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Minimax Estimation of a Bounded Normal Mean Vector

J. CALVIN BERRY

University of Southwestern Louisiana

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The problem of minimax estimation of a multivariate normal mean vector has received much attention in recent years. In this paper this problem is considered when the mean vector is restricted to a compact convex subset B of R^p. The cases of rectangular and spherical bounds are considered. The least favorable prior distributions and Bayes minimax estimator of the mean vector are obtained for the situation where B is a sphere of sufficiently small radius.

1. INTRODUCTION

This paper is concerned with the problem of minimax estimation of a bounded multivariate normal mean vector under quadratic loss. Let $\boldsymbol{\theta}$ denote the mean vector of a p-variate normal distribution with identity covariance matrix and assume that $\boldsymbol{\theta}$ is restricted to a compact convex subset B of R^p. Since the usual minimax estimators of $\boldsymbol{\theta}$ take on values outside of B with positive probability, they are neither admissible nor minimax when $\boldsymbol{\theta}$ is restricted to B. Inadmissibility is clear, since the truncated versions of these estimators, where the usual estimator is replaced by the closest value in B if the estimator is not in B, dominate the untruncated estimators in terms of risk. If an estimator which takes on values outside of B with positive probability were minimax, then its truncated version would be minimax as well. This leads to a contradiction, since the risk function is continuous in $\boldsymbol{\theta}$ and attains its maximum in B, but the risk of the truncated version of the estimator is strictly smaller than the risk of the untruncated estimator for $\boldsymbol{\theta}$ in B. A complete version of this argument for $p = 1$ is given on page 268 of Lehmann [1].

The univariate version of this problem has been considered by several authors. Casella and Strawderman [2] and Kempthorne [3] consider

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minimax estimation of a normal mean ϑ when $|\vartheta| \leqslant m$ and provide Bayes minimax estimators and least favorable priors for small m. Bickel [4] considers the same problem for large m and provides asymptotically minimax estimators and least favorable priors. Bickel also considers the multivariate problem for large m. DasGupta [5] considers a more general class of models and proves that the Bayes estimator corresponding to a prior supported on the boundary of the restricted parameter space is minimax, provided the restricted parameter space is small enough.

There are several ways in which the bounds $|\vartheta| \leqslant m$ can be extended to the vector mean $\boldsymbol{\theta}$, two such extensions will be considered here. For the rectangular bounds $|\vartheta_i| \leqslant m_i$ for $i = 1, ..., p$, the least favorable prior is obtained as the product of the appropriate univariate least favorable priors and the coordinates of the corresponding Bayes minimax estimator are the estimators of Casella and Strawderman [2] and Bickel [4]. For the remainder of this paper the spherical bounds $\|\boldsymbol{\theta}\| \leqslant m$ are considered. Equivariance and analyticity considerations guarantee that there is a unique least favorable prior supported on a finite number of spherical shells for which the corresponding Bayes estimator is minimax.

The main result of this paper is given in Section 3, where for small m a prior supported on a single spherical shell is shown to be least favorable and the corresponding Bayes estimator is shown to be minimax. The results of Casella and Strawderman [2] for $p = 1$ are obtained as a special case.

2. Preliminaries

Let $\mathbf{X}$ denote a p-variate normal random vector with mean vector $\boldsymbol{\theta}$ and identity covariance matrix. The mean vector will be assumed to be restricted so that $\|\boldsymbol{\theta}\| \leqslant m$ for some fixed $m > 0$.

Define the loss incurred when estimating $\boldsymbol{\theta}$ by $\boldsymbol{\delta}(\mathbf{X})$ as

$$L(\boldsymbol{\theta}, \boldsymbol{\delta}(\mathbf{X})) = \|\boldsymbol{\theta} - \boldsymbol{\delta}(\mathbf{X})\|^2,$$

with corresponding risk function

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}) = E\,\|\boldsymbol{\theta} - \boldsymbol{\delta}(\mathbf{X})\|^2.$$

Let $\pi(\boldsymbol{\theta})$ denote the uniform prior distribution on the surface of the sphere in p dimensions with center at the origin and radius m, i.e.,

$$\pi(\boldsymbol{\theta}) \propto \mathbf{1}_{\{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = m\}}(\boldsymbol{\theta}).$$

