = d_{j-1} d_i$ for all $0 \leqslant i < j \leqslant n+1$. The object $\Delta_{n+1}(\mathbf{A}._{\text{tr}})$ is, up to isomorphism, the subobject of the product $A_n \times A_n \times \overset{n+2}{\cdots} \times A_n$ whose elements are those $(n+2)$-tuples $(x_0, x_1, ..., x_{n+1})$ satisfying $d_i(x_j) = d_{j-1}(x_i)$, $0 \leqslant i < j \leqslant n+1$. And the morphisms $d_i: \Delta_{n+1}(\mathbf{A}._{\text{tr}}) \to A_n$ are the projections.

Let us note that there are degeneracy morphisms $s_j: A_n \to \Delta_{n+1}(\mathbf{A}._{\text{tr}})$, $0 \leqslant j \leqslant n$, determined by the identities

$$d_i s_j = s_{j-1} d_i \qquad \text{if} \quad i < j,$$
$$d_j s_j = d_{j+1} s_j = 1_{A_n}, \qquad \text{and}$$
$$d_i s_j = d_j s_{i-1} \qquad \text{if} \quad i > j+1,$$

in such a way that Scheme 2 is an $(n+1)$-truncated complex. Then, $\text{cosk}^n: \text{Tr}^n\,\text{Simpl}(\mathscr{D}) \to \text{Simpl}(\mathscr{D})$ is obtained by iterating simplicial kernel construction.



SCHEME 2

Given $\mathbf{A.}$ a simplicial object, its n-simplicial kernel is defined by $\Delta_n(\mathbf{A.}) = \Delta_n \operatorname{tr}^{n-1}(\mathbf{A.})$ if $n > 1$ and $\Delta_1(\mathbf{A.}) = A_0 x_{\Pi_0(\mathbf{A.})} A_0$, i.e., the kernel pair of the canonical epimorphism $A_0 \twoheadrightarrow \Pi_0(\mathbf{A.})$. The unity of the adjunction $\mathbb{d}.: \mathbf{A.} \to \operatorname{cosk}^n \operatorname{tr}^n(\mathbf{A.})$ consists of identities at dimensions $\leqslant n$, and the canonical morphism

$$\mathbb{d}_{n+1}: A_{n+1} \to \Delta_{n+1}(\mathbf{A.}); \qquad \mathbb{d}_{n+1}(x) = (d_0(x), d_1(x), ..., d_{n+1}(x))$$

at dimension $n+1$.

The endofunctor $\operatorname{Cosk}^n = \operatorname{cosk}^n \operatorname{tr}^n: \operatorname{Simpl}(\mathscr{D}) \to \operatorname{Simpl}(\mathscr{D})$, $n \geqslant 1$, is also called the *n-coskeleton functor.*

A simplicial object $\mathbf{A.}$ is called *aspherical* if the morphisms $\mathbb{d}_n: A_n \to \Delta_n(\mathbf{A.})$ are epimorphisms for all $n \geqslant 2$. The O-coskeleton functor $\operatorname{Cosk}^0: \operatorname{Simpl}(\mathscr{D}) \to \operatorname{Simpl}(\mathscr{D})$ is defined by

$$\operatorname{Cosk}^0(\mathbf{A.}) = \operatorname{cosk}^1(\Delta_1(\mathbf{A.}) \rightrightarrows A_0).$$

The above facts are what we will need to develop this section. More details about the above constructions can be found in [3].

The following proposition characterizes the homotopy groups $\Pi_n(\mathbf{A.})$ of a complex $\mathbf{A.}$ as $\Pi_0(\mathbf{A.})$-modules, by a universal property.

PROPOSITION 4. *Let* $\mathbf{A.}$ *be a simplicial object in* $\mathscr{D}$ *and* $n \geqslant 1$. *Then there exists a natural derivation* $\xi: \Delta_{n+1}(\mathbf{A.}) \to \Pi_n(\mathbf{A.})$, *such that the composition* $A_{n+1} \to^{\mathbb{d}_{n+1}} \Delta_{n+1}(\mathbf{A.}) \to^{\xi} \Pi_n(\mathbf{A.})$ *is zero, where* $\Pi_n(\mathbf{A.})$ *is considered as a* $\Delta_{n+1}(\mathbf{A.})$*-module via the canonical morphism* $\Delta_{n+1}(\mathbf{A.}) \to \Pi_0(\mathbf{A.})$.

