Convergence analysis of a monotone method for fourth-order semilinear elliptic boundary value problems

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Abstract

This work is concerned with the convergence of a monotone method for fourth-order semilinear elliptic boundary value problems. A comparison result for the rate of convergence is given. The global error is analyzed, and some sufficient conditions are formulated for guaranteeing a geometric rate of convergence.

Keywords: Fourth-order elliptic equations; Monotone method; Rate of convergence; Global error

1. Introduction

Boundary value problems of fourth-order differential equations arise frequently in applications, and have been given considerable attention. The earlier works are mostly devoted to two-point boundary value problems (cf. [1–8,10,12,19,21]). In recent years, attention has been given to fourth-order elliptic boundary value problems in multidimensional domains (cf. [11,13–15]). Let $\Omega$ be a simply connected...
bounded domain in $\mathbb{R}^n$, with a smooth boundary $\partial \Omega$. We consider a fourth-order semilinear elliptic boundary value problem of the form

$$\begin{align*}
\Delta(k(x)\Delta u) &= f(x, u, \Delta u), \quad x \in \Omega, \\
B[u] &= g_1(x), \quad B[k\Delta u] = g_2(x), \quad x \in \partial \Omega,
\end{align*}$$

(1.1)

where $\Delta$ is the Laplace operator and $B$ is the linear boundary operator given by

$$B[w] = w \quad \text{(Dirichlet type)}$$

(1.2)

or

$$B[w] = \frac{\partial w}{\partial v} + \beta(\cdot)w, \quad \beta(x) \geq 0 \text{ on } \partial \Omega, \quad \beta \in C^\infty(\partial \Omega) \quad \text{(Neumann or Robin type)},$$

(1.3)

with $\partial/\partial v$ denoting the outward normal derivative on $\partial \Omega$. It is assumed that $k(x)$ is a strictly positive $C^2$-function on $\overline{\Omega} \equiv \Omega \cup \partial \Omega$, $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R})$ and $g_i \in C^\infty(\partial \Omega)$ for $i = 1, 2$. When $n = 1$ and $k(x) \equiv 1$, problem (1.1) with the boundary condition (1.2) describes the static deflection of an elastic bending beam (with hinged ends) under a possible nonlinear loading (cf. [10,20]). It also describes the steady state of a prototype equation for phase transitions in condensed matter systems (cf. [9,22]). When $n = 2$, a physical interpretation of (1.1) is that it governs the static deflection of a plate under a lateral loading. Here $k(x)$ is the stiffness of the plate, $g_1(x)$ and $g_2(x)$ are possible boundary sources, and $f(x, u, \Delta u)$ is the loading function, which may depend on the deflection and the curvature of the plate (cf. [20]).

The literature dealing with the problem (1.1) is extensive, and most of the discussions are concerned with the existence, uniqueness, and multiplicity of solutions. However, the existence proof given in [15] is based on the monotone iterative method, which can be discretized and implemented numerically by some numerical methods. In [16] and [18], e.g., a finite difference–monotone iterative method is used to solve the problem (1.1) numerically. Nevertheless, we have not yet seen any global error analysis results in the literature. In practical implementation, it is very important to give a quantifiable rate of convergence: the faster, the more desirable. The purpose of this work is to carry out the error analysis of the monotone method for problem (1.1) and formulate some conditions for guaranteeing a geometric rate of convergence. The technique in this work can be extended to the other boundary value problems.

We state our main results in Section 2, and the proofs of the results are given in Section 3.

2. The main results

A pair of functions $\tilde{u}, \hat{u} \in C^4(\Omega) \cap C^2(\overline{\Omega})$ are called coupled upper and lower solutions of (1.1) if $\tilde{u} \geq \hat{u}$, $\Delta \tilde{u} \leq \Delta \hat{u}$ and

$$\begin{align*}
\left\{ \Delta(k(x)\Delta \tilde{u}) &\geq f(x, u, \Delta \tilde{u}), \quad \Delta(k(x)\Delta \hat{u}) \leq f(x, u, \Delta \hat{u}), \quad x \in \Omega, \quad \text{for all } \tilde{u} \leq u \leq \hat{u}, \\
B[\tilde{u}] &\geq g_1(x) \geq B[\hat{u}], \quad B[k\Delta \tilde{u}] \leq g_2(x) \leq B[k\Delta \hat{u}], \quad x \in \partial \Omega.
\end{align*}$$

(2.1)

