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The Index of Operators on Foliated Bundles

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We compute the equivariant cohomology Connes–Karoubi character of the index of elliptic operators along the leaves of the foliation of a flat bundle. The proof uses techniques similar to those developed in Algebraic Topology, in the study of non-commutative algebras. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let (V, π), $F \to V \xrightarrow{\pi} B$ be a smooth fiber bundle with fiber F of dimension q. We assume that (V, π) is endowed with a flat connection corresponding to an integrable subbundle $\mathscr{F} \subset TV$, of dimension $n = \dim(B)$, transverse at any point to the fibers of π. The pair $(V, \mathscr{F})$ is a foliation.

The purpose of this paper to study invariants of differential operators along the leaves of the above foliation. The index of an elliptic operator along the leaves of the foliation $\mathscr{F}$ is an element in the K-Theory group $K^0(\mathscr{F}) = K_0(\Psi^{-\infty}(\mathscr{F}))$ where $\Psi^{-\infty}(\mathscr{F})$ is the algebra of regularizing operators along the leaves. In the case of a foliated bundle there exists a Connes–Karoubi character $\mathrm{Ch}\colon K_0(\Psi^{-\infty}(\mathscr{F})) \to H_\Gamma^{*-q}(F, \mathcal{O})'$ to the dual of equivariant cohomology with twisted coefficients, where Γ is the fundamental group of the base B acting on the fiber F via holonomy, and $\mathcal{O}$ is the orientation sheaf. Our main theorem computes the Connes–Karoubi character of the index. This amounts to a proof of the "higher index theorem for foliations" in this special case. A very general higher index theorem for foliations can be found in [C3] and here we give a new proof of this theorem for flat bundles. Some very interesting results related to the results in this paper are contained in [CM2] where Diff-invariant structures are treated in detail. See also [C4]. In the even more special case of a family of elliptic operators our theorem recovers the computation of

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the Chern character of the index bundle of a family of elliptic operators [AS].

The problem that we consider was suggested by [DHK]. The proof of our theorem is based on the Cuntz–Quillen exact sequence [CQ] and the results in [BN] and [N3]. The second paper identifies the cyclic cohomology groups with geometric groups. The third paper provides us with the axiomatic setting necessary to deal with index problems in the framework of cyclic cohomology.

Let $P(\partial)=\sum a_\alpha \partial^\alpha$ be the local expression of an elliptic operator on the base B, acting between the sections of two vector bundles. We lift each vector field ∂_i on B to a vector field ∇_i on the total space V of our flat bundle. This will allow us to construct the lift $P(\nabla)$ which will be an example of an elliptic differential operator along the leaves of $\mathcal{F}$. In this rather degenerate case the invariants for $P(\nabla)$ reduce to invariants of $P(\partial)$. However not all operators that we consider arise in this way, actually very few do. The non-multiplicativity of the signature [A] is related to the phenomena that we investigate.

2. STATEMENT OF THE PROBLEM

Consider a smooth foliation $\mathcal{F}$ of smooth manifold V. All the structures that will be used in this paper will be smooth, i.e., C^∞, so that we shall omit "smooth" in the following. We think of the foliation $(V, \mathcal{F})$ as an integrable subbundle $\mathcal{F} \subset TV$. That is, $\mathcal{F}$ identifies with the tangent bundle to the foliation.

By considering only differentiations along the fibers of $\mathcal{F}$ one obtains longitudinal differential operators. In analogy with manifolds, one can proceed then to define longitudinal pseudodifferential operators, denoted $\Psi^p(\mathcal{F})$. A good reference to these constructions is [MoS]. An alternative description of these algebras for foliations coming from flat bundles is given in the next section.

The algebra $\Psi^{-\infty}(\mathcal{F})$ of regularizing operators along $\mathcal{F}$ is usually referred to as the "algebra of the foliation". If, by abuse of notation, we use the symbol $\mathcal{F}$ to denote the graph of the foliation $(V, \mathcal{F})$ as well [W] then $\Psi^{-\infty}(\mathcal{F})$ identifies with the algebra $C_c^\infty(\mathcal{F})$ of compactly supported smooth kernels on the graph.

We review here, in order to fix notation, the construction of the graph of the foliation $(V, \mathcal{F})$. It consists of equivalence classes of triples (x, y, γ), where $x, y \in V$ are on the same leaf and γ is a path from x to y completely contained in that leaf. The equivalence is given by "holonomy". The graph is a smooth manifold, usually non Hausdorff.

As in the classical case, the principal symbol induces an isomorphism

$$\sigma_0 \colon \Psi^0(\mathscr{F})/\Psi^{-1}(\mathscr{F}) \to C_c^\infty(S^*\mathscr{F})$$

where $S^*\mathscr{F}$ is the unit sphere bundle in the dual $\mathscr{F}^*$ of $\mathscr{F}$. The notion of asymptotic expansion generalizes as well, and this shows that a matrix of order 0 pseudodifferential operators is invertible modulo regularizing operators if and only if its principal symbol is invertible. From this we infer that σ_0 induces an isomorphism

$$\sigma_0 * : K_1^{\mathrm{top}}(\Psi^0(\mathscr{F})/\Psi^{-\infty}(\mathscr{F})) \to K_1^{\mathrm{top}}(C_c^\infty(S^*\mathscr{F})),$$

where K_i^{top} is the quotient of K_i^{alg} with respect to homotopy ($i=0$ or $i=1$).