Moreover, for any $\Pi_0(\mathbf{A.})$*-module* M *and any derivation* $\alpha: \Delta_{n+1}(\mathbf{A.}) \to M$ *with* $\alpha \mathbb{d}_{n+1} = 0$, *there exists a unique* $\Pi_0(\mathbf{A.})$*-module morphism,* f, *which makes the diagram*

$$\begin{array}{ccc} \Delta_{n+1}(\mathbf{A.}) & \xrightarrow{\ \xi\ } & \Pi_n(\mathbf{A.}) \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle f} & \\ M & & \end{array}$$

commutative. In other words, for any $\Pi_0(\mathbf{A.})$-module M, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\Pi_0(\mathbf{A.})}(\Pi_n(\mathbf{A.}), M) \xrightarrow{\xi^*} \mathrm{Der}(\Delta_{n+1}(\mathbf{A.}), M) \xrightarrow{d^*_{n+1}} \mathrm{Der}(A_{n+1}, M) \qquad \text{(I)}$$

is exact and natural (in **A.** *and M).*

Proof. Let us denote $\Theta_i: \Delta_{n+1}(\mathbf{A.}) \to \Delta_{n+1}(\mathbf{A.})$ as the map given by

$$\Theta_i(x_0, x_1, ..., x_{n+1}) = (x_0, x_1, ..., x_{n+1}) - s_i(x_i), \qquad 0 \leqslant i \leqslant n.$$

Observe, using the simplicial identities, that the element

$$\Theta_n \Theta_{n-1} \cdots \Theta_0(x_0, x_1, ..., x_{n+1})$$

has all its components, up to the last one, equal to zero. Then we write

$$\Theta_n \Theta_{n-1} \cdots \Theta_0(x_0, x_1, ..., x_{n+1}) = (0, 0, ..., 0, \Theta(x_0, x_1, ..., x_{n+1}))$$

for $\Theta(x_0, x_1, ..., x_{n+1})$ an element in $N_n(\mathbf{A.})$ and we define

$$\xi(x_0, x_1, ..., x_{n+1}) = [\Theta(x_0, x_1, ..., x_{n+1})],$$

where square bracket denotes equivalence class in $\Pi_n(\mathbf{A.})$.

If we consider $z = (x_0, x_1, ..., x_{n+1})$ any element in $\Delta_{n+1}(\mathbf{A.})$, the following facts have a straightforward proof:

(a) $z - \Theta_i(z) = s_i(x_i) = d_{n+1} s_i(x_i) \in d_{n+1}(A_{n+1})$ and therefore $z - (0, ..., 0, \Theta(z)) \in d_{n+1}(A_{n+1})$.

(b) If $a \in N_n(\mathbf{A.})$ is an element such that $z - (0, ..., 0, a) \in d_{n+1}(A_{n+1})$, then $\xi(z) = [a]$.

(c) For $0 \leqslant i \leqslant n$, $\xi \Theta_i = \xi$.

(d) For any $x \in A_n$ and $0 \leqslant i < n$,

$$\xi(d_{n+1} s_i(x) + z - d_{n+1} s_i(x)) = [s_i d_n(x) + \Theta(z) - s_i d_n(x)] = {}^{[d_0^n(x)]}\xi(z).$$

(e) For $1 \leqslant i \leqslant n+1$, consider B_i the ideal of $\Delta_{n+1}(\mathbf{A.})$ whose elements are in the form $(0, ..., 0, y_i, ..., y_{n+1})$. Then $\Theta_0(z) \in B_1$ and if $z \in B_i$, also $\Theta_i(z) \in B_{i+1}$.

To prove that ξ is a derivation, let us first see that the restriction of ξ to B_i are morphisms in $\mathscr{D}$, for all $1 \leqslant i \leqslant n+1$.

For $i = n+1$, the restriction ξ/B_{n+1} is just the composition of the two canonical projections $B_{n+1} \to N_n(\mathbf{A.})$ and $N_n(\mathbf{A.}) \to \Pi_n(\mathbf{A.})$, and therefore ξ/B_{n+1} is clearly a morphism.

Suppose now that ξ/B_i is a morphism. Then, for any elements $z=(x_0, x_1, ..., x_{n+1})$ and $z'=(x'_0, x'_1, ..., x'_{n+1})$ in B_{i-1}, we have

$$\begin{aligned}\xi(z+z') &= \xi(\Theta_{i-1}(z+z')) \\ &= \xi(\Theta_{i-1}(z)+\mathbb{d}_{n+1}s_{i-1}(x_i)+\Theta_{i-1}(z')-\mathbb{d}_{n+1}s_{i-1}(x_i)) \\ &= \xi(\Theta_{i-1}(z))+\xi(\mathbb{d}_{n+1}s_{i-1}(x_i)+\Theta_{i-1}(z')-\mathbb{d}_{n+1}s_{i-1}(x_i)) \\ &\qquad\qquad \text{(since } \xi/B_i \text{ is a morphism)} \\ &= \xi(\Theta_{i-1}(z))+\xi(\Theta_{i-1}(z'))=\xi(z)+\xi(z') \\ &\qquad\qquad \text{(using (d), since } d_0^n(x_i)=0).\end{aligned}$$

