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ORIGINAL ARTICLE

# Traveling wave solution by differential transformation method and reduced differential transformation method

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**Abstract** The present study further examines two recent semi-analytic methods, a reduced order of nonlinear differential transformation method (also called RDTM) and differential transformation method along with Pade approximation to discuss Jaulent–Miodek and coupled Whitham–Broer–Kaup equations. The basic ideas of these methods are briefly introduced and performance of the proposed methods for above mentioned equations is evaluated via comparing with exact solution. The results illustrate that the so-called DTM method, unlike RDTM, due to the presence of secular terms (similar to perturbation method), cannot be found practical for nonlinear partial differential equations (particularly in Acoustic and Wave propagation problems) even through utilizing Pade approximation; meanwhile, RDTM method, despite its simplicity and rapid convergence, assured a significant accuracy and great agreement, and thus it is fair to say that nonlinear problems together with Acoustic application which cannot be solved via Analytical methods, can be studied with reduced order of nonlinear differential transformation method.
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## 1. Introduction

In the applied sciences, almost every natural event can be modeled with nonlinear equations; this is due to nature of the phenomena in the world. Although it is easy to find solution of some problems by means of computers, it is still difficult to solve nonlinear equations either numerically or analytically in physics, engineering, chemistry and biology areas. Beside,

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solving these nonlinear problems is a great purpose which can make researchers encounter difficulties in finding exact solutions and in some cases has been quite untouched. The investigation of constructing traveling wave solutions has been a theme of intensive researches in the literature and in recent years, numerical analysis [1–3], has significantly been improved to be utilized for waves which have a special influence on the marine environment and eventually on the planet's climate.

Solving these nonlinear problems can make researchers encounter difficulties in finding the exact analytical solution [4–6], and thus it may guide authors to use various approximate analytical methods. Perturbation method is one of the most well-known methods for solving nonlinear equations, but since using the common perturbation method [7] is based on the existence of a small parameter, Therefore many different new methods have recently presented some ways to eliminate the small parameter, such as Parameter expansion Method [8–10], Variational Iteration Method [11–14], Homotopy Perturbation Method [15–20], Adomian Decomposition Method [21], Differential Transformation Method [22–27] and Reduced order of Differential Transformation Method [28]; most of them have good convergences and efficiency in weak nonlinear system but some of them are not good for strongly nonlinear system.

Despite the fact that it is always possible to apply numerical solutions to resolve various nonlinear equations, the accuracy and computational labor demanded for these methods greatly respond to their numerical capability and stability. Also, the benefit of analytical approximation is that they allow extensive comprehension of the nature and quality of nonlinear equations. Hence, it may not always be possible to derive analytical techniques of these equations. In such cases, semi-analytical solutions are used which yield series solutions. In these kinds of methods, the solutions are obtained in the form of series. Semi-analytical methods are based on finding the other terms of the series from given initial conditions for the problem being considered.

In this paper, a reduced order of nonlinear differential transformation method and classical DTM with Padé approximation of this method [29] are proposed to find semi-analytical approximation of Jaulent–Miodek equation and coupled Whitham–Broer–Kaup equations. The Padé approximation of Differential Transformation Method often yields better approximation than DTM, but this presentation demonstrated that RDTM gives even better approximation than both DTM and Padé approximation of DTM. So, the aim of this paper was to demonstrate efficiency of RDTM in weak and strongly nonlinear system in underwater wave equation.

## 2. Definitions and solution formulations

In this section basic definitions of two semi-analytical methods called DTM and RDTM are introduced as follows.

### 2.1. Basic idea of Two-dimensional DTM

To suggest the basic ideas of this method, a function of two variables $w(x,t)$ which can be outlined as a result of two single-variable function, i.e. $w(x,t) = \varphi(x)u(t)$, is considered. On the foundation of the characteristics of one-dimensional differential transform, the function $w(x,t)$ can be defined as follows:

$$w(x,t) = \sum_{i=0}^{\infty}\Phi(i)x^i \sum_{j=0}^{\infty}U(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W(i,j)x^i t^j \tag{1}$$

where $W(i,j) = \Phi(i)U(j)$, which is called the spectrum of $w(x,t)$. The basic definitions and fundamental operations for two-dimensional differential transform can be illustrated as follows:

Considering $w(x,t)$ as an analytical and continuously differentiable function with respect to time $t$ in the domain of interest, the following equation can be introduced as follows:

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{\substack{x=x_0\\ t=t_0}} \tag{2}$$

where the spectrum function $W(k,h)$ can be defined as transformed function.

Here, the lower case $w(x,t)$ stands for the original function and the upper case $w(k,h)$ represents the transformed function (which is also called T-function). The differential inverse transform of $W(k,h)$ can be illustrated in the following form:

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(t-t_0)^h \tag{3}$$

Substituting Eq. (2) into Eq. (3), $w(x,t)$ can be rewritten in the following form:

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{\substack{x=x_0\\ t=t_0}}(x-x_0)^k(t-t_0)^h \tag{4}$$

Here, by equaling $(x_0, t_0)$ as $(0,0)$, Eq. (4) can be illustrated in the following form:

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k t^h \tag{5}$$

where the function $w(x,t)$ can be represented by a finite series:

$$w(x,t) = \sum_{k=0}^{n}\sum_{h=0}^{m}W(k,h)x^k t^h + R_{a,b}(x,t) \tag{6}$$

where

$$R_{nm}(x,t) = \sum_{k=a+1}^{\infty}\sum_{h=b+1}^{\infty}w(k,h)x^k t^h \tag{7}$$

Here the values of $a$ and $b$ are obtained by convergence of the series coefficients.

The fundamental operations performed by utilizing the two-dimensional differential transform are illustrated in Table 1.