The most general form of the index problem for foliation is:

(FOL-ALG) Determine the algebraic K-theory boundary (index) map

$$\mathrm{Ind} \colon K_1^{\mathrm{alg}}(\Psi^0(\mathscr{F})/\Psi^{-\infty}(\mathscr{F})) \to K_0^{\mathrm{alg}}(\Psi^{-\infty}(\mathscr{F})).$$

(A very closed related problem is obtained by considering topological K-theory.)

The major difficulty is that we know very little about the K_0-group involved. (The K_1-groups are relatively easy to determine.)

Denote by $HC_i^{\mathrm{per}}(A)$, $i \in \mathbb{Z}/2\mathbb{Z}$, the periodic cyclic homology groups of an arbitrary complex algebra A, and by

$$\mathrm{Ch} \colon K_i^{\mathrm{alg}}(A) \to HC_i^{\mathrm{per}}(A)$$

the Connes–Karoubi character [C2, K, L, Q]. One way to avoid the above difficulty is to ask for the Connes–Karoubi character of the index.

(FOL-COH) Compute the composition

$$\mathrm{Ch} \circ \mathrm{Ind} \colon K_1^{\mathrm{top}}(C_c^\infty(S^*\mathscr{F})) \to HC_0^{\mathrm{per}}(\Psi^{-\infty}(\mathscr{F})).$$

A formula for $\mathrm{Ch} \circ \mathrm{Ind}$ will be called a “cohomological index theorem”.

The cohomological form of the problem is not just a simplification of the original problem, but it also brings a new perspective. This is because what we usually want is $\mathrm{Ch}(\mathrm{Ind}[u])$, not $\mathrm{Ind}[u]$ itself. Also this form of the problem makes the connection with the characteristic classes of foliations, as we shall see bellow.

The actual definition of the various cyclic homology groups will *not* be necessary for our purposes. What will matter will be that they exist, and that they satisfy certain general properties. This is very similar to the philosophy of Algebraic Topology, especially in the axiomatic approach due to Eilenberg and Steenrod.

Let us begin by explaining some of the constructions in a particular but suggestive case. Let A be an algebra and let $\tau\colon A \to \mathbb{C}$ be a trace (i.e., $\tau(xy) = \tau(yx)$). The map that associates to any idempotent $e = (e_{ij}) \in M_n(A)$ the number $\tau(e) = \Sigma\tau(e_{ii}) \in \mathbb{C}$ factors to a morphism

$$\tau_*\colon K_0^{\mathrm{alg}}(A) \to \mathbb{C}.$$

In general any trace τ defines a class $[\tau] \in HC^0_{\mathrm{per}}(A)$ and there exists a pairing $\langle\ ,\ \rangle$ between cycle homology and cyclic cohomology such that

$$\tau_*(e) = \langle \mathrm{Ch}([e]), [\tau] \rangle.$$

We shall call the elements of $HC^i_{\mathrm{per}}(A)$ higher traces.

Any homology invariant measure μ on a foliation $(V, \mathscr{F})$ determines a trace $\tau_\mu\colon \Psi^{-\infty}(\mathscr{F}) \to \mathbb{C}$. The quantity $\tau_\mu(\mathrm{Ind}[u])$ was determined by Connes in [C1]. In view of what we said above this amounts to a partial determination of $\mathrm{Ch}(\mathrm{Ind}[u])$.

We now very briefly review the most important properties of periodic cyclic (co)homology to be used in the following.

(1) The groups $HC_i^{\mathrm{per}}(A)$ and $HC^i_{\mathrm{per}}(A)$, $i \in \mathbb{Z}_2$ are covariant (resp. contravariant) functors on the category of complex locally convex algebras with continuous algebra morphisms. If $f\colon A \to B$ is an algebra morphism then we denote by f_* and, respectively, f^* the induced morphisms.

(2) There is a pairing $\langle -, - \rangle\colon HC_i^{\mathrm{per}}(A) \otimes HC^i_{\mathrm{per}}(A) \to \mathbb{C}$.

(3)

$$HC_i^{\mathrm{per}}(\mathbb{C}) \simeq \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \end{cases}$$

such that $HC^0_{\mathrm{per}}(\mathbb{C})$ is generated by the identity map (trace).

$$HC_i^{\mathrm{per}}(\mathbb{C}[\mathbb{Z}]) \simeq \mathbb{C} \qquad \text{for any} \quad i \in \mathbb{Z}/2\mathbb{Z},$$

where for any group Γ we denote by $\mathbb{C}[\Gamma]$ its group algebra.