Now, let z and z' be two arbitrary elements in $\Delta_{n+1}(\mathbf{A}.)$. Then

$$\begin{aligned}\xi(z+z') &= \xi(\Theta_0(z+z'))=\xi(\Theta_0(z)+\mathbb{d}_{n+1}s_0(x_0)+\Theta_0(z')-\mathbb{d}_{n+1}s_0(x_0)) \\ &= \xi(z)+{}^{[d_0^n(x_0)]}\xi(z'),\end{aligned}$$

analogously $\xi(z*z')=\xi(z)*z'+z*\xi(z')$ and $\xi(\omega(z))=\omega(\xi(z))$, therefore ξ is a derivation, which clearly verifies $\xi\mathbb{d}_{n+1}=0$.

Finally, given $\alpha\colon \Delta_{n+1}(\mathbf{A}.)\to M$ a derivation with $\alpha\mathbb{d}_{n+1}=0$, the unique morphism $f\colon \Pi_n(\mathbf{A}.)\to M$ such that $f\xi=\alpha$ is given by

$$f([a])=\alpha(0, ..., 0, a), \qquad \text{for all} \quad [a]\in \Pi_n(\mathbf{A}.). \quad ▮$$

As an immediate consequence of the above Proposition 4, a simplicial object is aspherical if and only if its homotopy groups at dimensions $\geqslant 1$ are all trivial.

PROPOSITION 5. *Let* $\mathbf{A}.$ *be a simplicial object in* $\mathscr{D}$, $n\geqslant 1$, *and* $\xi\colon \Delta_{n+1}(\mathbf{A}.)\to \Pi_n(\mathbf{A}.)$ *the derivation introduced in Proposition* 4. *Then* ξ *is an* $(n+1)$*-cocycle in* $Z^{n+1}(\mathrm{Cosk}^n(\mathbf{A}.), \Pi_n(\mathbf{A}.))$.

Moreover, the exact sequence (I) *in proposition* 4 *induces an exact sequence of abelian groups*

$$\mathrm{Hom}_{\Pi_0(\mathbf{A}.)}(\Pi_n(\mathbf{A}.), M)\xrightarrow{\xi^*} H^{n+1}(\mathrm{Cosk}^n(\mathbf{A}.), M)\xrightarrow{\mathbb{d}^*} H^{n+1}(\mathbf{A}., M) \qquad \text{(II)}$$

for all $\Pi_0(\mathbf{A}.)$*-modules* M.

Proof. Let us first prove that ξ is a cocycle, or equivalently that for all $(z_0, z_1, ..., z_{n+2})\in \mathrm{Cosk}^n(\mathbf{A}.)_{n+2}$ the alternating sum $\sum_{i=0}^{n+2}(-1)^i\,\xi(z_i)$ is zero.

Since $\mathrm{Cosk}^n(\mathbf{A}.)_{n+1}=\Delta_{n+1}(\mathbf{A}.)$, the elements z_i are in $\Delta_{n+1}(\mathbf{A}.)$. Then we can suppose $z_i=(x_{i0}, x_{i1}, ..., x_{in+1})$, with $x_{ij}\in A_n$, such that $x_{ij}=x_{ji-1}$, $0\leqslant j<i\leqslant n+1$. Now, any simplicial object in $\mathscr{D}$ satisfies Kan's condition [7] and therefore for any z_i, $0\leqslant i\leqslant n+1$, there exists $w_i\in A_{n+1}$ such that

$d_j(w_i)=x_{ij}$, $0\leqslant j\leqslant n$, and also there exists $w\in A_{n+2}$ such that $d_i(w)=w_i$. Consequently the element $d_{n+2}(w)\in A_{n+1}$ has $d_i d_{n+2}(w)=d_{n+1}(w_i)$, and the elements $a_i=d_{n+1}(w_i)-x_{in+1}$ are in $N_n(\mathbf{A}.)$ for all $0\leqslant i\leqslant n+1$.

Using now condition (b) (in the proof of Proposition 4) we have

$$\xi(z_i)=[a_i], \qquad 0\leqslant i\leqslant n+1,$$

and

$$\begin{aligned} d_{n+2}(w) &= (d_{n+1}(w_0), d_{n+1}(w_1), ..., d_{n+1}(w_{n+1})) \\ &= (a_0+x_{0n+1}, a_1+x_{1n+1}, ..., a_{n+1}+x_{n+1n+1}) \\ &= (a_0+x_{n+20}, a_1+x_{n+21}, ..., a_{n+1}+x_{n+2n+1}) \\ &= (a_0, a_1, ..., a_{n+1})+z_{n+2}. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \xi(\mathbb{d}_{n+1} d_{n+2}(w))=\xi((a_0, a_1, ..., a_{n+1})+z_{n+2}) \\ &= \xi(a_0, a_1, ..., a_{n+1})+\xi(z_{n+2}) \qquad (\text{since } d_0^n(a_i)=0) \\ &= \sum_{i=0}^{n+2} (-1)^i \xi(z_i) \qquad (\text{since } a_i\in N_{n+1}(\mathbf{A}.)), \end{aligned}$$

and so ξ is a cocycle.