### 2.2. Basic idea of two-dimensional RDTM

Similarly on previous section, considering $w(x,t) = \varphi(x)u(t)$, based on the properties of one-dimensional differential transform, the function $w(x,t)$ can be obtained as follows:

$$w(x,t) = \sum_{i=0}^{\infty}\Phi(i)x^i \sum_{j=0}^{\infty}U(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W(i,j)x^i t^j \tag{8}$$

**Table 1** The fundamental operations of two-dimensional differential transform method and reduced order of such method.

| Original function | Reduced transformed function | 2D Transformed function |
|---|---|---|
| $W(x,t) = U(x,t) \pm v(x,t)$ | $W_k(x) = U_k(x) \pm V_k(x)$ | $W(k,h) = U(k,h) \pm V(k,h)$ |
| $W(x,t) = cu(x,t)$ | $W_k(x) = cU_k(x)$ | $W(k,h) = cU(k,h)$ |
| $W(x,t) = \frac{\partial}{\partial x}u(x,t)$ | $W_k(x) = \frac{\partial}{\partial x}U_k(x)$ | $W(k,h) = (k+1)U(k+1,h)$ |
| $W(x,t) = \frac{\partial}{\partial t}u(x,t)$ | $W_k(x) = (k+1)U_{k+1}(x)$ | $W(k,h) = (h+1)U(k,h+1)$ |
| $W(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$ | $W_k(x) = \frac{(k+s)!}{K!}\frac{\partial^r}{\partial x^r}U_{k+s}(x)$ | $W(k,h) = \frac{(k+r)!(h+s)!}{k!h!}U(k+r,h+s)$ |
| $W(x,t) = u(x,t)v(x,t)$ | $W_k(x) = \sum_{r=o}^{k} U_r(x)V_{k-r}(x)$ | $W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$ |
| $W(x,t) = x^m t^n$ | $W_k(x) = x^m\delta(k-n) = \begin{cases} x^m, k=n \\ 0, otherwise \end{cases}$ | $W(k,h) = \delta(k-m,h-n) = \begin{cases} 1, k=m, h=n \\ 0, otherwise \end{cases}$ |

where $w(i,j) = \Phi(i)U(j)$, which is also called the spectrum of $w(x,t)$.

The reduced transformed function of $w(x,t)$ can be yield in the following spectrum function:

$$W_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=t_0} \tag{9}$$

where $w(x,t)$ is analytical function in the domain of interest. As mentioned in previous section, the lowercase $w(x,t)$ stands for the original function and the uppercase $w(x)$ represents the reduced transformed function. The differential inverse transform of $w(x)$ can be written in the following form:

$$w(x,t) = \sum_{k=o}^{\infty} W_k(x)(t-t_0)^k \tag{10}$$

Substituting Eq. (9) into Eq. (10) $w(x,t)$ can be rewritten in the following form:

$$w(x,t) = \sum_{k=o}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=t_0}(t-t_0)^k \tag{11}$$

The fundamental operations performed by utilizing the two-dimensional reduced two-dimensional differential transform are illustrated in Table 1.

## 3. Results and discussion

To examine the proposed technique, two typical examples are used in traveling wave phenomena. The first example is coupled Whitham–Broer–Kaup equation that it is a wave equation in shallow water and the second example is the Jaulent–Miodek equations.

### *3.1. Whitham–Broer–Kaup equations*

Coupled Whitham–Broer–Kaup (CWBK) equations [31] describe the propagation of shallow water waves (see Fig. 1) with different dispersion relations:

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \beta\frac{\partial^2 u}{\partial x^2} = 0, \\ &\frac{\partial v}{\partial t} + \frac{\partial(uv)}{\partial x} + \alpha\frac{\partial^3 u}{\partial x^3} - \beta\frac{\partial^2 v}{\partial x^2} = 0, \end{aligned} \tag{12a}$$

With the initial conditions of

$$\begin{aligned} &u(x,0) = u_0(x,t) = \lambda - 2k\left(\alpha+\beta^2\right)^{0.5}\coth\left[k(x+x_0)\right] \\ &v(x,0) = v_0(x,t) = -2k^2\left(\alpha+\beta(\alpha+\beta^2)^{0.5}+\beta^2\right)\operatorname{csch}^2[k(x+x_0)] \end{aligned} \tag{12b}$$

When $\alpha = 0$ and $\beta = 0.5$, the WBK equations are reduced to the modified Boussinesq (MB) equations. While $\alpha = 1$ and $\beta = 0$, the WBK equations are reduced to the approximate long-wave (ALW) equations in shallow water.

Now both methods of DTM and RDTM will be applied on the WBK equations. As it will be presented below, the DTM converts Eq. (12) to a two parameter recursive equation, while the RDTM converts this equation to a one-parameter recursive equation.

#### *3.1.1. Differential Transformation Method (DTM)*

Considering Eq. (3) utilizing Two-Dimensional operators listed in Table 1, the transform of Eq. (12) can be written in the following form:

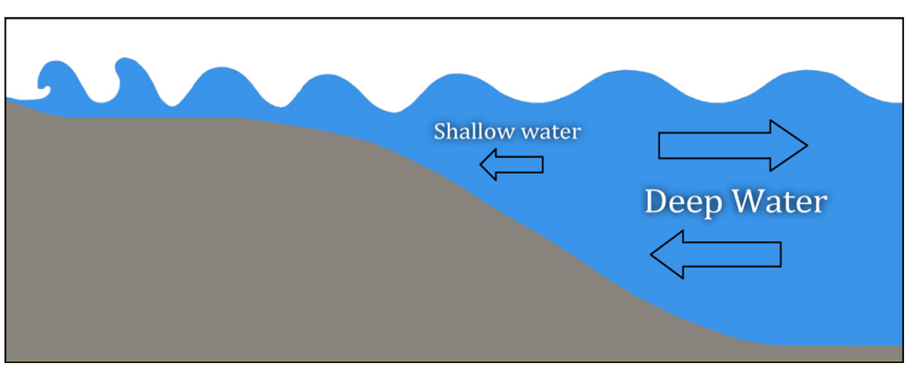



**Fig. 1** Schematics of shallow water.

$$u(k,h+1)=-\frac{1}{h+1}\left(\sum_{r=0}^{k}\sum_{s=0}^{h}u(r,h-s)(k-r+1)u(k-r+1,s)+(k+1)v(k+1,h)+\beta(k+2)(k+1)u(k+1,h)\right) \tag{13a}$$