(4) Consider a separated étale groupoids $\mathscr{G}$ [BN], that is, $\mathscr{G}$ is a small category together with manifold structures on $\mathscr{G}^{(0)} \overset{\mathrm{def}}{=} Ob(\mathscr{G})$ and $\mathscr{G}^{(1)} \overset{\mathrm{def}}{=} Mor(\mathscr{G})$ such that all morphisms are invertible, all structural maps are smooth and the domain and range are local diffeomorphisms. Let $B\mathscr{G}$ be the geometric realization of the nerve of $\mathscr{G}$ (this is Grothendieck's classifying space of the topological category $\mathscr{G}$). Also denote by $\mathscr{O}(\mathscr{G})$ the complex orientation sheaf of $B\mathscr{G}$ (this is defined because $\mathscr{G}$ is étale) and denote

by q the common dimension of $\mathscr{G}^{(0)}$ and $\mathscr{G}^{(1)}$. The main result of [BN] establishes the existence of an injective map

$$\Phi\colon \bigoplus_{i+j\equiv q(2)} H^j(B\mathscr{G}, \mathscr{O}(\mathscr{G})) \to HC^i_{\mathrm{per}}(C_c^\infty(\mathscr{G})),$$

where $C_c^\infty(\mathscr{G})$ is endowed with the natural topology and the convolution product. The morphism Φ is multiplicative and functorial with respect to *étale* morphisms [N3] that are one-to-one on units. We will call Φ *the geometric map*. The morphism Φ is multiplicative, $\Phi(\xi\times\zeta)=\Phi(\xi)\otimes\Phi(\zeta)$ where $\times$ is the external product in cohomology and $\otimes$ is the external product in periodic cyclic cohomology [N2, N3].

(4′) We are going to make more explicit the constructions of (4) in the case of interest for us. Let Γ be a discrete group acting on a manifold X. Define $\mathscr{G}$ by $\mathscr{G}^{(0)}=X$, $\mathscr{G}^{(1)}=X\times\Gamma$ with $domain(x,\gamma)=x$, $range(x,\gamma)=\gamma\cdot x$. Then $C_c^\infty(\mathscr{G})$ is the (algebraic) crossed product algebra and $B\mathscr{G}=(X\times E\Gamma)/\Gamma=$ "the homotopy quotient $X//\Gamma$", here $\Gamma\to E\Gamma\to B\Gamma$ is the universal principal Γ-bundle. The map Φ of (4) becomes an injective map

$$\Phi\colon \bigoplus_{i+j\equiv \dim(X)(2)} H^j(X//\Gamma, \mathscr{O}(X)) \to HC^i_{\mathrm{per}}(C^\infty(X)\rtimes\Gamma)$$

(this is more precise form of some results in [N1]). If X is orientable and Γ preserves the orientation, then the left hand side reduces to equivariant cohomology: $H^*_\Gamma(X)=H^*(X//\Gamma)$. Moreover Φ is an isomorphism if Γ acts freely. Here $\mathscr{O}(M)$ denotes the complexified orientation sheaf on the smooth manifold M.

(5) (Excision) Any two-sided ideal I of a complex algebra A gives rise to a periodic six-term exact sequence of periodic sequence of periodic cyclic cohomology groups

$$\cdots\to HC^i_{\mathrm{per}}(A)\to HC^i_{\mathrm{per}}(I)\xrightarrow{\partial} HC^{i+1}_{\mathrm{per}}(A/I)\to HC^{i+1}_{\mathrm{per}}(A)\to\cdots$$

$i\in\mathbb{Z}_2$ similar to the topological K-theory exact sequence. Thus periodic cyclic cohomology defines a generalized cohomology theory for algebras [CQ, CQ1, CQ2]. This boundary map is multiplicative: If B is an other algebra and we denote by $\partial_{A\otimes B}$ the boundary map for the exact sequence corresponding to $I\otimes B\subset A\otimes B$ then we have $\partial_{A\otimes B}(x\otimes y)=\partial(x)\otimes y$ for any $x\in HC^*_{\mathrm{per}}(I)$ and any $y\in HC^*_{\mathrm{per}}(B)$. Here $\otimes$ denotes also the external product in cyclic cohomology.

(6) There is a functorial morphism

$$\mathrm{Ch}\colon K_i^{\mathrm{alg}}(A)\otimes\mathbb{C}\to HC_i^{\mathrm{per}}(A),\qquad i\in\{0,1\}$$

called the Connes–Karoubi character [C2, K] which is onto for $A = \mathbb{C}$ or $A = \mathbb{C}[\mathbb{Z}]$. For $A = C^\infty(X)$, Ch coincides with the classical Chern character up to rescaling [MiS].

(7) The boundary map in algebraic K-theory and periodic cyclic cohomology are compatible in the following sense

$$\langle \mathrm{Ind}[u], \xi \rangle = \langle \mathrm{Ch}[u], \partial \xi \rangle$$

for any $u \in K_1^{\mathrm{alg}}(A/I)$ and $\xi \in HC^0_{\mathrm{per}}(I)$ [N3]. Here we have denoted for simplicity $\langle \mathrm{Ch}(\mathrm{Ind}[u]), \xi \rangle = \langle \mathrm{Ind}[u], \xi \rangle$.