Consider now the group homomorphism

$$\xi^*: \mathrm{Hom}_{\Pi_0(\mathbf{A}.)}(\Pi_n(\mathbf{A}.), M)\to \mathrm{Der}(\Delta_{n+1}(\mathbf{A}.), M);$$

since ξ is a cocycle, ξ^* takes any homomorphism to a cocycle and so it induces a group homomorphism

$$\xi^*: \mathrm{Hom}_{\Pi_0(\mathbf{A}.)}(\Pi_n(\mathbf{A}.), M)\to H^{n+1}(\mathrm{Cosk}^n(\mathbf{A}.), M).$$

Then we have the sequence

$$\mathrm{Hom}_{\Pi_0(\mathbf{A}.)}(\Pi_n(\mathbf{A}.), M)\xrightarrow{\xi^*} H^{n+1}(\mathrm{Cosk}^n(\mathbf{A}.), M)\xrightarrow{\mathbb{d}^*} H^{n+1}(\mathbf{A}., M)$$

in which clearly $\mathbb{d}^*\xi^*=0$, since $\xi\mathbb{d}_{n+1}=0$.

Moreover, if a cocycle $\alpha: \Delta_{n+1}(\mathbf{A}.)\to M$ represents an element in $H^{n+1}(\mathrm{Cosk}^n(\mathbf{A}.), M)$ which is taken to zero by $\mathbb{d}^*$, then there exists a derivation $\beta: A_n\to M$ such that

$$\alpha\mathbb{d}_{n+1}=\sum_{i=0}^{n+1} (-1)^i \beta d_i.$$

But, since $\mathrm{Cosk}^n(\mathbf{A}.)_n=A_n$ and $d_i=d_i\mathbb{d}_{n+1}$ (where d_i's denote the face operators at dimension $n+1$ of $\mathbf{A.}$ and $\operatorname{Cosk}^n(\mathbf{A.})$, respectively), $\alpha - \sum_{i=0}^{n+1} (-1)^i \beta d_i$ is a cocycle in $Z^{n+1}(\operatorname{Cosk}^n(\mathbf{A.}), M)$. This cocycle represents the same element in $H^{n+1}(\operatorname{Cosk}^n(\mathbf{A.}), M)$ that α does, and it is taken by $\mathbb{d}_{n+1}^*$ to the zero derivation in $\operatorname{Der}(A_{n+1}, M)$. So, by the exactness of sequence (I), there exists a $\Pi_0(\mathbf{A.})$-module homomorphism $f: \Pi_n(\mathbf{A.}) \to M$ such that $\xi^*(f) = \alpha - \sum_{i=0}^{n+1} (-1)^i \beta d_i$ and therefore α represents an element in the image of ξ^*. ∎

Finally we obtain the announced five term exact sequence.

Proposition 6. *Let $\mathbf{A.}$ be a simplicial object, $A = \Pi_0(\mathbf{A.})$, and M an A-module in $\mathcal{D}$. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow H^n(\operatorname{Cosk}^n(\mathbf{A.}), M) \xrightarrow{\mathbb{d}^*} H^n(\mathbf{A.}, M) \xrightarrow{\chi} \operatorname{Hom}_A(\Pi_n(\mathbf{A.}), M)$$
$$\swarrow \xi^*$$
$$H^{n+1}(\operatorname{Cosk}^n(\mathbf{A.}), M) \xrightarrow{\mathbb{d}^*} H^{n+1}(\mathbf{A.}, M),$$

for all $n \geqslant 1$.

Proof. The connecting morphism χ is defined as follows:
Given a cocycle $\alpha: A_n \to M$ in $Z^n(\mathbf{A.}, M)$, consider the derivation

$$\beta = \sum_{i=0}^{n+1} (-1)^i \alpha d_i: \Delta_{n+1}(\mathbf{A.}) \to M.$$

Then $\beta \mathbb{d}_{n+1} = 0$ and therefore, using Proposition 4, there exists a unique morphism of $\Pi_0(\mathbf{A.})$-modules $f: \Pi_n(\mathbf{A.}) \to M$ such that $f\xi = \beta$. Now it is clear that β depends only of the class of α in $H^n(\mathbf{A.}, M)$ and so we can define the image, by χ, of such a class as f, i.e., $\chi[\alpha] = f$.

After Proposition 5, we only need to prove the exactness of the sequence in the three first points:

— $\mathbb{d}^*$ is indeed an inclusion since $\operatorname{tr}^n(\mathbb{d}.) = 1_{\operatorname{tr}^n(\mathbf{A.})}$.