$$v(k,h+1)=-\frac{1}{(h+1)}\left(\sum_{r=0}^{k}\sum_{s=0}^{h}u(r+1,h-s)(r+1)v(k-r,s)-\beta(k+2)(k+1)v(k+2,h)+\alpha(k+3,h)+\sum_{r=0}^{k}\sum_{s=0}^{h}u(r,h-s)(k-r+1)v(k-r+1,s)\right) \tag{13b}$$

$$\begin{aligned}
&u(0,0)=0.005-0.4149258884(\alpha+\beta^2)^{0.5}\\
&u(1,0)=0.006081746400(\alpha+\beta^2)^{05}\\
&u(2,0)=-0.001261737006(\alpha+\beta^3)^{0.5}\\
&u(3,0)-0.0001806737216(\alpha+\beta^2)^{0.5}\\
&u(4,0)=-0.00002065994220(\alpha+\beta^2)^{0.5}\\
&u(5,0)=0.000002093429365(\alpha+\beta^2)^{0.5}
\end{aligned} \tag{13c}$$

$$\begin{aligned}
v(1,0)&=0.002523474022\alpha+0.007573474079\beta(\alpha+\beta^2)^{0.5}\\
&\quad+0.002523474022\beta^2\\
v(2,0)&=-0.0005420211700\alpha-0.0005420211700\beta(\alpha+\beta^2)^{0.5}\\
&\quad-0.0005420211700\beta^2\\
v(3,0)&=0.00008263976976\alpha+00000863976976\beta(\alpha+\beta^2)^{0.5}\\
&\quad+0.00008263976976\beta^2\\
v(4,0)&=-0.00001046714690\alpha-0.00001046714690\beta(\alpha+\beta^2)^{0.5}\\
&\quad-0.00001046714690\beta^2\\
v(5,0)&=-0.000001222229302\alpha+0.000001222229302\beta(\alpha+\beta^2)^{0.5}\\
&\quad+0.000001222229302\beta^2
\end{aligned} \tag{13d}$$

where $\alpha$ and $\beta$ are arbitrary constants.

Here the DTM approximation solution in a series form of shallow water equations can be written as follows after choosing $=0.5,\ \beta=1$:

$$\begin{aligned}
u(x,t)&=\underbrace{0.0002212792139}_{u(3,0)}x^3-0.00002530315824x^4\\
&\quad+0.000002563916878x^5-5.025409912\times10^{-7}x^5t^2\\
&\quad+\underbrace{0.000002827122765}_{u(4,2)}x^4t^2+0.0006792907213xt\\
&\quad-0.0002057444618x^2t+0.00003896481753x^3t\\
&\quad-0.000006051774292x^4t+8.62620126210^{-7}x^5t\\
&\quad+0.0004959168039t^2-0.00008208341845t^3\\
&\quad-0.00001455891934x^3t^2+0.00006587410293x^2t^2\\
&\quad-0.0002342208235xt^2+0.00003403452497xt^3\\
&\quad-0.0000095083716x^2t^3+0.000002148905879x^3t^3\\
&\quad-4.250256060\times10^{-7}x^4t^3+7.629724908\times10^{-8}x^5t^3\\
&\quad-0.5031783537-0.000037242944t+0.00744858771x
\end{aligned} \tag{14a}$$

$$\begin{aligned}
v(x,t)&=-0.001476869403x^2+0.0002251722889x^3\\
&\quad-0.00002852030483x^4+0.000003330263022x^5\\
&\quad-1.508745804\times10^{-7}x^5t^2+6.658000824\times10^{-7}x^4t^2\\
&\quad-\underbrace{0.00023826289}_{v(1,1)}xt+0.0000671301294x^2t\\
&\quad-0.00001365032476x^3t+0.000002396636403x^4t\\
&\quad-3.87567658\times10^{-7}x^5t-0.00001738960181t^2\\
&\quad\underbrace{+0.00000778131186}_{v(0,3)}t^3-0.000002407799495 4x^3t^2\\
&\quad+0.00000620559122x^2t^2-0.00000702137988xt^2\\
&\quad-0.000002162077099xt^3+5.586065149\times10^{-7}x^2t^3\\
&\quad-1.436558993\times10^{-7}x^3t^3+3.65453209\times10^{-8}x^4t^3\\
&\quad-8.887616723\times10^{-9}x^5t^3-0.01657120728\\
&\quad+0.000297539711t+0.006875822899x
\end{aligned} \tag{14b}$$

*3.1.2. Reduced Differential Transformation Method (RDTM)*

Regarding to Eq. (12b) and the properties of RDTM, initial conditions can be rewritten as follows:

$$\begin{aligned}
&u_0(x)=0.005-0.4(\alpha+\beta^2)^{0.5}\coth(0.2(x+10))\\
&v_0(x)=-0.08\left(\alpha+\beta(\alpha+\beta^2)^{0.5}+\beta^2\right)\operatorname{csch}^2(0.2(x+10))
\end{aligned} \tag{15}$$

Considering Eq. (12a) and utilizing the basic properties of the reduced differential transformation listed in Table 2, the transformed form of Eq. (12a) can be rewritten as follows:

$$u_{k+1}(x)=-\frac{1}{(k+1)}\left(\beta\frac{\partial^2}{\partial x^2}u_k(x)+\frac{d}{dx}v_k(x)+\sum_{r=0}^{k}\left(\frac{\partial}{\partial x}u_r(x)u_{k-r}(x)\right)\right) \tag{16a}$$

$$v_{s+1}=-\frac{1}{(s+1)}\left(\alpha\frac{d^3}{dx^3}u_s(x)-\beta\frac{d^2}{dx^2}v_s(x)+\sum_{r=0}^{s}\left(\frac{d}{dx}u_r(x)v_{s-r}(x)\right)+\sum_{r=0}^{s}\left(u_{s-r}(x)\frac{d}{dx}v_r(x)\right)\right) \tag{16b}$$