We now go back to our foliation $(V, \mathscr{F})$.

A complete transversal $N \subset V$ is a submanifold of dimension $q =$ the codimension of $\mathscr{F}$ which is transverse to the leaves and which intersects each leaf. Complete transversal always exist but they are usually not compact and not connected. The choice of a transversal N determines an étale groupoid by restriction: $\mathscr{G}(N) = \mathscr{F}|_N$. Explicitly $\mathscr{G}^{(0)}(N) = N$, $\mathscr{G}^{(1)}(N) = \{[x, y, \gamma)] \in \mathscr{F}, x, y \in N\}$.

The equivalence relation is easier to describe in this case. A path γ from x to y can be covered by distinguished coordinate patches and hence defines a diffeomorphism $\varphi_\gamma \colon N_x \to N_y$ from a neighborhood of x in N to a neighborhood of y. Then $[(x, y, \gamma)] = [(x_1, y_1, \gamma_1)]$ if and only if $x = x_1$, $y = y_1$ and $\varphi_\gamma = \varphi_{\gamma_1}$ in a (possibly smaller) neighborhood of x. It is known that the choice of N is not important because of Morita equivalence, which gives that $B\mathscr{G}(N)$ is homotopy equivalent to $B\mathscr{F}$ [Ha].

There is a map

$$HC^i_{\mathrm{per}}(C^\infty_c(\mathscr{G}(N)) \to HC^i_{\mathrm{per}}(\Psi^{-\infty}(\mathscr{F})))$$

which is also given by Morita equivalence and an inclusion

$$j \colon \Psi^{-\infty}(\mathscr{F}) \hookrightarrow C^\infty_c(\mathscr{G}(N)) \hat{\otimes} \mathscr{R}$$

(where $\mathscr{R} = \Psi^{-\infty}(\mathbb{R}^p)$, $p = \dim \mathscr{F}$) as a full corner [BGR].

Denote by q the codimension of $\mathscr{F} = \dim(N)$.

LEMMA 1. *The morphism*

$$\Phi_0 \colon H^i(B\mathscr{F}, \mathscr{O}(\mathscr{F})) \to HC^{i+q}_{\mathrm{per}}(\Psi^{-\infty}(\mathscr{F}))$$

given by the composition

$$H^i(B\mathscr{F}, \mathscr{O}(\mathscr{F})) \simeq H^i(B\mathscr{G}(N), \mathscr{O}) \xrightarrow{\Phi} HC^{i+q}_{\mathrm{per}}(C^\infty_c(\mathscr{G}(N)))$$
$$\to HC^{i+q}_{\mathrm{per}}(\Psi^{-\infty})(\mathscr{F})$$

does not depend on the choice of N.

Proof. For any complete transversal N the morphism

$$j: \Psi^{-\infty}(\mathcal{F}) \hookrightarrow C_c^\infty(\mathcal{G}(N)) \hat{\otimes} \mathcal{R}$$

depends on a partition of unity in such a way that any two such morphisms are conjugated by an inner automorphism.

Let N_1 and N_2 be two complete transversals. Choose a third transversal N' not intersecting N_1 and N_2. By considering $N_1' = N_1 \cup N'$ and $N_2' = N_2 \cup N'$ one can reduce the problem to the case when $N_1 \subset N_2$ and then use the remark in the beginning. ∎

Consider the continuous map $f: V \to B\mathcal{F}$ which classifies $\mathcal{F}$ [Ha]. Denote by $p: S^*\mathcal{F} \to V$ the canonical projection. The canonical orientation of the leaves of $S^*\mathcal{F}$ gives an identification of $\mathcal{O}(S^*\mathcal{F})$ with $(f \circ p)^* \mathcal{O}(\mathcal{F})$. We use this identification to define a morphism

$$(f \circ p)^*: H^*(B\mathcal{F}, \mathcal{O}(\mathcal{F})) \to H^*(S^*\mathcal{F}, \mathcal{O}(S^*\mathcal{F})).$$

In the following statement we are going to use the notation

$$\Phi_0: H^i(B\mathcal{F}, \mathcal{O}(\mathcal{F})) \to H^{i+q}_{\text{per}}(\Psi^{-\infty}(\mathcal{F}))$$

for the morphism defined in the previous lemma. Also Ind will denote the boundary map in topological K-theory (as in FOL-TOP), and $\mathcal{T}(\mathcal{F}^*)$ will be the Todd class of $\mathcal{F}^* = \operatorname{Hom}_{\mathbb{R}}(\mathcal{F}, \mathbb{C})$.

Index formula problem. Let $\mathcal{F}$ be a foliation of dimension n and codimension q, and let $f: V \to B\mathcal{F}$, $p: S^*\mathcal{F} \to V$ and Φ_0 be as above. Then for any $u \in K^1(S^*\mathcal{F})$ and $\xi \in H^{2m}(B\mathcal{F}, \mathcal{O}(\mathcal{F}))$ we have the following index formula

$$\langle \operatorname{Ind}[u], \Phi_0(\xi) \rangle = \frac{(-1)^n}{(2\pi i)^m} \langle \operatorname{ch}[u]\, p^*(\mathcal{T}(\mathcal{F}^*)\, f^*(\xi)), [S^*\mathcal{F}] \rangle$$

where ch is the classical Chern character.