— Given $[\alpha] \in H^n(\mathbf{A.}, M)$, $\chi[\alpha]\xi = 0$ if and only if $\sum_{i=0}^{n+1} (-1)^i \alpha d_i = 0$ and this is just the condition of $[\alpha]$ being an element in $H^n(\operatorname{Cosk}^n(\mathbf{A.}), M)$. Therefore we have exactness in $H^n(\mathbf{A.}, M)$.

— For any $\alpha \in Z^n(\mathbf{A.}, M)$ we have $\xi^*\chi[\alpha] = [\chi[\alpha]\xi] = [\sum_{i=0}^{n+1} (-1)^i \alpha d_i] = 0$. Conversely, if $f: \Pi_n(\mathbf{A.}) \to M$ is a morphism of A-modules such that $\xi^*(f) = [f\xi] = 0$, there exists a derivation $\alpha: A_n \to M$ with $f\xi = \sum_{i=0}^{n+1} (-1)^i \alpha d_i$. Then $\alpha \in Z^n(\mathbf{A.}, M)$ since $\xi \mathbb{d}_{n+1} = 0$ and clearly $\chi[\alpha] = f$. So the sequence is exact in $\operatorname{Hom}_A(\Pi_n(\mathbf{A.}), M)$. ∎

4. Cohomology and Baer Invariants

Here we obtain the exact sequences (S_n) as a consequence of the exact sequences in Proposition 6.

Given an object A in $\mathscr{D}$, a simplicial object $\mathbf{F}.$ in $\mathscr{D}$ is said to be a *resolution* of A if $\mathbf{F}.$ is aspherical and $\Pi_0(\mathbf{F}.) = A$.

A resolution $\mathbf{F}.$ of A is said *free* if all F_i are free and the free generators are stable under the degeneracy operators; i.e., if $X_i \subseteq F_i$ are the sets of free generators, then $s_j(X_i) \subseteq X_{i+1}$ for all $i \geqslant 0$ and $0 \leqslant j \leqslant i$.

Note that, for any object A, the cotriple resolution of A, $\mathbb{G}.(A)$, is always a free resolution. And also that any truncated free resolution of A can be extended, step by step, to a free resolution.

The comparison theorem asserts that given $\mathbf{F}.$, a free resolution of A, $\mathbf{B}.$, an aspherical simplicial object, and $f\colon A \to \Pi_0(\mathbf{B}.)$, a morphism, there exists a simplicial morphism $\mathbf{f}.\colon \mathbf{F}. \to \mathbf{B}.$ (lifting of f), unique up to homotopy, such that $\Pi_0(\mathbf{f}.) = f$. This theorem allows us to define the cohomology groups of an object A in $\mathscr{D}$, with coefficients in an A-module M, as the cohomology of any free resolution $\mathbf{F}.$ of A

$$\mathscr{D}^n(A, M) = H^n(\mathbf{F}., M), \qquad n \geqslant 0.$$

General properties of this cohomology as well as the fact that it particularizes to those well known, such as Eilenberg–Mac Lane cohomology for groups, Shukla cohomology for associative algebras or André–Quillen cohomology for commutative algebras, can be found in [1, 2, 13, 14].

Let $T\colon \mathscr{D} \to \mathscr{D}'$ be an arbitrary functor between varieties of Ω-groups. The nth *left derived functor* of T, $L_n T\colon \mathscr{D} \to \mathscr{D}'$, $n \geqslant 0$, is defined by

$$L_n T(A) = \Pi_n(T(\mathbf{F}.)) \qquad \text{and} \qquad L_n T(f) = \Pi_n(T(\mathbf{f}.)),$$

for any object A and any morphism $f\colon A \to B$ in $\mathscr{D}$, where $\mathbf{F}.$ is a free resolution of A and $\mathbf{f}.$ is a lifting of f, from a free resolution of A to a free resolution of B.

Note that if T preserves colimits, then $L_0 T \cong T$. Moreover, by Proposition 3, $L_n T(A)$ is in a canonical way a $T(A)$-module in $\mathscr{D}'$. (For general properties of these functors, see [7, 12]).

Let us now consider $\mathscr{B}$ a variety of $\mathscr{D}$, with associated quotient functor U. The nth *Baer invariant of* an object A in $\mathscr{D}$ *with respect to the variety* $\mathscr{B}$, denoted by $\mathscr{B}_n(A)$, $n \geqslant 1$, is defined to be the value of the nth left derived of U in A,

$$\mathscr{B}_n(A) = L_n U(A), \qquad n \geqslant 1.$$

Since the functor U preserves colimits, the Baer invariants $\mathscr{B}_n(A)$ are, in a natural way, $U(A)$-modules in $\mathscr{B}$ and therefore A-modules in $\mathscr{D}$, via the canonical projection $A \twoheadrightarrow U(A)$. This last fact has been observed individually for $n=1$ in many particular contexts, for example in varieties of groups or associative algebras ([8, 9, 11]), where $\mathscr{B}_1(A)$ is isomorphic to the first Baer invariant in the sense of Fröhlich [4]. Our method gives nevertheless a unified treatment to the problem of making $\mathscr{B}_n(A)$ an A-module for all $n \geqslant 1$.