**Table 2** Comparing absolute errors for $u(x,t)$ when $c_2=b_0=0.01,\ t=10$.

| $x$ | $\lvert dtm-exact\rvert$ | $\lvert dtm/pade-exact\rvert$ | $\lvert rdtm-exact\rvert$ | Exact |
|---|---|---|---|---|
| 0 | 0.00000344152 | 0.0000033792009 | 0.0000006916518 | −0.005524782308 |
| 5 | 0.000011599401 | 0.00001736913714 | 0.00000947459432 | −0.00383672316 |
| 10 | 0.000929522797 | 0.000844025707 | 0.0000050840587 | 0.000972165045 |
| 15 | 0.01733900160 | 0.004713077007 | 0.0000007964337 | 0.000921118068 |
| 20 | 0.01234103442 | 0.00942064088 | 0.0000000899743 | 0.001818283294 |

Considering the initial conditions presented in Eq. (15) the following solutions can be obtained:

$$u_1(x) = \frac{-\left(5\times10^{-13}(60.\cosh(0.2x+2)+9.79795897\sinh(0.2x+2.))\right)}{\left((\cosh(0.2x+2.)^2-1.)\sinh(0.2x+2.)\right)}$$

$$u_2(x) = \frac{\begin{pmatrix}-2\times10^{14}(500\cosh(0.2x+2)^3-2\cosh(0.2x+2)^2\sinh(0.2x+2))) \\ +2.4495\times10^7\cosh(0.2x+2.)-3\sinh(0.2x+2.))\end{pmatrix}}{((\cosh(0.2x+2)^2-1)\sinh(0.2x+2.))}$$

$$u_3(x) = \frac{\begin{pmatrix}10^{-14}(-300.\cosh(0.2x+2.)-16330.\sinh(0.2x+2.)) \\ +1200.\cosh(0.2x+2.)^3+2000.\cosh(0.2x+2.)^7 \\ -2000.\cosh(0.x+2.)^5-1.\cosh(0.2x+2.)^6\sinh(0.2x+2.))) \\ +32662.\cosh(0.2x+2.)^4\sinh(0.2x+2.)) \\ -16331.\cosh(0.2x+2.)^2\sinh(0.2x+2.))\end{pmatrix}}{\left((\cosh(0.2x+2.)^2-1)^3\cdot\sinh(0.2x+2.)\right)} \tag{17a}$$

$$v_1(x) = \frac{\begin{pmatrix}2\times10^{-13}(15\cosh(0.2x+2.)^4-5-110\cosh(0.2x+2.)^2 \\ -2.179795896\times10^9\cosh(0.2x+2.)\sinh(0.2x+2.))\end{pmatrix}}{(\cosh(.2x+2)^2-1)^2}$$

$$v_2(x) = \frac{\begin{pmatrix}10^{-14}(2\cosh(0.2x+2)^3\sinh(0.2x+2)+580\cosh(0.2x+2)^6 \\ -21798080\times10^7+3\cosh(0.2x+2.)\sinh(0.2x+2.) \\ -2.1796660\times10^7\cosh(0.2x+2.)^2+4.3594260\times10^7\cosh(0.2x+2.)^4))\end{pmatrix}}{(\cosh(.2x+2)^2-1)^2} \tag{17b}$$

$$v_3(x) = \frac{\begin{pmatrix}10^{-14}\cosh(0.2x+2.)-58126.\sinh(.2x+2.)2\cosh(0.2x+2.)^6\sinh(.2x+2.)) \\ +29069.\cosh(0.2x+2.)^4\sinh(0.2x+2)-60.\cosh(0.2x+2.)) \\ +29069.\cosh(0.2x+2.)^4\sinh(0.2x+2.)-560.\cosh(0.2x+2.)^3 \\ +200.\cosh(0.2x+2.)^5+120.\cosh(0.2x+2.)^7))\end{pmatrix}}{\left(\cosh(0.2x+2.)^2-1.\right)^4}$$

pRegarding the basic definition of RDTM the two terms of RDTM approximation solution in a series form of Broer–Kaup equations are written as follows:

$$u(x,t) = \sum_{k=o}^{3} U_k(x)t^k \tag{18a}$$

$$v(x,t) = \sum_{k=o}^{3} V_k(x)t^k \tag{18b}$$

In the figure mentioned below, a comparison between the proposed methods demonstrates that DTM cannot be assumed as accurate method to show real behavior of the system. Indeed, like classical perturbation method, terms that cause exponential divergence in trajectories can be found. However, since the Whitham–Broer–Kaup equation can be categorized as weak formed of nonlinear equations, Pade approximation can have great effect in solution optimization (see Fig. 2).

### 3.2. Jaulent–Miodek equations

In order to assess the accuracy of Two-dimensional DTM and RDTM for solving nonlinear Jaulent–Miodek equations, the following procedure will be considered:

$$\begin{aligned} &u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_xv_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_xv^2 = 0 \\ &v_t + v_{xxx} - 6u_xv - 6uv_x - \frac{15}{2}v_xv^2 = 0 \end{aligned} \tag{19a}$$

With the initial conditions of

$$\begin{aligned} &u(x,0) = \frac{c_2}{4} - \frac{1}{4}b_0^2 - \frac{1}{2}b_0\cdot\sqrt{c_2}\cdot\mathrm{sech}(\sqrt{c_2x}) - \frac{3}{4}\cdot c_2\cdot\mathrm{sech}^2(\sqrt{c_2x}) \\ &v(x,0) = b_0\cdot 2\cdot\sqrt{c_2}\cdot\mathrm{sech}(\sqrt{c_2x}) \end{aligned} \tag{19b}$$

$b_0$, $c_2$ are constant, and, respectively $c_2 = b_0 = 0.01$.