The posterior distribution of $\boldsymbol{\theta}$ is given by

$$\pi(\boldsymbol{\theta} \mid \mathbf{x}) \propto \exp\{\mathbf{x}'\boldsymbol{\theta}\}\, \mathbf{1}_{\{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = m\}}(\boldsymbol{\theta}).$$

Setting $r=\|\mathbf{x}\|$, $\boldsymbol{\phi}=\boldsymbol{\theta}/m$, and $\boldsymbol{\mu}=\mathbf{x}/\|\mathbf{x}\|=\mathbf{x}/r$, yields

$$\pi(\boldsymbol{\theta} \mid \mathbf{x}) \propto \exp\{m\|\mathbf{x}\|\, \boldsymbol{\phi}'\boldsymbol{\mu}\}\, \mathbf{1}_{\{\boldsymbol{\phi}:\|\boldsymbol{\phi}\|=1\}}(\boldsymbol{\phi}),$$

hence, the posterior distribution of $\boldsymbol{\phi}=\boldsymbol{\theta}/m$ is the p-variate von Mises distribution with concentration parameter mr and mean direction vector $\boldsymbol{\mu}=\mathbf{x}/r$.

The Bayes estimator of $\boldsymbol{\theta}$, $\boldsymbol{\delta}_\pi(\mathbf{x})$, is the posterior mean vector, thus

$$\boldsymbol{\delta}_\pi(\mathbf{x})=E\{\boldsymbol{\theta} \mid \mathbf{x}\}=mE\{\boldsymbol{\phi} \mid \mathbf{x}\}=A(mr)(m/r)\mathbf{x},$$

where

$$A(mr)=\frac{I_{p/2}(mr)}{I_{p/2-1}(mr)},$$

and $I_v(z)$ denotes the modified Bessel function of the first kind of order v.

Remark. A series representation and an integral representation of $I_v(z)$ are

$$\begin{aligned} I_v(z) &= \sum_{k=0}^{\infty} \frac{(z/2)^{2k+v}}{k!\,\Gamma(v+k+1)} \\ &= \left(\frac{(z/2)^v}{2\sqrt{\pi}\,\Gamma(v+1/2)}\right)\int_0^{2\pi} \sin^{2v}\alpha \exp\{z\cos\alpha\}\, d\alpha. \end{aligned}$$

Remark. When $p=1$, $\delta_\pi(x)=m\tanh(mx)$, the estimator obtained by Casella and Strawderman [2]. When $p=2$, $\boldsymbol{\delta}_\pi(\mathbf{x})=(mI_1(mr)/rI_0(mr))\mathbf{x}$, and when $p=3$, $\boldsymbol{\delta}_\pi(\mathbf{x})=(m/r)(\coth(mr)-1/mr)\mathbf{x}$.

This estimation problem and Bayes estimator are equivariant for rigid rotations, hence the risk function $R(\boldsymbol{\theta}, \boldsymbol{\delta}_\pi)=E\,\|\boldsymbol{\theta}-\boldsymbol{\delta}_\pi(\mathbf{X})\|^2$ is constant for fixed $\|\boldsymbol{\theta}\|^2$.

The Bayes estimator $\boldsymbol{\delta}_\pi(\mathbf{X})$ can be expressed in the form

$$\boldsymbol{\delta}_\pi(\mathbf{X})=\mathbf{X}+\mathbf{g}(\mathbf{X}), \qquad \text{where} \quad \mathbf{g}(\mathbf{X})=[A(mr)(m/r)-1]\mathbf{X}.$$

Stein's unbiased estimator of this risk function, $\lambda(\mathbf{X})$, is given by

$$\lambda(\mathbf{X})=p+\|\mathbf{g}(\mathbf{X})\|^2+2\sum_{i=1}^{p}\frac{\partial}{\partial x_i}g_i(\mathbf{X}),$$

where $g_i(\mathbf{X})$ denotes the ith coordinate of $\mathbf{g}(\mathbf{X})$,

$$\|\mathbf{g}(\mathbf{x})\|^2=\left[\left(\frac{m}{r}\right)A(mr)-1\right]^2 r^2=m^2A^2(mr)-2mrA(mr)+r^2$$

$$\frac{\partial}{\partial x_i}g_i(\mathbf{x})=\left(\left[\left(\frac{m}{r}\right)A(mr)-1\right]+\left[m^2A'(mr)-\left(\frac{m}{r}\right)A(mr)\right]\right)\left(\frac{x_i^2}{r^2}\right).$$