The above formula, if correct, would identify the morphism ℓ in Connes' higher index theorem for foliations [C3].

3. THE INDEX THEOREM FOR FOLIATED BUNDLES

In case $\mathcal{F}$ is actually a fiber bundle the formula in the above problem becomes the formula for the Chern character of the index-bundle of a family. If ξ corresponds to a homology invariant measure this is Connes' theorem explained in the previous section. For invariant forms in general it is a theorem of Heitsch [He]. Previous results for flat bundles were obtained in [DHK, J, MN] and for foliations in general but particular cocyles in [NT]. Except for families of elliptic operators, the proofs of all the other circumstances when the index theorem for foliations is known are based on heat-kernels. Our proof marks a departure from this approach.

Consider a smooth fiber bundle (V, π), $0 \to F \to V \xrightarrow{\pi} B \to 0$, with fiber F of dimension q. Assume that (V, π) is endowed with a flat connection that we interpret as an integrable subbundle $\mathscr{F} \subset TV$, of dimension $n = \dim(B)$, transverse at any point to the fibers. The pair $(V, \mathscr{F})$ defines a foliation.

Fix $b_0 \in B$ and denote by $\Gamma = \pi_1(B, b_0)$. Flat fiber bundles $(V\pi, \mathscr{F})$ with fiber $F = \pi^{-1}(b_0)$, $b_0 \in B$, are in a one-to-one correspondence with group morphisms $\Gamma \to \mathrm{Diff}(F)$. Denote by $(\gamma, x) \to \gamma x$ the corresponding action. Then $V = \tilde{B} \times_\Gamma F$ where Γ acts on the right by deck transformation on the universal covering space $\tilde{B}$ of B.

Denote by $\Psi_c^p = \Psi_c^p(\tilde{B})$ the space of compactly supported order p classical pseudodifferential operators on $\tilde{B}$. Compactly supported means that their Schwartz kernel is a compactly supported distribution on $\tilde{B} \times \tilde{B}$.

Our proof of the index theorem for flat bundles will use an alternative description of the various algebras associated to the foliation in terms of certain crossed products. If the groups Γ acts on an algebra A_0 then the *algebraic crossed product* $A_0 \rtimes \Gamma$ is by definition the linear span of formal products $\{a\gamma, a \in A_0, \gamma \in \Gamma\}$ with the product rule $(a\gamma)(a'\gamma') = a\gamma(a')\gamma\gamma'$.

The group Γ acts on the spaces $\Psi_c^p = \Psi_c^p(\tilde{B})$ so we obtain exact sequences

$$0 \to \Psi_c^{-\infty} \rtimes \Gamma \to \Psi_c^0 \rtimes \Gamma \to (\Psi_c^0 / \Psi_c^{-\infty}) \rtimes \Gamma \to 0$$

$$0 \to \Psi_c^{-1} \rtimes \Gamma \to \Psi_c^0 \rtimes \Gamma \xrightarrow{\sigma_0} C_c^\infty(S^*\tilde{B}) \rtimes \Gamma \to 0$$

where σ_0 is the principal symbol map and $S^*\tilde{B} \subset T^*\tilde{B}$ is the cosphere bundle of $\tilde{B}$.

By a standard procedure we enlarge the algebra Ψ_c^0 to include all $(n+1)$-summable Schatten–von Neumann operators. Explicitly, denote by

$$C_{n+1} = C_{n+1}(L^2(\tilde{B})) = \{T, tr(T^*T)^{(n+1)/2} < \infty\}$$

where T denotes a bounded operator on $L^2(\tilde{B})$ and tr is the usual trace.

It is a simple known fact that $\Psi_c^{-1} \subset C_{n+1}$ and that $\Psi_c^0 \cap C_{n+1} \subset \Psi_c^{-1}$. This tells us that if we define $E = E(\tilde{B}) = \psi_c^0 + C_{n+1}$ we have an exact sequence

$$0 \to C_{n+1} \to E \xrightarrow{\sigma_0} C_c^\infty(S^*\tilde{B}) \to 0$$

considered also in [N2, N3].

Fix $2m \geqslant n+1$ and denote by $Tr_m \in HC^{2m}(C_{n+1})$ the continuous cocycle

$$Tr_m(a_0, ..., a_{2m}) = \frac{(-1)^m\, m!}{(2m)!}\, tr(a_0, \cdots a_m).$$

The normalization factor is chosen such that $Tr_m|_{C_1} = S^m tr$, where S is the Connes periodicity operator. By abuse of notation we denote by $Tr \in HC^0_{\mathrm{per}}(C_{n+1})$ the class of Tr_m for any $2m \geqslant n+1$.

Denote by $H^*(X)$ the $\mathbb{Z}/2\mathbb{Z}$-periodic complex cohomology groups of the a manifold X.