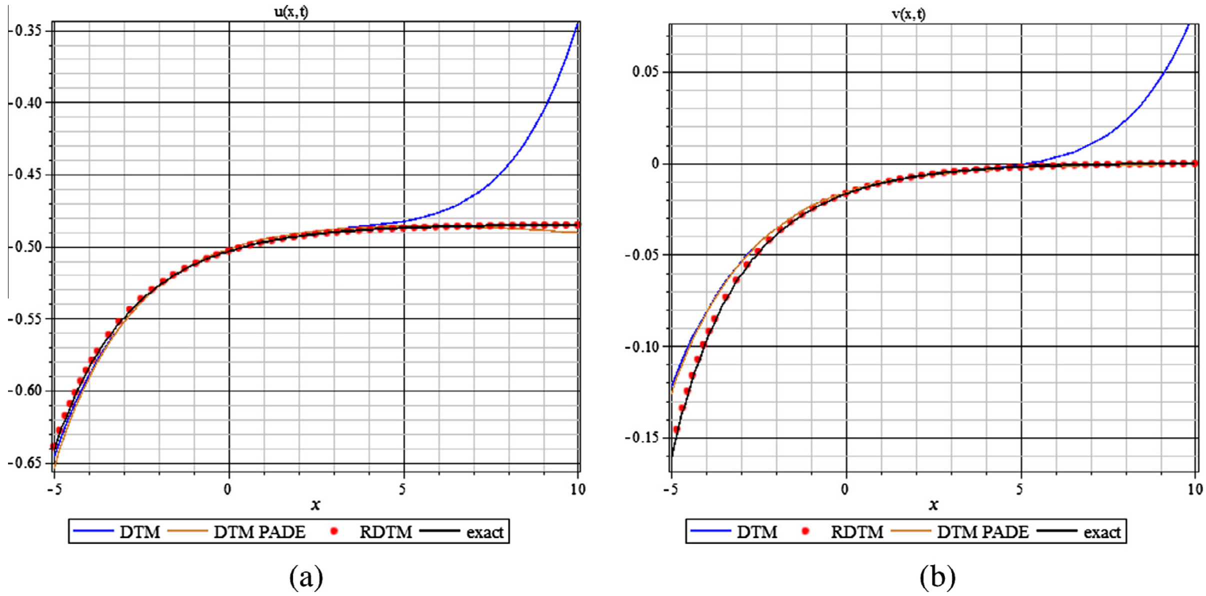



**Fig. 2** Charts (a) and (b) show the comparison between results of these methods and exact at $t = 2$.

Now both methods of DTM and RDTM will be applied on the CJM equations. As it will be presented below, the DTM converts Eq. (24) to a two parameter recursive equation, while the RDTM converts this equation to a one-parameter recursive equation.

### 3.2.1. Differential Transformation Method (DTM)

Considering Eq. (3) and utilizing Two-Dimensional operators listed in Table 1, the transform of Eq. (24) can be written in the following form:

$$\begin{aligned}U(k,h+1) = \frac{1}{h+1}\Bigg( &(k+3)(k+2)(k+1)U(k+3,h) \\ &+\frac{3}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}V(r,h-s)(k-r+1)(k-r+2) \\ &\times(k-r+3)V(k-r+3,s)+\frac{9}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}r \\ &+1V(r+1,h-s)(k-r+1)(k-r+2) \\ &\times U(k-r+2,s)-6\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1) \\ &\times U(k-r+1,s)-6\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U(r,h-s-p) \\ &\times V(t,s)(k-r-t+1)V(k-r-t+1,p) \\ &-\frac{3}{2}\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(r+1)U(r+1,h-s-p) \\ &\times V(t,s)V(k-r-t,p)\Bigg) \end{aligned} \tag{20a}$$

$$\begin{aligned}V(k,h+1) = -\frac{1}{h+1}\Bigg( &(k+3)(k+2)(k+1).V(k+3,h) \\ &-6\sum_{r=0}^{k}\sum_{s=0}^{h}V(r,h-s)(k-r+1,s) \\ &-6\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)V(k-r+1,s) \\ &-\frac{15}{2}\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(r+1)V(r+1,h-s-p) \\ &\times V(t,s)V(k-r-t,p)\Bigg) \end{aligned} \tag{20b}$$

Here the DTM approximation solution in a series form can be written as follows:

$$\begin{aligned}u(x,t) = &-0.005525-0.0000000316508756t^2+2.226288000 \\ &*10^{-20}t^3+0.000001236333333xt-1.27*10^{-19}xt^2 \\ &+0.000000000095xt^3+0.000077x^2 \\ &+0.00000000305x^2t^2-2.994678336*10^{-21}x^2t^3 \\ &+0.00000001733x^3t-1.54204000*10^{-20}x^3t^2 \\ &-3.2620688*10^{-12}x^3t^3-0.000000510416x^4 \\ &-5.2*10^{-20}x^4t-8.22*10^{-11}x^4t^2 \\ &+7.9650*10^{-23}x^4t^3-5.030216466*10^{-10}x^5t \\ &+5.623*10^{-22}x^5t^2+3.7887*10^{-14}x^5t^3+2.87*10^{-9}x^6 \\ &-3.22*10^{-22}x^6t+1.243*10^{-12}x^6t^2-4.653*10^{-25}x^6t^3\end{aligned} \tag{21a}$$

$$v(x,t) = 0.11 - 0.00000585624t^2 - 3.682*10^{-20}t^3 + 0.000036366666xt - 2.8*10^{-18}xt^2 - 0.0000000054xt^3 - 0.0005x^2 - 0.000000115x^2t^2 - 3.25*10^{-20}x^2t^3 - 9.21*10^{-7}x^3t - 4.488*10^{-20}x^3t^2 + 0.000000009x^3t^3 + 0.000002083333x^4 + 1.4*10^{-18}x^4t + 2.035*10^{-9}x^4t^2 + 1.4*10^{-18}x^4t^1 + 2.96*10^{-21}x^4t^3 + 1.099*10^{-8}x^5t + 1.2676*10^{-20}x^5t^2 - 2.16*10^{-13}x^5t^3 - 8.47*10^{-9}x^6 + 4.584*10^{-20}x^6t - 2.42*10^{-11}x^6t^2 - 9.031*10^{-23}x^6t^3 \quad (21b)$$