Following [15], we let $v = -k\Delta u$ and write (1.1) in the equivalent form

$$\begin{align*}
-vu = v/k, \quad -\Delta v &= f(x, u, -v/k), \quad x \in \Omega, \\
B[u] &= g_1(x), \quad B[v] = -g_2(x), \quad x \in \partial \Omega.
\end{align*}$$

(2.2)

It is obvious that $u$ is a solution of (1.1) if and only if $(u, v)$ is a solution of (2.2). The monotone method for (1.1) is based on the coupled system (2.2). Let $\tilde{v} = -k\Delta \tilde{u}$ and $\hat{v} = -k\Delta \hat{u}$. It is easy to see from
(2.1) that the pair \((\hat{u}, \hat{v}), (\hat{u}, \hat{v})\) satisfy the relation
\[
\begin{aligned}
-\Delta \hat{u} &\geq v/k, \quad -\Delta \hat{v} \geq f(x, u, -\hat{v}/k), \quad x \in \Omega, \\
-\Delta \hat{u} \leq v/k, \quad -\Delta \hat{v} \leq f(x, u, -\hat{v}/k), \quad x \in \Omega, \\
\text{for all } (\hat{u}, \hat{v}) \leq (u, v) \leq (\hat{u}, \hat{v}), \\
B[\hat{u}] \geq g_1(x) \geq B[\hat{v}], \quad B[\hat{v}] \geq -g_2(x) \geq B[\hat{v}], \quad x \in \partial \Omega.
\end{aligned}
\]

(2.3)

Let \(\tilde{u} = (\hat{u}, \hat{v})\) and \(\hat{u} = (\hat{u}, \hat{v})\). We define the sectors
\[
S(\hat{u}, \hat{u}) = \{(u, v); u, v \in C^2(\Omega), (\hat{u}, \hat{u}) \leq (u, v) \leq (\hat{u}, \hat{u})\},
\]
\[
(\hat{u}, \hat{u}) = \{(u, v); u, v \in C^2(\Omega), \tilde{u} \leq (u, v) \leq \hat{u}\}.
\]

The monotone method for (2.2) depends on the monotone property of the function \(f(x, u, v)\) and a pair of nonnegative constants \((\gamma_1^*, \gamma_2^*)\) (both are nonzero when \(\beta(x)\) in (1.3) is identically zero). Specifically, we have:

(i) if \(f(x, u, v)\) is monotone nondecreasing in \(u\) for all \((x, u, v) \in \Omega \times S(\hat{u}, \hat{u})\), we let
\[
(u^{(0)}, v^{(0)}) = (\hat{u}, \hat{v}) \quad \text{(resp. } (u^{(0)}, v^{(0)}) = (\hat{u}, \hat{v})\text{)}
\]

and denote by \((\tilde{u}^{(m)}, \tilde{v}^{(m)})\) (resp. \((\tilde{u}^{(m)}, \tilde{v}^{(m)})\) \((m = 1, 2, \ldots)\) the corresponding iterations from
\[
\begin{aligned}
(\gamma_1^* - \Delta)u^{(m)} &= \gamma_1^* u^{(m-1)} + v^{(m-1)}/k, \quad x \in \Omega, \\
(\gamma_2^* - \Delta)v^{(m)} &= \gamma_2^* v^{(m-1)} + \frac{f(x, u^{(m-1)}, -v^{(m-1)})}{k}, \quad x \in \Omega, \\
B[u^{(m)}] &= g_1(x), \quad B[v^{(m)}] = -g_2(x), \quad x \in \partial \Omega.
\end{aligned}
\]

(2.5)

(ii) if \(f(x, u, v)\) is monotone nonincreasing in \(u\) for all \((x, u, v) \in \Omega \times S(\hat{u}, \hat{u})\), we let
\[
\{\tilde{u}^{(0)}, \tilde{v}^{(0)}\} = (\hat{u}, \hat{v}), \quad \{u^{(0)}, v^{(0)}\} = (\hat{u}, \hat{v})
\]