Application of standard Bessel function properties yields

$$A'(z)=\frac{d}{dz}A(z)=1+\left(\frac{1-p}{z}\right)A(z)-A^2(z); \tag{2.1}$$

therefore,

$$\begin{aligned}\sum_{i=1}^{p}\frac{\partial}{\partial x_i}g_i(\mathbf{x})&=\sum_{i=1}^{p}\left[\left[\left(\frac{m}{r}\right)A(mr)-1\right]\right.\\&\quad\left.+\left[m^2A'(mr)-\left(\frac{m}{r}\right)A(mr)\right]\left(\frac{x_i^2}{r^2}\right)\right]\\&=m^2-p-m^2A^2(mr),\end{aligned}$$

and Stein's unbiased estimator of the risk is given by

$$\lambda(\mathbf{X})=2m^2+r^2-p-2mrA(mr)-m^2A^2(mr). \tag{2.2}$$

The squared length of $\mathbf{X}$, r^2, is distributed as a noncentral chi-square random variable with p degrees of freedom and noncentrality parameter $\|\boldsymbol{\theta}\|^2$. Thus $E(r^2)=p+\|\boldsymbol{\theta}\|^2$ and

$$R(\boldsymbol{\theta},\boldsymbol{\delta}_\pi)=2m^2+\|\boldsymbol{\theta}\|^2-E\{2mrA(mr)+m^2A^2(mr)\}, \tag{2.3}$$

where the expectation is with respect to $r^2\sim\chi'^2_{p,\|\boldsymbol{\theta}\|^2}$.

3. Main Results

Since $R(\boldsymbol{\theta},\boldsymbol{\delta}_\pi)$ is constant for fixed $\|\boldsymbol{\theta}\|$ and depends on $\boldsymbol{\theta}$ only through $\|\boldsymbol{\theta}\|$, there is no loss of generality in taking $\boldsymbol{\theta}=(\vartheta,0,...,0)'$. With this choice of $\boldsymbol{\theta}$ the risk function is given by

$$R(\boldsymbol{\theta},\boldsymbol{\delta}_\pi)=R(\vartheta,\boldsymbol{\delta}_\pi)=2m^2+\vartheta^2-E\{2mrA(mr)+m^2A^2(mr)\}.$$

The proofs of Lemma 3.1 and Lemma 3.2 use a sign change argument based on Theorem 3 and Corollary 2 of Karlin [6].

LEMMA 3.1. *If* $\mathbf{X}\sim N_p(\boldsymbol{\theta},\mathbf{I})$ *with* $\|\boldsymbol{\theta}\|\leqslant m$, *then*

$$\max_{-m\leqslant\vartheta\leqslant m}R(\vartheta,\boldsymbol{\delta}_\pi)=\max\{R(0,\boldsymbol{\delta}_\pi),R(m,\boldsymbol{\delta}_\pi)\};$$

i.e., the maximum of the risk function $R(\boldsymbol{\theta},\boldsymbol{\delta}_\pi)$ *is attained for* $\|\boldsymbol{\theta}\|=0$ *or for* $\|\boldsymbol{\theta}\|=m$.

Proof. Consider the derivative of $R(\vartheta, \boldsymbol{\delta}_\pi)$ with respect to ϑ,

$$\frac{d}{d\vartheta} R(\vartheta, \boldsymbol{\delta}_\pi) = 2\vartheta - \int \cdots \int [2mrA(mr) + m^2A^2(mr)] \left(\frac{1}{2\pi}\right)^{p/2}$$

$$\times (x_1 - \vartheta) \exp\left\{-\left[\frac{(x_1-\vartheta)^2 + \sum_{i=2}^p x_i^2}{2}\right]\right\} d\mathbf{x}.$$