As a direct consequence of Proposition 6 we have

PROPOSITION 7. *Let $\mathscr{B}$ be a variety of $\mathscr{D}$, A an object in $\mathscr{D}$, and M a $U(A)$-module in $\mathscr{B}$. Then for any free resolution* $\mathbf{F}.$ *of A in $\mathscr{D}$ and any $n \geqslant 1$ there exists a natural exact sequence of abelian groups*

$$0 \to H^n(\mathrm{Cosk}^n(U(\mathbf{F}.)), M) \to \mathscr{D}^n(A, M) \to \mathrm{Hom}_A(\mathscr{B}_n(A), M) \to H^{n+1}(\mathrm{Cosk}^n(U(\mathbf{F}.)), M) \to \mathscr{D}^{n+1}(A, M).$$

where M is considered as an A-module in $\mathscr{D}$ via the canonical projection $A \twoheadrightarrow U(A)$.

Proof. Consider the simplicial object $U(\mathbf{F}.)$ in $\mathscr{B}$ and the $\Pi_0(U(\mathbf{F}.)) = U(A)$ module M. Since $\Pi_n(U(\mathbf{F}.)) = \mathscr{B}_n(A)$, by Proposition 6, there exists a natural exact sequence of abelian groups

$$0 \to H^n(\mathrm{Cosk}(U(\mathbf{F}.)), M) \to H^n(U(\mathbf{F}.), M) \to \mathrm{Hom}_A(\mathscr{B}_n(A), M) \to H^{n+1}(\mathrm{Cosk}^n(U(\mathbf{F}.)), M) \to H^{n+1}(U(\mathbf{F}.), M)$$

for any $n \geqslant 1$. But using the adjunction $\mathscr{B} \leftrightarrows^{U}_{\text{inclusion}} \mathscr{D}$, $U \dashv$ inclusion, we obtain an isomorphism of cocomplexes $\mathrm{Der}(U(\mathbf{F}.), M) \cong \mathrm{Der}(\mathbf{F}., M)$ and therefore isomorphisms of abelian groups

$$H^n(U(\mathbf{F}.), M) \cong H^n(\mathbf{F}., M) = \mathscr{D}^n(A, M), \qquad n \geqslant 1. \quad \blacksquare$$

The following lemma will allow us to identify the first and fourth terms in the sequence of Proposition 7, in the special cases of $n=1$ or $\mathscr{B}_r(A) = 0$ for $1 \leqslant r \leqslant n-1$, with the cohomology groups in the subvariety $\mathscr{B}$, $\mathscr{B}^n(U(A), M)$ and $\mathscr{B}^{n+1}(U(A), M)$, respectively.

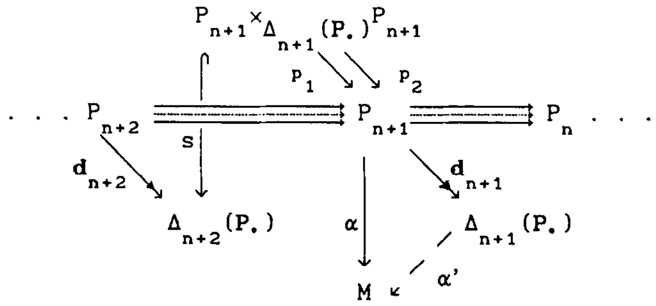
LEMMA 8. *Let* $\mathbf{P}.$ *be a $\mathscr{B}$-free resolution of an object B in $\mathscr{B}$ and M a B-module in $\mathscr{B}$. Then for any $n \geqslant 0$, the morphism* $\mathbb{d}.: \mathbf{P}. \to \mathrm{Cosk}^n(\mathbf{P}.)$ *induces isomorphisms*

$$\mathscr{B}^i(A, M) \cong H^i(\mathrm{Cosk}^n(\mathbf{P}.), M),$$

for all $0 \leqslant i \leqslant n+1$ and $n \geqslant 0$.

Proof. Since $\mathbb{d}.$ consists of the identity morphisms at dimensions $\leqslant n$ and $\mathbb{d}_{n+1}: P_{n+1} \to \Delta_{n+1}(\mathbf{P}.)$ is an epimorphism, it is clear that $\mathscr{B}^i(A, M) \cong H^i(\mathrm{Cosk}^n(\mathbf{P}.), M)$ for all $0 \leqslant i \leqslant n$. Let us see that $\mathbb{d}.$ induces also an isomorphism for $i = n+1$.