### *3.2.2. Reduced Differential Transformation Method (RDTM)*

Considering Eq. (10) and utilizing the basic properties of the reduced differential transformation listed in Table 1, the transformed form of Eq. (24) can be rewritten as follows:

$$u_{k+1}(x) = \frac{-1}{(k+1)}\left(+\frac{\partial^3}{\partial x^3}u_k(x) + \frac{3}{2}\sum_{r=0}^{k}\left(\frac{\partial^3}{\partial x^3}v_r(x)\cdot v_{k-r}(x)\right) + \frac{9}{2}\sum_{r=0}^{k}\left(\frac{\partial}{\partial x}v_r(x)\cdot\frac{\partial^2}{\partial x^2}v_{k-r}(x)\right) - 6\sum_{r=0}^{k}\left(u_r(x)\frac{\partial}{\partial x}u_{k-r}(x)\right) - 6\sum_{m=0}^{k}u_{k-m}(x)\sum_{l=0}^{m}\left(\frac{\partial}{\partial x}v_{m-l}(x)\cdot v_l(x)\right) - \frac{3}{2}\sum_{m=0}^{k}\frac{\partial}{\partial x}u_{k-m}(x)\sum_{l=0}^{m}v_l(x)\cdot v_{m-l}(x)\right) \quad (22a)$$

$$v_{s+1}(x) = -\frac{1}{(s+1)}\left(+\frac{\partial^3}{\partial x^3}v_s(x) - 6\cdot\sum_{r=0}^{s}\frac{\partial}{\partial x}u_r(x)\cdot v_{s-r}(x) - 6\sum_{r=0}^{s}\left(u_{s-r}(x)\frac{\partial}{\partial x}v_r(x)\right) - \frac{15}{2}\sum_{m=0}^{s}\frac{\partial}{\partial x}v_{s-m}(x)\sum_{l=0}^{m}v_l(x)\cdot v_{m-l}(x)\right) \quad (22b)$$

Considering the initial conditions presented in Eq. (19b) the following solutions can be obtained.

Regarding the basic definition of RDTM the two terms of RDTM approximation solution in a series form of Jaulent–Miodek equation equations are written as follows:

$$u(x,t) = \frac{1}{\cosh(0.1x)^{14}}(2*10^{-19}(-2.5*10^{15}\cosh(0.1x)^{13} + 1.2375*10^{16}\cosh(0.1x)^{14} - 3.75*10^{16}\cosh(0.1x)^{12} - 1.125*10^{14}\sinh(0.1x)t\cosh(0.1x)^9 + 1.2875*10^{12}\sinh(0.1x)t\cosh(0.1x)^{12} + 3.7875*10^{13}\sinh(0.1x)t\cosh(0.1x)^{11} - 2.625*10^{13}\sinh(0.1x)t\cosh(0.1x)^{10} + 1.203805*10^8\sinh(0.1x)t^3\cosh(0.1x)^{11} + 3.73858435*10^{11}\sinh(0.1x)t^3\cosh(0.1x)^9 + 1.93698855*10^{10}\sinh(0.1x)t^3\cosh(0.1x)^{10} + 56000\sinh(0.1x)t^3\cosh(0.1x)^{12} - 6.20303888*10^{12}\sinh(0.1x)t^3\cosh(0.1x)^7 - 4.900043304*10^{11}\sinh(0.1x)t^3\cosh(0.1x)^8 + 1.921786988*10^{13}\sinh(0.1x)t*\cosh(0.1x)^5 + 1.789542647*10^{12}\sinh(0.1\,x)t^3\cosh(0.1x)^6 - 8.17167418*10^{12}t^4 + 2.60606275*10^{10}t^2\cosh(0.1x) - 3.29281250*10^8t^2\cosh(0.1x) + 1.342263656*10^{13}t^2\cosh(0.1x)^{10} + 1.752227312*10^{12}t^2\cosh(0.1x)^{11} - 5.51469375*10^{13}t^2\cosh(0.1x)^8 - 8.7441375*10^{12}t^2\cosh(0.1x)^9 - 4.107169404*10^{11}t^4\cosh(0.1x)^8 + 4.29981662*10^{12}t^4\cosh(0.1x)^6 + 2.848915952*10^{11}t^4\cosh(0.1x)^7 + 1.84595*10^5t^4\cosh(0.1x)^{12} - 2.03673633*10^{10}t^4\cosh(0.1x)^9 + 1.36898025*10^8t^4\cosh(0.1x)^{11} + 5.95991855*10^9t^4\cosh(0.1x)^{10} - 39.t^4\cosh(0.1x)^{13} - 1.439397676*10^{13}t^4\cosh(0.1x)^4 - 1.105811235*10^{12}t^4\cosh(0.1x)^5 - 1.52211375*10^{12}\sinh(0.1x)t^3\cosh(0.1x)^4 - 1.497377812*10^{13}\sinh(0.1x)t^3\cosh(0.1x)^3 + 1.565767818*10^{12}t^4\cosh(0.1x)^3 - 7.288171975*10^{11}t^4\cosh(0.1x) + 7.599375*10^{12}t^2\cosh(0.1x)^7 + 4.4465625*10^{13}t\cosh(0.1x)^6 + 1.863769604*10^{13}t^4\cosh(0.1x)^2)) \quad (23a)$$

$$v(x,t) = \frac{1}{\cosh(0.1x)^{13}}(2*10^{-19}(5*10^{16}\cosh(0.1x)^{13} + 5*10^{17}\cosh(0.1x)^{12} + 3.75*10^{15}\sinh(0.1x)t\cosh(0.1x)^9 - 2.275*10^{14}\sinh(0.1x)t\cosh(0.1x)^{11} + 7.5*10^{14}\sinh(0.1x)t\cosh(0.1x)^{10} - 7.6325*10^6\sinh(0.1x)t^3\cosh(0.1x)^{11} + 2.8886716*10^{11}\sinh(0.1x)t*\cosh(0.1x)^9 + 2.4802575*10^9\sinh(0.1x)t^3\cosh(0.1x)^{10} - 8.908625638*10^{12}\sinh(0.1x)t^3\cosh(0.1x)^7 - 3.719945925*10^{11}\sinh(0.1x)t^3\cosh(0.1x)^8 + 3.567654562*10^{13}\sinh(0.1x)t^3\cosh(0.1x)^5 + 2.25349425*10^{12}\sinh(0.1x)t^3\cosh(0.1x)^6 - 1.275936553*10^{13}t^4 + 5.130625*10^{10}t^2\cosh(0.1x)^{12} + 3.984613750*10^{13}t\cosh(0.1x)^{10} + 1.5135*10^{12}t^2\cosh(0.1x)^{11}$$