Integration by parts with respect to x_1 yields

$$\frac{d}{d\vartheta} R(\vartheta, \boldsymbol{\delta}_\pi) = 2\vartheta - \int \cdots \int 2C(m, r) \left(\frac{1}{2\pi}\right)^{p/2}$$

$$\times \exp\left\{-\left[\frac{(x_1-\vartheta)^2 + \sum_{i=2}^p x_i^2}{2}\right]\right\} d\mathbf{x},$$

where

$$2C(m, r) = \frac{\partial}{\partial x_1} [2mrA(mr) + m^2A^2(mr)].$$

Application of (2.1) yields

$$C(m, r) = m\left[A(mr) + [mr + m^2A(mr)]\right.$$

$$\left.\times\left[1 + \left(\frac{1-p}{mr}\right) A(mr) - A^2(mr)\right]\right]\left(\frac{x_1}{r}\right).$$

Transforming $x_1, x_2, ..., x_p$ to the spherical coordinates $r, \beta_1, ..., \beta_{p-1}$, where $r \geqslant 0$, $0 < \beta_1 \leqslant 2\pi$, and $0 < \beta_i \leqslant \pi$ for $i = 2, ..., p-1$ yields

$$x_1 = r \cos \beta_1,$$

$$x_i = r \cos \beta_i \prod_{j=1}^{i-1} \sin \beta_j \qquad \text{for} \quad i = 2, ..., p-1,$$

$$x_p = r \prod_{j=1}^{p-1} \sin \beta_j.$$

The Jacobian, J, of this transformation is given by

$$J = r^{p-1} \prod_{j=1}^{p-1} \sin^{p-1-j} \beta_j.$$

Hence,

$$\frac{d}{d\vartheta} R(\vartheta, \boldsymbol{\delta}_\pi) = 2\vartheta - 2\left(\frac{1}{2\pi}\right)^{p/2} \left(\prod_{j=2}^{p-1} \int_0^\pi \sin^{p-1-j} \beta_j \, d\beta_j\right)$$
$$\times \int_0^\infty C(m, r) \exp\left\{-\left[\frac{r^2+\vartheta^2}{2}\right]\right\} r^{p-1}$$
$$\times \int_0^{2\pi} \cos\beta_1 \sin^{p-2}\beta_1 \exp\{\vartheta r \cos\beta_1\} \, d\beta_1 \, dr. \tag{3.1}$$

Application of the identity

$$\int_0^\pi \sin^k \beta \, d\beta = \frac{\sqrt{\pi}\,\Gamma((k+1)/2)}{\Gamma((k+2)/2)} \qquad \text{for} \quad k = 0, 1, \ldots$$

yields

$$\prod_{j=2}^{p-1} \int_0^\pi \sin^{p-1-j} \beta_j \, d\beta_j = \frac{\pi^{(p-1)/2}}{\Gamma((p-1)/2)}. \tag{3.2}$$

Integration by parts with respect to β_1 and standard Bessel function properties yield

$$\int_0^{2\pi} \cos\beta_1 \sin^{p-2}\beta_1 \exp\{\vartheta r \cos\beta_1\} \, d\beta_1$$
$$= \int_0^{2\pi} \left[\left(\frac{p-3}{\vartheta r}\right) \sin^{p-4}\beta_1 - \left(\frac{p-2}{\vartheta r}\right) \sin^{p-2}\beta_1\right] \exp\{\vartheta r \cos\beta_1\} \, d\beta_1$$
$$= \left[\frac{2\sqrt{\pi}(p-3)\,\Gamma((p-3)/2)}{(\vartheta r/2)^{(p-4)/2}}\right]\left[I_{(p-4)/2}(\vartheta r) - \left(\frac{p-2}{\vartheta r}\right) I_{(p-2)/2}(\vartheta r)\right]$$
$$= \left[\frac{2^{(p-2)/2}\sqrt{\pi}(p-3)\,\Gamma((p-3)/2)}{(\vartheta r)^{(p-2)/2}}\right] I_{p/2}(\vartheta r). \tag{3.3}$$

Combining (3.1)–(3.3) yields

$$\frac{d}{d\vartheta} R(\vartheta, \boldsymbol{\delta}_\pi) = 2\vartheta\left(1 - \int_0^\infty \left(\frac{C(m, r)}{r}\right)\left[\left(\frac{r^{p/2} I_{p/2}(\vartheta r)}{2\vartheta^{p/2}}\right)\right.\right.$$
$$\left.\left. \times \exp\left\{-\left[\frac{r^2+\vartheta^2}{2}\right]\right\}\right] dr^2\right).$$

As a function of r^2 the expression in the large square brackets is the noncentral χ^2 density with $p+2$ degrees of freedom and noncentrality parameter ϑ^2, hence

$$\frac{d}{d\vartheta} R(\vartheta, \boldsymbol{\delta}_\pi) = 2\vartheta E\left([1 - m^2 A'(mr)] - \left[\left(\frac{m}{r}\right) A(mr)[m^2 A'(mr) + 1] \right] \right), \tag{3.4}$$

where the expectation is with respect to $r^2 \sim \chi'^2_{p+2,\vartheta^2}$.