Let $\alpha: P_{n+1} \to M$ be a cocycle in $Z^{n+1}(\mathbf{P}., M)$ and consider Scheme 3, where the morphism $s: P_{n+1} \times_{\Delta_{n+1}(\mathbf{P}.)} P_{n+1} \to \Delta_{n+2}(\mathbf{P}.)$ is defined by $s(x, y) = (0,..., 0, x, y)$.



SCHEME 3

Then α coequalizes (p_1, p_2). In fact, let $(x, y) \in P_{n+1} \times_{\Delta_{n+1}(\mathbf{P}.)} P_{n+1}$ and consider $s(x, y) \in \Delta_{n+2}(\mathbf{P}.)$. Since $\mathbf{P}.$ is aspherical, there exists $z \in P_{n+2}$ such that $\mathbb{d}_{n+2}(z) = s(x, y)$. But α is a cocycle and therefore

$$0 = \sum_{i=0}^{n+2} (-1)^i \alpha d_i(z) = \alpha(x) - \alpha(y).$$

Consequently, since $\mathbb{d}_{n+1}: P_{n+1} \twoheadrightarrow \Delta_{n+1}(\mathbf{P}.)$ is an epimorphism, there exists $\alpha': \Delta_{n+1}(\mathbf{P}.) \to M$ such that $\alpha' \mathbb{d}_{n+1} = \alpha$. Moreover, since α is a cocycle and $\mathbb{d}.: \mathbf{P}. \to \mathrm{Cosk}^n(\mathbf{P}.)$ is epic, we have that α' is a cocycle in $Z^{n+1}(\mathrm{Cosk}^n(\mathbf{P}.), M)$.

This construction defines a group homomorphism from $H^{n+1}(\mathbf{P}., M) = \mathscr{B}^{n+1}(A, M)$ to $H^{n+1}(\mathrm{Cosk}^n(\mathbf{P}.), M)$, which is an inverse of $\mathbb{d}^*$: $H^{n+1}(\mathrm{Cosk}^n(\mathbf{P}.), M) \to H^{n+1}(\mathbf{P}., M)$. ∎

We can establish now the main result in this paper:

THEOREM 9. *Let $\mathscr{B}$ be a variety of $\mathscr{D}$, A an object in $\mathscr{D}$, and M a $U(A)$-module in $\mathscr{B}$. Then there exists an exact sequence of abelian groups*

$$0 \to \mathscr{B}^1(U(A), M) \to \mathscr{D}^1(A, M) \to \mathrm{Hom}_A(\mathscr{B}_1(A), M)$$
$$\to \mathscr{B}^2(U(A), M) \to \mathscr{D}^2(A, M). \qquad (\mathrm{S}_1)$$

Moreover, if $\mathscr{B}_i(A)=0$ for all $1\leqslant i\leqslant n-1$, there also exists an exact sequence of abelian groups

$$0\to\mathscr{B}^n(U(A),M)\to\mathscr{D}^n(A,M)\to\mathrm{Hom}_A(\mathscr{B}_n(A),M)$$
$$\to\mathscr{B}^{n+1}(U(A),M)\to\mathscr{D}^{n+1}(A,M). \qquad (\mathrm{S}_n)$$

Proof. Let $\mathbf{F}.$ be a free resolution of A in $\mathscr{D}$. Then $U(F_1)\rightrightarrows U(F_o)$ is always a free truncated resolution of $U(A)$ in $\mathscr{B}$. Let $\mathbf{G}.$ be an extension of $U(F_1)\rightrightarrows U(F_o)$ to a free resolution of $U(A)$ in $\mathscr{B}$.

Using Lemma 8, we have isomorphisms for $i=1$ and 2,

$$\mathscr{B}^i(U(A),M)=H^i(\mathbf{G}.,M)\cong H^i(\mathrm{Cosk}^1(\mathbf{G}.),M)=H^i(\mathrm{Cosk}^1(U(\mathbf{F}.)),M).$$

Then the sequence S_1 is just that in Proposition 7 for $n=1$.

Let us note now that, for $n>1$, the n-truncated simplicial object $\mathrm{tr}^n(U(\mathbf{F}.))$ is a truncated free resolution of $U(A)$ in $\mathscr{B}$ if and only if $\Pi_i(U(\mathbf{F}.))=\mathscr{B}_i(A)$ is zero for all $1\leqslant i\leqslant n-1$. An in this case, the process used to obtain (S_1) can be applied to obtain the sequence (S_n). ∎

As an immediate consequence we have

COROLLARY 10. *Let $\mathscr{B}$ be a variety of $\mathscr{D}$ and A an object in $\mathscr{D}$. Then, for $r\geqslant 1$, the following conditions are equivalent:*

(i) *$\mathscr{B}_i(A)=0$ for all $1\leqslant i\leqslant r$.*

(ii) *$\mathscr{B}^i(U(A),M)\cong\mathscr{D}^i(A,M)$ for all $1\leqslant i\leqslant r$, and the morphism*

$$\mathscr{B}^{r+1}(U(A),M)\to\mathscr{D}^{r+1}(A,M)$$

is a monomorphism, for any $U(A)$-module M in $\mathscr{B}$.