and denote by \((\tilde{u}^{(m)}, \tilde{v}^{(m)}), (\tilde{u}^{(m)}, \tilde{v}^{(m)})\) the corresponding iterations from
\[
\begin{aligned}
(\gamma_1^* - \Delta)\tilde{u}^{(m)} &= \gamma_1^* \tilde{u}^{(m-1)} + \tilde{v}^{(m-1)}/k, \quad x \in \Omega, \\
(\gamma_2^* - \Delta)\tilde{v}^{(m)} &= \gamma_2^* \tilde{v}^{(m-1)} + \frac{f(x, \tilde{u}^{(m-1)}, -\tilde{v}^{(m-1)})}{k}, \quad x \in \Omega, \\
(\gamma_1^* - \Delta)\tilde{u}^{(m)} &= \gamma_1^* \tilde{u}^{(m-1)} + v^{(m-1)}/k, \quad x \in \Omega, \\
(\gamma_2^* - \Delta)\tilde{v}^{(m)} &= \gamma_2^* \tilde{v}^{(m-1)} + \frac{f(x, \tilde{u}^{(m-1)}, -\tilde{v}^{(m-1)})}{k}, \quad x \in \Omega, \\
B[\tilde{u}^{(m)}] &= B[\tilde{u}^{(m)}] = g_1(x), \quad B[\tilde{v}^{(m)}] = B[\tilde{v}^{(m)}] = -g_2(x), \quad x \in \partial \Omega.
\end{aligned}
\]

(2.7)

Henceforth, we assume that \((\gamma_1, \gamma_2)\) are a pair of constants such that
\[
\begin{aligned}
&g_v(x, u, v) \geq -\gamma_2, \quad \forall (x, u, v) \in \Omega \times (\tilde{u}, \tilde{u}), \quad \text{where } g(x, u, v) = f(x, u, -v/k), \ g_v \equiv \partial g/\partial v; \\
&\gamma_i \geq 0 \quad (i = 1, 2) \quad \text{and is not zero when } \beta(x) \text{ in (1.3) is identically zero.}
\end{aligned}
\]

\(H_1\)

In addition, we introduce the following notation:
\[
\lambda_0 = \text{the smallest eigenvalue of the eigenvalue problem: } \begin{cases}
\Delta \phi + \lambda \phi = 0 & \text{in } \Omega, \\
B[\phi] = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\[
\begin{aligned}
\overline{k} &= \max_{x \in \Omega} k(x), & \underline{k} &= \min_{x \in \Omega} k(x), \\
M_1 &= \max_{(x, u, v) \in Q} \left| f_u(x, u, v) \right|, & m_1 &= \min_{(x, u, v) \in Q} \left| f_u(x, u, v) \right|, \\
M_2 &= \max_{(x, u, v) \in Q} \left[ -k^{-1}(x) f_f(x, u, v) \right], & m_2 &= \min_{(x, u, v) \in Q} \left[ -k^{-1}(x) f_f(x, u, v) \right].
\end{aligned}
\]
where $Q = \Omega \times S(\bar{u}, \bar{v})$. Obviously, $\lambda_0 \geq 0$, and $\lambda_0 = 0$ if and only if $B$ corresponds to the homogeneous Neumann boundary condition. We have the following convergence theorem.

**Theorem 2.1** (See [15]). Let $\bar{u}$ and $\bar{v}$ be a pair of coupled upper and lower solutions of (1.1), and let hypothesis (H1) hold. For each type of monotonicity of $f$, the sequences $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ and $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ described in (2.4)–(2.7) with $(\gamma_1^*, \gamma_2^*) = (\gamma_1, \gamma_2)$ satisfy the monotone property

\[(\bar{u}, \bar{v}) \leq (\bar{u}^{(m)}, \bar{v}^{(m)}) \leq (\bar{u}^{(m+1)}, \bar{v}^{(m+1)}) \leq (\bar{u}^{(m+1)}, \bar{v}^{(m+1)})\]

\[\leq (\bar{u}, \bar{v}), \quad m = 0, 1, 2, \ldots,\]

(2.9)

and there exist functions $(u, v)$ and $(\bar{u}, \bar{v})$ satisfying $(u, v) \leq (\bar{u}, \bar{v})$ such that the sequences $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ and $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ converge to $(\bar{u}, \bar{v})$ and $(u, v)$, respectively, and any solution $(u, v)$ of (2.2) in $(\bar{u}, \bar{v})$ satisfies $(u, v) \leq (u, v) \leq (\bar{u}, \bar{v})$. Moreover, we have:

(i) if $f(x, u, v)$ is monotone nondecreasing in $u$ for all $(x, u, v) \in \Omega \times S(\bar{u}, \bar{v})$, then both $(\bar{u}, \bar{v})$ and $(u, v)$ are solutions of (2.2) in $(\bar{u}, \bar{v})$, and in addition if

\[\text{either } \lambda_0(\lambda_0 - M_2) > M_1/\bar{k} \text{ or } \lambda_0(\lambda_0 - m_2) < m_1/\bar{k}, \quad (H_2)\]

then $(\bar{u}, \bar{v}) = (u, v) = (u^*, v^*)$ and $(u^*, v^*)$ is the unique solution of (2.2) in $(\bar{u}, \bar{v})$;

(ii) if $f(x, u, v)$ is monotone nonincreasing in $u$ for all $(x, u, v) \in \Omega \times S(\bar{u}, \bar{v})$ and satisfies the condition (H2), then $(\bar{u}, \bar{v}) = (u, v) = (u^*, v^*)$ and $(u^*, v^*)$ is the unique solution of (2.2) in $(\bar{u}, \bar{v})$.

Although convergence is guaranteed in Theorem 2.1, the rate and global error are not known in general. We now give a comparison result for the rate of convergence and a global error analysis result. In addition, we formulate some conditions that are sufficient for guaranteeing a geometric rate of convergence. Throughout the work, we use $\|u\|_{\frac{s}{\Omega}}$ to denote the norm of a scalar function $u$ in the Sobolev space $H^s(\Omega)$, and use $\|(u_1, u_2, \ldots, u_n)\|_{\frac{s}{\Omega}} (n = 2$ or $4)$ to denote the norm of a vector-valued function $(u_1, u_2, \ldots, u_n)$ in the product space $H^s(\Omega) \times \cdots \times H^s(\Omega)$ taken $n$ times:

\[\|(u_1, u_2, \ldots, u_n)\|_{\frac{s}{\Omega}}^2 = \|u_1\|_{\frac{s}{\Omega}}^2 + \|u_2\|_{\frac{s}{\Omega}}^2 + \cdots + \|u_n\|_{\frac{s}{\Omega}}^2.\]

(2.10)

**Theorem 2.2.** Let the conditions of Theorem 2.1 be satisfied. Let $(\gamma_1', \gamma_2')$ be a pair of constants satisfying $\gamma_1' \geq \gamma_1$ and $\gamma_2' \geq \gamma_2$. Denote by $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ and $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ the sequences from (2.4)–(2.7) with $(\gamma_1', \gamma_2') = (\gamma_1, \gamma_2)$ and $(\gamma_1^*, \gamma_2^*) = (\gamma_1', \gamma_2')$, respectively. Then we have

\[(\bar{u}^{(m)}, \bar{v}^{(m)}) \leq (\bar{u}^{(m)}, \bar{v}^{(m)}), \quad (\bar{u}^{(m)}, \bar{v}^{(m)}) \geq (\bar{u}^{(m)}, \bar{v}^{(m)}), \quad m = 1, 2, \ldots.\]

(2.11)

The comparison result (2.11) shows that the rate of convergence of iterations from (2.4)–(2.7) depends on the constants $(\gamma_1^*, \gamma_2^*)$: the smaller $(\gamma_1^*, \gamma_2^*)$, the faster the convergence.

**Theorem 3.2.** Let the conditions of Theorem 2.1 be satisfied. Denote by $\{(\bar{u}^{(m)}, \bar{v}^{(m)})\}$ the sequences from (2.4)–(2.7) with $(\gamma_1^*, \gamma_2^*) = (\gamma_1, \gamma_2)$, and denote by $(u^*, v^*)$ the unique solution of (2.2) in $(\bar{u}, \bar{v})$. Let $(\hat{e}_1^{(m)}, \hat{e}_2^{(m)}) = (\bar{u}^{(m)} - u^*, \bar{v}^{(m)} - v^*)$ and $(\hat{e}_1^{(m)}, \hat{e}_2^{(m)}) = (u^* - \bar{u}^{(m)}, v^* - \bar{v}^{(m)})$. Then we have:

(i) if

\[\max(M_1, M_2 + 1/\bar{k}) < \lambda_0,\]

(2.12)
then
\[ \| \varepsilon_1^{(m)} + \varepsilon_2^{(m)} + \varepsilon_1^{(m)} + \varepsilon_2^{(m)} \|_{0, \Omega} \leq \rho_1^m \| \varepsilon_1^{(0)} + \varepsilon_2^{(0)} + \varepsilon_1^{(0)} + \varepsilon_2^{(0)} \|_{0, \Omega}, \quad m = 1, 2, \ldots, \] (2.13)

where
\[ \rho_1 = \frac{\max(\gamma_1, \gamma_2) + \max(M_1, M_2 + 1/k)}{2\lambda_0 + \max(\gamma_1, \gamma_2) - \max(M_1, M_2 + 1/k)} < 1; \] (2.14)