Lemma 2. *We have that Φ induces an isomorphism*

$$H^{*-1}(S^*B) \simeq HC^*_{\mathrm{per}}(C_c^\infty(S^*\tilde{B}) \rtimes \Gamma), \qquad i \in \mathbb{Z}/2\mathbb{Z}.$$

Proof. Since Γ acts freely on the oriented odd dimensional manifold $S^*\tilde{B}$ and $(S^*\tilde{B})/\Gamma = S^*B$ then we have that the map Φ is an isomorphism, see [BN]. ∎

Recall [N3] that there is a $HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$ module structure on $HC^*_{\mathrm{per}}(A \rtimes \Gamma)$ induced by the $\mathbb{C}[\Gamma]$-coalgebra structure of $A \rtimes \Gamma$

$$A \rtimes \Gamma \ni a\gamma \xrightarrow{\delta} a\gamma \otimes \gamma \in (A \rtimes \Gamma) \otimes \mathbb{C}[\Gamma],$$

thus $xy = \delta^*(x \otimes y)$ if $y \in HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$ and $x \in HC^*_{\mathrm{per}}(A \rtimes \Gamma)$.

Denote by $g\colon S^*B \to B\Gamma$ the classifying map of the covering $\Gamma \to S^*\tilde{B} \to S^*B$. Also recall [C2] that $H^*(B\Gamma)$ is a direct summand of $HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$. Denote by

$$r_0\colon HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma]) \to H^*(B\Gamma)$$

the natural projection. It is a ring morphism satisfying $r_0 \circ \Phi = id$ where $\Phi\colon H^*(B\Gamma) \to HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$ is the geometric map.

Lemma 3. *The $HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$-module structure of $HC^*_{\mathrm{per}}(C_c^\infty(S^*\tilde{B}) \rtimes \Gamma)$ is given, using the isomorphism of the previous lemma, by $g^* \circ r_0$: $\Phi(\zeta)\,\xi = \Phi(\zeta g^*(r_0(\xi)))$ for any $\zeta \in H^{*-1}(S^*B)$ and $\xi \in HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$. In particular $\Phi(\zeta)\,\Phi(\eta) = \Phi(\zeta g^*(\eta))$.*

Proof. We know that the action of $HC^*_{\mathrm{per}}(\mathbb{C}[\Gamma])$ factors through r_0 because Γ acts without fixed points on $S^*\tilde{B}$. This shows that we can assume $\xi = \Phi(\eta)$ for $\eta = r_0(\xi)$, $\eta \in H^*(B\Gamma)$. The module structure is obtained using the multiplicativity of Φ and the fact that the composition

$$S^*B = S^*\tilde{B}//\Gamma \to S^*\tilde{B}//(\Gamma \times \Gamma) = S^*B \times B\Gamma$$

corresponding to δ is $id \times g$, by definition. We then have

$$\Phi(\zeta)\,\Phi(\eta) = \delta^*(\Phi(\zeta) \otimes \Phi(\eta)) = \Phi(\zeta g^*(\eta)). \quad \blacksquare$$

PROPOSITION. *Suppose the graph of $(V, \mathscr{F})$ is separated where V, F, B, and $E = E(\tilde{B})$ are as above. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc}
0 \to & \Psi^{-1}(\mathscr{F}) & \to & \Psi^0(\mathscr{F}) & \to & C_c^\infty(S^*\mathscr{F}) & \to 0 \\
& \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & \\
0 \to & C_{n+1} \hat{\otimes} C_c^\infty(F) \rtimes \Gamma & \to & (E \hat{\otimes} C_0^\infty(F)) \rtimes \Gamma & \to & C_c^\infty(S^*\tilde{B} \times F) \rtimes \Gamma & \to 0
\end{array}$$

where α is an isomorphism onto $e(E \hat{\otimes} C_0^\infty(F)) \rtimes \Gamma)\, e$ for some idempotent e and β induces an isomorphism in cyclic cohomology.

Proof. This is the promised equivalent definition of the various algebras associated to $(V, \mathscr{F})$ in the particular case of a foliated bundle. The idempotent e is defined using a standard argument based on partitions of unity as follows.

Choose a partition of unity on B subordinated to a finite trivializing cover $(U_i)_{i=1,N}$ of $\tilde{B}$. We can find smooth real functions φ_i, $\sum_i \varphi_i^2 = 1$, with support in U_i. Choose *disjoint* open sets $V_i \subset \tilde{B}$ on which the projection $\tilde{B} \to B$ is a diffeomorphism. Denote by $\gamma_{i,j}$ the corresponding 2-cocyle with values in Γ and by $\tilde{\varphi}_i$ the lifts of φ_i to functions supported on V_i, so that $\tilde{\varphi}_i \tilde{\varphi}_j = 0$ if $i \neq j$. Then $e = \sum_{i,j} \tilde{\varphi}_i \gamma_{i,j} \tilde{\varphi}_j$ is the idempotent e we were looking for. Note that if B is not compact $\gamma_{i,j}$ are going to be locally constant functions on $U_i \cap U_j$, not constant in general, so e will have entries in the multiplier algebra of $E(\tilde{B})$ and not in $E(\tilde{B})$ itself. $\blacksquare$

Consider now the exact sequence

$$0 \to C_{n+1} \rtimes \Gamma \to E \rtimes \Gamma \to C_c^\infty(S^*\tilde{B}) \rtimes \Gamma \to 0.$$

We want to identify the Cuntz–Quillen boundary map

$$\partial_{E \rtimes \Gamma}\colon HC^*_{\mathrm{per}}(C_{n+1} \rtimes \Gamma) \to HC^{*+1}_{\mathrm{per}}(C_c^\infty(S^*\tilde{B}) \rtimes \Gamma) \simeq H^*(S^*B)$$

of this exact sequence.