Watson [7] shows that $A'(z)$ is positive and nondecreasing for $z \geqslant 0$, hence, $1 - m^2 A'(mr)$ is increasing for $r \geqslant 0$, and $m^2 A'(mr) + 1$ is positive and decreasing for $r \geqslant 0$. For $r \geqslant 0$, $(m/r)\, A(mr)$ is positive and can be expressed in the form

$$\begin{aligned}\left(\frac{m}{r}\right) A(mr) &= \left(\frac{m}{r}\right) \frac{I_{p/2}(mr)}{I_{(p-2)/2}(mr)} \\ &= \left(\frac{m^2}{2}\right) \frac{\sum_{k=0}^{\infty} ((mr/2)^{2k+p/2-1}/k!\, \Gamma(p/2+k+1))}{\sum_{k=0}^{\infty} ((mr/2)^{2k+p/2-1}/k!\, \Gamma(p/2+k))}.\end{aligned}$$

The derivative of $(m/r)\, A(mr)$ with respect to r can be expressed as a fraction with a positive denominator and with the numerator given by

$$\begin{aligned}&\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mr/2)^{2i+2j+p-3}(p+2j)(i-j)}{i!\, j!\, \Gamma(p/2+i+1)\, \Gamma(p/2+j+1)} \\ &\quad = \sum_{i<j} \frac{(mr/2)^{2i+2j+p-3}(-2(i-j)^2)}{i!\, j!\, \Gamma(p/2+i+1)\, \Gamma(p/2+j+1)} < 0;\end{aligned}$$

hence, $(m/r)\, A(mr)$ is a decreasing function of r. Combining these results, the integrand in (3.4) is seen to be the difference of an increasing function of r and a decreasing function of r; hence, this integrand changes sign at most once as r^2 varies from 0 to ∞, Notice that this integrand is either strictly positive or has a single sign change from negative to positive as r^2 varies from 0 to ∞.

The results of Karlin indicate that the expectation changes sign at most once as ϑ^2 varies from 0 to ∞ and that this sign change is in the same order as the sign change in the integrand. Hence, the risk function changes sign at most three times as ϑ varies from $-\infty$ to ∞ and the sign changes are in the order $- + - +$.

The risk function is an even function of ϑ; therefore, if the risk function has an extremum for $\vartheta \neq 0$, then the extremum must be a minimum. Hence, for $-m \leqslant \vartheta \leqslant m$, the risk function attains its maximum for $\vartheta = 0$ or $\vartheta = \pm m$.

LEMMA 3.2. *There exists a unique* $m_0 > 0$ *such that* $R(m, \boldsymbol{\delta}_\pi) \geqslant R(0, \boldsymbol{\delta}_\pi)$ *for all* $-m_0 \leqslant m \leqslant m_0$, *i.e.,* $R(\boldsymbol{\theta}, \boldsymbol{\delta}_\pi) \geqslant R(\mathbf{0}, \boldsymbol{\delta}_\pi)$ *for all* $\boldsymbol{\theta}$ *such that* $\|\boldsymbol{\theta}\| = m \leqslant m_0$.

Proof. Application of (3.2) with $\boldsymbol{\theta} = (\vartheta, 0, ..., 0)'$ yields

$$R(\vartheta, \boldsymbol{\delta}_\pi) = 2m^2 + p + 2\vartheta^2 - E_{\vartheta^2}\{[mA(mr) + r]^2\},$$

thus

$$\begin{aligned} D(m) &= R(m, \boldsymbol{\delta}_\pi) - R(0, \boldsymbol{\delta}_\pi) \\ &= 2m^2 + E_0\{[mA(mr) + r]^2\} - E_{m^2}\{[mA(mr) + r]^2\}, \end{aligned}$$

where E_{ϑ^2} denotes expectation with respect to $r^2 \sim \chi'^2_{p,\vartheta^2}$. Straightforward algebra yields

$$D(m) = 2m^2 + E_{m^2}\left[[mA(mr) + r]^2 \times \left(\frac{e^{m^2/2} m^{(p-2)/2}}{2^{(p-2)/2} \Gamma(p/2)\, I_{(p-2)/2}(mr)} - 1 \right) \right].$$