(ii) if
\[ \max(M_1, M_2 + 1/k) + \max(M_1 + M_2, 1/k) < 2\lambda_0, \] (2.15)

then
\[ \| (\varepsilon_1^{(m)}, \varepsilon_2^{(m)}, \varepsilon_1^{(m)}, \varepsilon_2^{(m)}) \|_{0, \Omega} \leq \rho_2^m \| (\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \varepsilon_1^{(0)}, \varepsilon_2^{(0)}) \|_{0, \Omega}, \quad m = 1, 2, \ldots, \] (2.16)

where
\[ \rho_2 = \frac{\max(\gamma_1, \gamma_2) + \max(M_1, M_2 + 1/k)}{2\lambda_0 + \max(\gamma_1, \gamma_2) - \max(M_1 + M_2, 1/k)} < 1; \] (2.17)

(iii) if \( f(x, u, v) \) is monotone nondecreasing in \( u \) for all \( (x, u, v) \in \Omega \times S(\hat{u}, \hat{v}) \), and if
\[ \lambda_0(\lambda_0 - M_2) > (M_1 + 1/k)^2/4, \] (2.18)

then there exists a positive constant \( \rho < 1 \) independent of \( m \) such that
\[ \| (\varepsilon_1^{(m)}, \varepsilon_2^{(m)}) \|_{0, \Omega} \leq \rho^m \| (\varepsilon_1^{(0)}, \varepsilon_2^{(0)}) \|_{0, \Omega}, \quad \| (\xi_1^{(m)}, \xi_2^{(m)}) \|_{0, \Omega} \leq \rho^m \| (\xi_1^{(0)}, \xi_2^{(0)}) \|_{0, \Omega}, \\
m = 1, 2, \ldots. \] (2.19)

Any one of the error estimates (2.13), (2.16) and (2.19) guarantees a geometric rate of convergence of iterations from (2.4)–(2.7).

3. Proofs of the main results

Proof of Theorem 2.2. We prove (2.11) only for the case where \( f(x, u, v) \) is monotone nonincreasing in \( u \) for all \( (x, u, v) \in \Omega \times S(\hat{u}, \hat{v}) \). The proof for monotone nondecreasing \( f \) is similar.

Let \( (\overline{w}^{(m)}, \overline{z}^{(m)}) = (\overline{w}^{(m)} - \overline{u}^{(m)}, \overline{v}^{(m)} - \overline{u}^{(m)}) \) and \( (\underline{w}^{(m)}, \underline{z}^{(m)}) = (\underline{u}^{(m)} - \underline{u}^{(m)}, \underline{v}^{(m)} - \underline{v}^{(m)}) \). Then by (2.7),

\[
\begin{align*}
(\gamma_1 - \Delta)\overline{w}^{(m)} &= \gamma_1(\overline{w}^{(m-1)} - \overline{w}^{(m-1)}) + (\gamma_1 - \gamma_1)(\overline{w}^{(m-1)} - \overline{w}^{(m-1)}) + (\overline{v}^{(m-1)} - \overline{v}^{(m-1)})/k, \\
(\gamma_2 - \Delta)\overline{z}^{(m)} &= (\gamma_2 - \gamma_2)(\overline{v}^{(m-1)} - \overline{v}^{(m-1)}) + F_{\sigma}(x, \overline{u}^{(m-1)}, \overline{v}^{(m-1)}) - F_{\sigma}(x, \overline{u}^{(m-1)}, \overline{v}^{(m-1)}), \\
(\gamma_1 - \Delta)\underline{w}^{(m)} &= \gamma_1(\underline{w}^{(m-1)} - \underline{w}^{(m-1)}) + (\gamma_1 - \gamma_1)(\underline{w}^{(m-1)} - \underline{w}^{(m-1)}) + (\underline{v}^{(m-1)} - \underline{v}^{(m-1)})/k, \\
(\gamma_2 - \Delta)\underline{z}^{(m)} &= (\gamma_2 - \gamma_2)(\underline{v}^{(m-1)} - \underline{v}^{(m-1)}) + F_{\sigma}(x, \underline{u}^{(m-1)}, \underline{v}^{(m-1)}) - F_{\sigma}(x, \underline{u}^{(m-1)}, \underline{v}^{(m-1)}), \\
B[\overline{w}^{(m)}] &= B[\overline{z}^{(m)}] = B[\underline{w}^{(m)}] = B[\underline{z}^{(m)}] = 0,
\end{align*}
\] (3.1)