Let us finally observe that when the object A is $\mathscr{B}$-perfect, $\mathscr{B}$-splitting, or, in general, it verifies $\mathscr{B}^n(U(A),-)=0$, $n\geqslant 2$, then there is a short exact sequence of Groups

$$0\to\mathscr{B}^1(U(A),M)\to\mathscr{D}^1(A,M)\to\mathrm{Hom}_A(\mathscr{B}_1(A),M)\to 0.$$

If in addition we suppose $\mathscr{B}_i(A)=0$ for all $1\leqslant i\leqslant n-1$, then $\mathscr{D}^r(A,M)=0$ for all $2\leqslant r\leqslant n-1$ and $\mathscr{D}^n(A,M)\cong\mathrm{Hom}(\mathscr{B}_n(A),M)$. These facts generalize other already known in cohomology of groups, in fact:

Suppose $\mathscr{D}$ is the variety of all groups and $\mathscr{B}$ the subvariety of abelian groups. Then for any group A and any abelian group M, the cohomology in $\mathscr{D}$ is just the Eilenberg–Mac Lane cohomology of groups, $\mathscr{D}^n(A,M)\cong H^{n+1}(A,M)$, $n\geqslant 1$ (see [2]), and the cohomology in the subvariety is

$\mathscr{B}^n(U(A), M) \cong \mathrm{Ext}^n(A_{ab}, M)$. Now, since for any free resolution $\mathbf{F}.$ of A there are natural isomorphisms

$$U(F_n) \cong (F_n)_{ab} \cong I(F_n) \otimes_{Z(F_n)} \mathbb{Z},$$

where $I(F_n)$ is the ideal augmentation, the nth Baer invariant of A is just the integral homology of A at dimension $n+1$,

$$\mathscr{B}_n(A) \cong H_{n+1}(A, \mathbb{Z}).$$

Therefore, Theorem 9 states now that there is a short exact sequence of abelian groups

$$0 \to \mathrm{Ext}^1(A_{ab}, M) \to H^2(A, M) \to \mathrm{Hom}(H_2(A, \mathbb{Z}), M) \to 0$$

and if, moreover, the homology groups $H_i(A, \mathbb{Z})$ are trivial for $2 \leqslant i \leqslant n$, then $H^r(A, M)$ is trivial for $3 \leqslant r \leqslant n$ and

$$H^{n+1}(A, M) \cong \mathrm{Hom}(H_{n+1}(A, \mathbb{Z}), M).$$

References

1. M. André, "Méthode simpliciale en algèbre homologique et algèbre commutative," Lecture Notes in Mathematics, Vol. 32, Springer-Verlag, Berlin/New York, 1967.
2. M. Barr and J. Beck, "Homology and Standard Construction," Lecture notes in Mathematics, Vol. 80, Springer-Verlag, Berlin/New York, 1969.
3. J. Duskin, Simplicial methods and the interpretation of the triple cohomology, *Mem. Amer. Math. Soc.* **163** (1975).
4. A. Fröhlich, Baer invariants of algebras, *Trans. Amer. Math. Soc.* **109** (1963), 221–224.
5. P. J. Higgins, Groups with multiple operators, *Proc. London Math. Soc.* **6** (1956).
6. P. J. Higgins, Baer invariant and the Birkhoff–Witt theorem, *J. Algebra* **11** (1969), 469–482.
7. F. Keune, Homotopical algebra and algebraic K-theory, Thesis, University of Amsterdam, 1972.
8. C. R. Leedham-Green and S. Mackay, Baer invariant, isologisms, varietal laws and homology, *Acta Math.* **137** (1976), 99–150.
9. A. S. T. Lue, Baer invariant and extensions relative to a variety, *Proc. Cambridge Philos. Soc.* **63** (1967), 569–578.
10. A. S. T. Lue, Connected algebras of groups relative and universal coverings, *Comment. Math. Helv.* **48** (1973), 70–375.
11. A. S. T. Lue, Cohomology of groups relative to a variety, *J. Algebra* **69** (1981), 155–174.
12. K. Modi, Simplicial methods and the homology of groups, Thesis, University of London, 1976.
13. G. Orzech, Obstruction theory in algebraic categories, I-II, *J. Pure Appl. Algebra* **2** (1972), 287–340.
14. M. Tierney and W. Vogel, Simplicial resolution and derivated functors, *Math. Z.* **111** (1969), 1–14.