The integrand in this expression for $D(m)$ is equal to zero when $2^{(p-2)/2}\Gamma(p/2)\, I_{(p-2)/2}(mr) = e^{m^2/2} m^{(p-2)/2}$ and has at most one sign change, from positive to negative, as r^2 varies from 0 to ∞. The results of Karlin indicate that $D(m)$ changes sign at most once as m^2 varies from 0 to ∞ and that this sign change is from positive to negative. Hence, there exists a unique value $m_0 > 0$ such that $D(m) \geqslant 0$ for all $-m_0 \leqslant m \leqslant m_0$ and the lemma is established.

THEOREM 3.1. *If* $\mathbf{X} \sim N_p(\boldsymbol{\theta}, \mathbf{I})$ *with* $\|\boldsymbol{\theta}\| \leqslant m$ *and if* m_0 *denotes the value determined in Lemma* 3.2, *then* $\boldsymbol{\delta}_\pi(\mathbf{x})$ *is the Bayes minimax estimator of* $\boldsymbol{\theta}$ *with respect to* π *under quadratic loss and* π *is the least favorable prior, provided* $|m| \leqslant m_0$.

Proof. The risk function of $\boldsymbol{\delta}_\pi$ is constant for fixed $\|\boldsymbol{\theta}\|$, therefore, the Bayes risk of $\boldsymbol{\delta}_\pi$ with respect to π is $r(\pi, \boldsymbol{\delta}_\pi) = R(m, \boldsymbol{\delta}_\pi)$. Application of Lemma 3.1 and Lemma 3.2 shows that the risk of $\boldsymbol{\delta}_\pi$ attains its maximum on the support of π, hence, $\boldsymbol{\delta}_\pi$ is minimax and π is least favorable, provided $m \leqslant m_0$. Notice that $R(m, \boldsymbol{\delta}_\pi)$ is the minimax value.

Remark. For $p = 1$, Casella and Strawderman [2] report that $m_0 \approx 1.05674$. When $p = 2$, $m_0 \approx 1.53499$, and when $p = 3$, $m_0 \approx 1.90799$.

TABLE I

Selected Minimax Risk Values

$p=2$										
m	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.53499		
risk	0.039	0.148	0.305	0.482	0.655	0.806	0.927	0.989		
$p=3$										
m	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	1.90799
risk	0.039	0.152	0.321	0.26	0.746	0.961	1.158	1.330	1.473	1.538

Selected values of the minimax risk for $p=2$ and $p=3$ are provided in Table I.

4. Comments

In Section 3 the least favorable prior and Bayes minimax estimator were exhibited for small m. For the univariate problem with $m > m_0$ Casella and Strawderman [2] demonstrated that a prior supported on three points is the least favorable, subject to an upper bound on m. For the general p case, this three point prior corresponds to a prior which is a mixture of a point mass at the origin and the uniform prior π of Section 2 and 3. It seems reasonable to conjecture that this prior is least favorable for $m > m_0$, subject to an upper bound on m. The complexity of this prior and the corresponding Bayes estimator complicate the verification of this conjecture.

The results of this paper can be applied to certain situations where the bounds on $\boldsymbol{\theta}$ are elliptical. For example, consider a p-variate normal random vector $\mathbf{X}$ with mean vector $\boldsymbol{\theta}$ and known positive definite covariance matrix $\boldsymbol{\Sigma}$, with $\boldsymbol{\theta}$ restricted so that $\boldsymbol{\theta}'\Sigma^{-1}\boldsymbol{\theta} \leqslant m^2$. Setting $\mathbf{Y} = \boldsymbol{\Sigma}^{-1/2}\mathbf{X}$ yields a p-variate normal random vector with mean vector $\boldsymbol{\gamma} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\theta}$ and identity convariance matrix, where $\|\boldsymbol{\gamma}\| \leqslant m$. Extensions to the general problem with elliptical bounds are complicated by the need for asymmetrical least favorable priors.

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