where \( F_{\sigma}(x, u, v) = \sigma v + f(x, u, -v/k) \) for some constant \( \sigma \). Since the initial iterations are the same, we have from (3.1), the monotone property (2.9) and the comparison principle for the Laplace operator (see [17]) that \( \overline{w}^{(1)} \geq 0, \overline{z}^{(1)} \geq 0, \underline{w}^{(1)} \geq 0 \) and \( \underline{z}^{(1)} \geq 0 \), which prove (2.11) for \( m = 1 \). Using
the monotone nonincreasing property of $f$ in $u$, the monotone property (2.9) and hypothesis (H1), an induction argument shows that the relation (2.11) holds for all $m = 1, 2, \ldots$ 

\[ \Box \]

**Proof of Theorem 2.3.** We first note that, by the monotone property (2.9), $\xi_i^{(m)} \geq 0$ and $\xi_i^{(m)} \geq 0$ for $i = 1, 2$. Let $\gamma = \max(\gamma_1, \gamma_2)$. In view of the comparison result in Theorem 2.2, it suffices to prove the conclusions for the case of $(\gamma^*_1, \gamma^*_2) = (\gamma, \gamma)$.

**Proof of (i).** Consider the case where $f(x, u, v)$ is monotone nonincreasing in $u$ for all $(x, u, v) \in \Omega \times S(\tilde{u}, \tilde{u})$. By (2.2) and (2.7),

\[
\begin{align*}
(\gamma - \Delta)\xi_1^{(m)} &= \gamma \xi_1^{(m-1)} + \xi_2^{(m-1)} / k, & x \in \Omega, \\
(\gamma - \Delta)\xi_2^{(m)} &= \gamma \xi_2^{(m-1)} + f(x, u^{(m-1)}, -\tilde{v}^{(m-1)}) / k - f(x, u^*, -v^*) / k, & x \in \Omega, \\
(\gamma - \Delta)\xi_1^{(m)} &= \gamma \xi_1^{(m-1)} + \xi_2^{(m-1)} / k, & x \in \Omega, \\
(\gamma - \Delta)\xi_2^{(m)} &= \gamma \xi_2^{(m-1)} + f(x, u^*, -v^*) / k - f(x, \tilde{u}^{(m-1)}, -\tilde{v}^{(m-1)}) / k, & x \in \Omega, \\
B[\xi_1^{(m)}] = B[\xi_2^{(m)}] &= 0, & x \in \partial \Omega.
\end{align*}
\]

Let $e^{(m)} = \xi_1^{(m)} + \xi_2^{(m)} + \xi_1^{(m)} + \xi_2^{(m)}$. Summing the above relations, applying the mean-value theorem and using the notation in (2.8) yields

\[ (\gamma - \Delta)e^{(m)} \leq (\gamma + \max(M_1, M_2 + 1/k))e^{(m-1)}. \]

Multiplying (3.3) by $e^{(m)}$ and integrating by parts, we obtain

\[
-\int_{\partial \Omega} \frac{\partial e^{(m)}}{\partial v} e^{(m)} d\sigma + \int_{\Omega} |\nabla e^{(m)}|^2 dx + \gamma \int_{\Omega} |e^{(m)}|^2 dx \\
\leq (\gamma + \max(M_1, M_2 + 1/k)) \int_{\Omega} e^{(m)} e^{(m-1)} dx.
\]

Since

\[ \lambda_0 = \inf_{v \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} |v|^2 dx} \quad \text{(Dirichlet type)}, \quad \lambda_0 = 0 \quad \text{(Neumann type)}, \]

\[ \lambda_0 = \inf_{v \in H^1(\Omega)} \left[ \left( \int_{\partial \Omega} \beta |v|^2 d\sigma + \int_{\Omega} |\nabla v|^2 dx \right) / \int_{\Omega} |v|^2 dx \right] \quad \text{(Robin type)}, \]

we obtain from (3.4) that

\[ (\lambda_0 + \gamma) \int_{\Omega} |e^{(m)}|^2 dx \leq (\gamma + \max(M_1, M_2 + 1/k)) \int_{\Omega} e^{(m)} e^{(m-1)} dx \\
\leq [(\gamma + \max(M_1, M_2 + 1/k))/2] \int_{\Omega} (|e^{(m)}|^2 + |e^{(m-1)}|^2) dx. \]

This implies $\|e^{(m)}\|_{0, \Omega} \leq \rho_1^m \|e^{(0)}\|_{0, \Omega}$ which proves (2.13). The proof for the case of monotone nondecreasing $f$ is similar.