Observe that since Γ acts by inner automorphisms on C_{n+1} we have that $C_{n+1} \rtimes \Gamma \cong C_{n+1} \otimes \mathbb{C}[\Gamma]$.

Define the *Index characteristic class* $\mathscr{I}(M) \in H^{\mathrm{even}}(S^*M)$ of a smooth manifold M to be rescaled Todd class of T^*M, $\mathscr{I}(M)_{2k} = (2\pi\imath)^{n-k}\, p^*\mathscr{T}(T^*M \otimes \mathbb{C})_{2k}$ where $\mathscr{T}$ is the usual Todd class and $n = \dim(M)$.

LEMMA 4. *Let* $\xi \in H^j(B\Gamma)$, $n = \dim(B)$, *then*

$$\partial_{E \rtimes \Gamma}(Tr \otimes \Phi(\xi)) = (-1)^n \, \Phi(\mathcal{I}(B) \, g^*(\xi)) \in HC^{j+1}(C_c^\infty(S^*B))$$

where $\mathcal{I}$ *is the Index class and* $\Phi\colon H^*(S^*B) \to HC^{*+1}(C_c^\infty(S^*B))$ *is as in the previous section.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \to & C_{n+1} & \to & E(B) & \to & C_c^\infty(S^*B) & \to 0 \\
 & \uparrow{\scriptstyle \alpha_0'} & & \uparrow{\scriptstyle \alpha'} & & \uparrow{\scriptstyle =} & \\
0 \to & e(C_{n+1} \rtimes \Gamma)\, e & \to & \mathfrak{A} & \to & C_c^\infty(S^*B) & \to 0 \\
 & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & \\
0 \to & C_{n+1} \rtimes \Gamma & \to & E(\tilde{B}) \rtimes \Gamma & \to & C_c^\infty(S^*\tilde{B}) \rtimes \Gamma & \to 0
\end{array}$$

where $\mathfrak{A} = e(E(\tilde{B}) \rtimes \Gamma)\, e$, for an idempotent e implementing the Morita equivalence, and α is the inclusion. The idempotent e is defined as in the previous proposition. The morphisms α' is defined using the natural representation of $E(\tilde{B}) \rtimes \Gamma$ on $L^2(\tilde{B})$ given by the fact that the action of Γ is implemented by inner automorphisms. This shows that the restriction α_0' is given by the composition

$$e(C_{n+1} \rtimes \Gamma)\, e \to C_{n+1} \rtimes \Gamma \simeq C_{n+1} \otimes \mathbb{C}[\Gamma] \xrightarrow{1 \otimes \chi} C_{n+1}$$

where $\chi\colon \mathbb{C}[\Gamma] \to \mathbb{C}$ is the augmentation morphism $\gamma \to 1$.

The above commutative diagram has the property that

$$\beta^*\colon HC^*_{\text{per}}(C_c^\infty(S^*\tilde{B}) \rtimes \Gamma) \to HC^*_{\text{per}}(C_c^\infty(S^*B))$$

is the isomorphism of Lemma 2 and that α_0^* is an isomorphism as well. Also, if we regard the augmentation morphism $\chi\colon \mathbb{C}[\Gamma] \to \mathbb{C}$ as a trace (an element of cyclic cohomology) then $\alpha_0^*(Tr \otimes \chi) = \alpha'^*_0(Tr)$, and χ is the identity of $HC^*_{\text{per}}(\mathbb{C}[\Gamma])$. This gives that $(Tr \otimes \chi)\, \Phi(\xi) = Tr \otimes \Phi(\xi)$. The boundary ∂_{AS} of the top exact sequence is determined by the Atiyah–Singer formula which gives, using the results of [N3], $\partial_{AS}(Tr) = (-1)^n \, \Phi(\mathcal{I}(B))$. Denote by ∂_2 the boundary map of the middle exact sequence. Using the $HC^*_{\text{per}}(\mathbb{C}[\Gamma])$-linearity on ∂_2 and $\partial_{E \rtimes \Gamma}$, see [Ni3], we obtain

$$\begin{aligned}
\beta^* \circ \partial_{E \rtimes \Gamma}(Tr \otimes \Phi(\xi)) &= \partial_2 \circ \alpha_0^*(Tr \otimes \chi)\, \Phi(\xi) = \partial_2 \circ \alpha_0'^*(Tr)\, \Phi(\xi) \\
&= \partial_{AS}(Tr)\, \Phi(\xi)
\end{aligned}$$

Since $\partial_{AS}(Tr)\, \Phi(\xi) = (-1)^n \, \Phi(\mathcal{I}(B))\, \Phi(\xi) = (-1)^n \, \Phi(\mathcal{I}(B)\, g^*(\xi))$, using the formula from the previous lemma, the result follows. ∎

It is interesting to observe that the mere existence of the top commutative diagram in the previous proof implies a theorem of Atiyah and Singer [A]. The lemma is equivalent to the higher index theorem for coverings of Connes and Moscovici [CM1, N3].