$$-1.817025*10^{14}t^2\cosh(0.1x)^8$$
$$-6.77025*10^{12}t^2\cosh(0.1x)^9$$
$$-2.939789592*10^{11}t^4\cosh(0.1x)^8$$
$$+4.525726444*10^{12}t^4\cosh(0.1x)^6$$
$$+2.346510165*10^{11}t^4\cosh(0.1x)^7-6400.t^4\cosh(0.1x)^{12}$$
$$-8.70814199*10^9t^4\cosh(0.1x)^9$$
$$+2.7465011*10^6t^4\cosh(0.1x)^{11}$$
$$+1.5363349*10^9t^4\cosh(0.1x)$$
$$-1.843515477*10^{13}t^4\cosh(0.1x)^4$$
$$-1.203548872*10^{12}t^4\cosh(0.1x)^5$$
$$-2.42505*10^{12}\sinh(0.1x)t^3\cosh(0.1x)^4$$
$$-3.1968*10^{13}\sinh(0.1x)t^3\cosh(0.1x)^3$$
$$+1.990444746*10^{12}t^4\cosh(0.1x)^3$$
$$-1.023166856*10^{12}t^4\cosh(0.1x)$$
$$+5.625*10^{12}t^2\cosh(0.1x)^7+1.54125*10^{14}t^2\cosh(0.1x)^6$$
$$+2.687748267*10^{13}t^4\cosh(0.1x)^2)) \quad (23b)$$

In Fig. 3 the comparison between these two methods and the exact one will be illustrated through the figures and tables as follows. As it is shown below, the comparison between RDTM and DTM shows the significant difference between the numerical results of these methods which indicates the accuracy of RDTM. In prior works a new after treatment method called Padé Approximation of DTMis studied to reduce the error of such method.

Through the solution of Whitham–Broer–Kaup equation, it is found that due to the weakness of nonlinearity, Pade approximation can have positive effect in precision, yet in the figure below it is shown that in spite of this approximation, divergence will occur in curve moving far from origin.

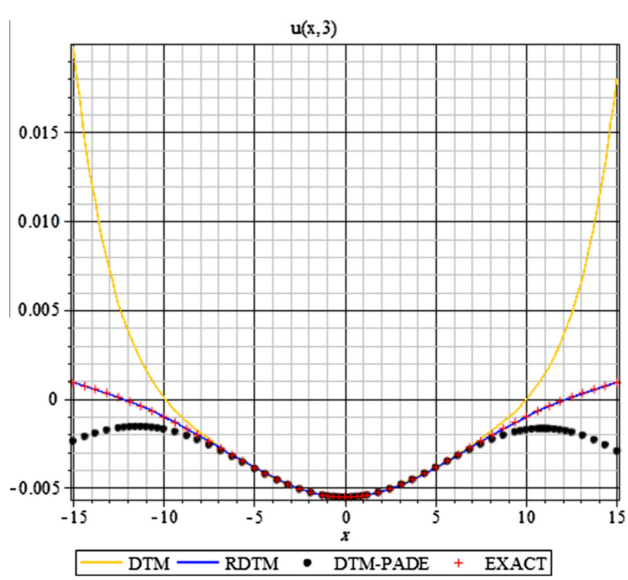
In the following figure, it is clearly illustrated that by passing further in time RDTM remains accurate and precise in comparison with exact solution (see Fig. 4).

In order to have a better evaluation between proposed methods, Tables 2 and 3 are presented to demonstrate the variations in solutions within different times and places which confirm previous results.

## 4. Conclusion

In this paper, the basic ideas of two-dimensional differential transformation method (DTM), its Padé approximate and its reduced form (RDTM) have been presented and applied to obtain numeric/analytical solutions of Jaulent–Miodek coupled equations and coupled Whitham–Broer–Kaup equations. The established results exhibit that RDTM is really accurate, capable and convergent technique and it compares extremely well with the exact solution.

Comparing various results obtained in this study, clearly demonstrated that in different nonlinear problems as well as Wave propagation and Acoustic applications in ocean engineering which consist of coupled equations with strong nonlinearity, DTM can have serious inaccuracy which leads to incorrect prediction of response, while RDTM positively confirmed great applicability in equations with both weak and strong nonlinearity, where the convergence of the result in the first 3–4 terms is completely precise. Using finite element method (FEM), boundary element method (BEM) and CFD are very common in ocean engineering but present results show that the Reduced differential transformation method (RDTM) could be good alternative.



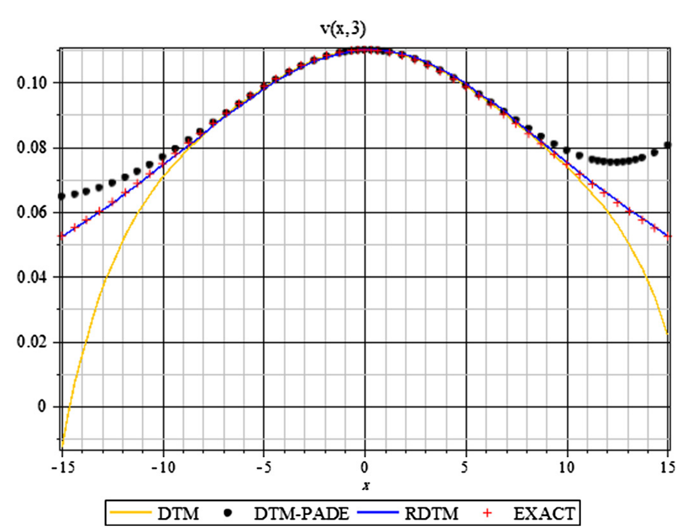




**Fig. 3** (a) and (b) The comparison between results of these methods for $u(x,t)$ and $v(x,t)$ respectively when $c_2 = b_0 = 0.01$, $t = 3$.