**Proof of (ii).** Consider only the case where $f(x, u, v)$ is monotone nonincreasing in $u$ for all $(x, u, v) \in \Omega \times S(\tilde{u}, \tilde{u})$. Multiplying the corresponding equation in (3.2) by $\xi_i^{(m)}$ and $\xi_i^{(m)} (i = 1, 2)$,
using the same argument for (3.6) and then summing the resulting relations, we get

\[
(\lambda_0 + \gamma) \int_{\Omega} \sum_{i=1}^{2} (|e_i^{(m)}|^2 + |e_i^{(m-1)}|^2) dx \leq [(\gamma + \max(M_1 + M_2, 1/k))/2] \times \int_{\Omega} \sum_{i=1}^{2} (|e_i^{(m)}|^2 + |e_i^{(m-1)}|^2) dx + [(\gamma + \max(M_1, M_2 + 1/k))/2] \times \int_{\Omega} \sum_{i=1}^{2} (|e_i^{(m-1)}|^2 + |e_i^{(m-1)}|^2) dx.
\]

From this relation, we conclude (2.16).

Proof of (iii). Let \((e_1^{(m)}, e_2^{(m)}) = (e_1^{(m)}, e_2^{(m)})\) or \((e_1^{(m)}, e_2^{(m)})\). An argument similar to that in the proof of (i) leads to

\[
(\lambda_0 + \gamma) \int_{\Omega} (|e_1^{(m)}|^2 + |e_2^{(m)}|^2) dx \leq \int_{\Omega} (e_1^{(m)}, e_2^{(m)}) A (e_1^{(m-1)}, e_2^{(m-1)})^T dx,
\]

where

\[
A = \begin{bmatrix} \gamma & 1/k \\ M_1 & \gamma + M_2 \end{bmatrix}.
\]

A simple matrix calculation shows that the \(l^2\)-norm of \(A\) is given by

\[
\|A\|_2 = \frac{1}{2} \left( \sqrt{(2\gamma + M_2)^2 + (M_1 - 1/k)^2} + \sqrt{M_2^2 + (M_1 + 1/k)^2} \right).
\]

We have that \((\lambda_0 + \gamma)^{-1} \|A\|_2 < 1\) if and only if

\[
4M_2\gamma + M^* + 2\sqrt{M_2^2 + (M_1 + 1/k)^2} \sqrt{(2\gamma + M_2)^2 + (M_1 - 1/k)^2} < 8\lambda_0\gamma + 4\lambda_0^2,
\]

where \(M^* = 2M_2^2 + (M_1 - 1/k)^2 + (M_1 + 1/k)^2\). Since the condition (2.18) is equivalent to \(4M_2 + 4\sqrt{M_2^2 + (M_1 + 1/k)^2} < 8\lambda_0\), there exists a sufficiently large \(\overline{\gamma}\) such that for all \(\gamma \geq \overline{\gamma}, (3.8)\), i.e., \((\lambda_0 + \gamma)^{-1} \|A\|_2 < 1\), holds. Therefore for any \(\gamma \geq \overline{\gamma}\), we have from (3.7) that there exists a positive constant \(\rho < 1\) independent of \(m\) such that

\[
\int_{\Omega} (|e_1^{(m)}|^2 + |e_2^{(m)}|^2) dx \leq \rho \|e_1^{(m)}, e_2^{(m)}\|_{0,\Omega} \|e_1^{(m-1)}, e_2^{(m-1)}\|_{0,\Omega}, \quad m = 1, 2, \ldots
\]

which implies

\[
\|e_1^{(m)}, e_2^{(m)}\|_{0,\Omega} \leq \rho^m \|e_1^{(0)}, e_2^{(0)}\|_{0,\Omega}, \quad m = 1, 2, \ldots.
\]

If \(\gamma < \overline{\gamma}\), we have from Theorem 2.2 that the above estimate is also true. This proves (2.19). □

References