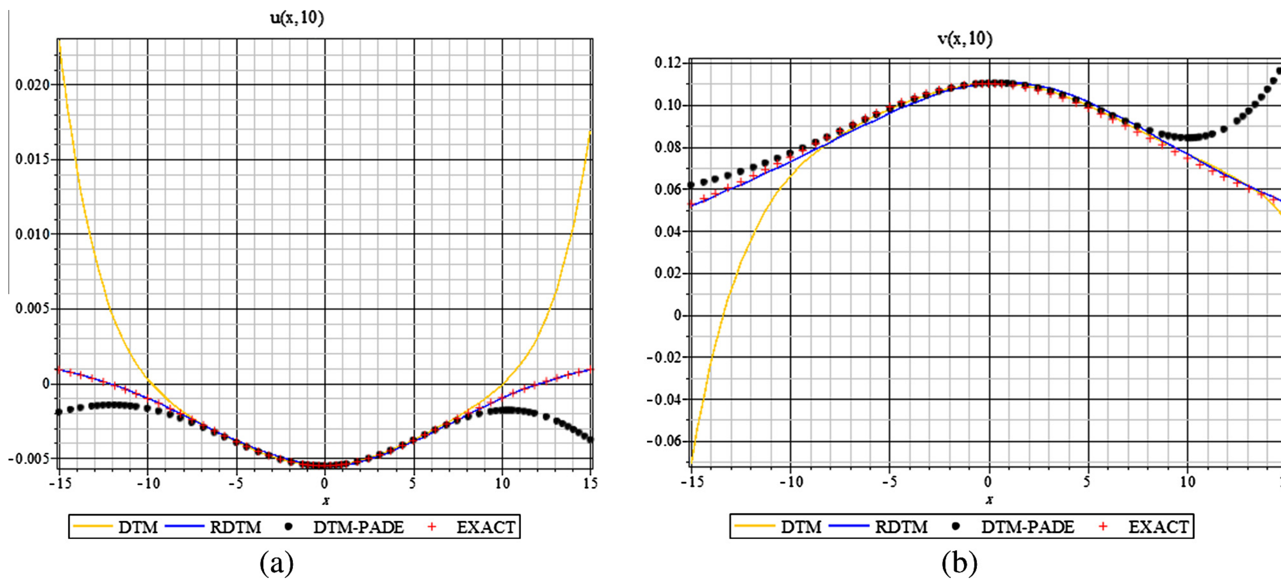



**Fig. 4** (a) and (b) The comparison between results of these methods for $u(x,t)$ and $v(x,t)$ respectively when $c_2 = b_0 = 0.01,\ t = 10$.

**Table 3** Comparing absolute errors for $v(x,t)$ when $c_2 = b_0 = 0.01,\ t = 10$.

| $x$ | $\lvert dtm - exact\rvert$ | $\lvert dtm/pade - exact\rvert$ | $\lvert rdtm - exact\rvert$ | Exact |
|---|---|---|---|---|
| 0 | 0.00028402898 | 0.000284028931 | 0.000006700998 | 0.1099985955 |
| 5 | 0.00128437969 | 0.001419815780 | 0.000293679592 | 0.098846397708 |
| 10 | 0.00200490799 | 0.00964398844 | 0.000202903099 | 0.07454399109 |
| 15 | 0.00746206532 | 0.06898341749 | 0.000081341524 | 0.05230605412 |
| 20 | 0.1765371557 | 0.2099535314 | 0.000026816697 | 0.03644473553 |

## Appendix A. Basic definition of Padé Approximant

In this section, a new method which is after treatment technique will be presented. A Padé approximant is the ratio of two polynomials gathered from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Padé approximants can be written in the following form:

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)} \tag{24}$$

where $P_L(x)$ is polynomial of degree at most $L$ and $Q_M(x)$ is a polynomial of degree at most $M$.

Coefficient of $P_L(x)$ and $Q_M(x)$ can be determined by the following formal power series:

$$y(x) = \sum_{i=1}^{\infty} a_i x^i \tag{25a}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O\left(x^{L+M+1}\right) \tag{25b}$$

Due to the fact that the numerator and denominator can be clearly multiplied by a constant and $[L/M]$ can be leaved unchanged, the normalization condition will be required as follows:

$$Q_M(0) = 1.0 \tag{26}$$

Finally, by imposing that $P_L(x)$ and $Q_M(x)$ have non-common factors, If the coefficient of $P_L(x)$ and $Q_M(x)$ be given by:

$$\begin{cases} P_L(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_L x^L, \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \ldots q_M x^M \end{cases} \tag{27}$$

Then by substituting $Q_M(x)$ into Eq. (8), the coefficient equations can be linearized.

$$\begin{cases} a_{L+1} + a_L q_1 + \ldots + a_{L-M} q_M = 0, \\ q_{L+2} + q_{L+1} q_1 + \ldots + a_{L-M+2} q_M = 0, \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + \ldots + a_L q_M = 0, \end{cases} \tag{28}$$

$$\begin{cases} a_0 = p_0, \\ a_0 + a_0 q_1 + \ldots = p_1 \\ a_L + a_{L-1} q_1 + \ldots + a_0 q_L = p_L \end{cases} \tag{29}$$

In order to resolve these equations, by computing Eq. (11) and substituting that into Eq. (12) the explicit formula can be successfully given from the unknowns. And if Eqs. (11) and (12) are nonsingular, then these equations can be obtained as follows:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\begin{bmatrix} a_{L-M+1} a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L a_{L+1} & \cdots & a_{L+M} \\ x^M x^{M-1} & & 1 \end{bmatrix}} \tag{30}$$

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