Suppose now that the foliation $(V, \mathscr{F})$ defined at the beginning of this section by a flat connection of the bundle $\pi: V \to B$ has a separated graph. This is equivalent to the following condition: the only $\gamma \in \Gamma$ for which F^{γ} has a nonempty interior are those γ that act trivially on F. Here F denotes the fiber of $V \to B$ as before.

Consider now the exact sequence

$$0 \to C_{n+1} \hat{\otimes} C_c^{\infty}(F) \rtimes \Gamma \to E(\mathscr{F}) \xrightarrow{\sigma_0} C_c^{\infty}(S^*\mathscr{F}) \to 0$$

induced by the morphism β in the previous proposition. Denote by

$$\partial_{\mathscr{F}}: HC^*_{\mathrm{per}}(C_{n+1} \hat{\otimes} (F) \rtimes \Gamma) \to HC^{*+1}_{\mathrm{per}}(C_c^{\infty}(S^*\mathscr{F})) = H^{*+q}(S^*\mathscr{F}, \mathcal{O}(S^*\mathscr{F}))$$

where q is the codimension of $\mathscr{F}$, and $\mathcal{O}(S^*\mathscr{F})$ is the orientation sheaf of $S^*\mathscr{F}$.

In the particular case of $(V, \mathscr{F})$ that we are studying, the classifying space $B\mathscr{F}$ of the graph of the foliation coincides, up to homotopy, with the homotopy quotient $F//\Gamma$. The map $f: V \to B\mathscr{F}$ can be described as the second component in the composition $(id, f): V = (\tilde{B} \times F)/\Gamma \to (\tilde{B} \times F)//(\Gamma \times \Gamma) = B \times (F//\Gamma)$. This map satisfies $\mathcal{O}(S^*\mathscr{F}) \simeq p^*f^*\mathcal{O}(F)$ where the isomorphism depends on a choice of the orientation of the leaves of $S^*\mathscr{F}$, and $p: S^*\mathscr{F} \to V$ is the projection. We orient the leaves of $S^*\mathscr{F}$ as the boundary of an almost complex manifold.

Also recall from the previous section that there exist maps

$$\Phi: H^*(F//\Gamma, \mathcal{O}(F)) \to HC^{*+q}_{\mathrm{per}}(C_c^{\infty}(F) \rtimes \Gamma)$$

and

$$\Phi_0: H^*(B\mathscr{F}, \mathcal{O}(\mathscr{F})) \to HC^{*+q}_{\mathrm{per}}(\Psi^{-\infty}(\mathscr{F}))$$

where $q = \dim(F)$. In the case we discuss now, that of a foliated bundle, these two maps are related by $Tr \otimes \Phi(\xi) = \Phi_0(\xi)$.

Denote by $\pi_0: S^*\mathscr{F} \to S^*B$ the projection covering $\pi: V \to B$ and let $\mathscr{I}(\mathscr{F}) = \pi_0^*(\mathscr{I}(B)) \in H^*(S^*\mathscr{F})$ be the Index class of $\mathscr{F}$.

THEOREM. *Let $\xi \in H^*(F//\Gamma, \mathcal{O}(F))$ and $u \in K_1(C^{\infty}(S^*\mathscr{F}))$ be an invertible symbol, also let $f: V \to B\mathscr{F} = F//\Gamma$ and $p: S^*\mathscr{F} \to V$ be as above, $f_0 = f \circ p$. Then*

$$\langle \Phi_0(\xi), \mathrm{Ind}[u] \rangle = (-1)^n \langle \mathrm{Ch}[u]\, \mathscr{I}(\mathscr{F})\, f_0^*(\xi), [S^*\mathscr{F}] \rangle$$

where $\mathrm{Ind}\colon K_1^{\mathrm{alg}}(C^\infty(S^*\mathscr{F})) \to K_0^{\mathrm{alg}}(C_{n+1} \otimes C^\infty(F) \rtimes \Gamma)$ *is the boundary map in algebraic K-theory*, $n = \dim(B)$ *and the cup product is defined using* $\mathcal{O}(S^*\mathscr{F}) \simeq f_0^* \mathcal{O}(F)$.

Proof. Denote $\Gamma_2 = \Gamma \times \Gamma$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & C_{n+1} \otimes C_c^\infty(F) \rtimes \Gamma & \to & (E \otimes C_c^\infty(F)) \rtimes \Gamma & \to & C_c^\infty(S^*\widetilde{B} \times F) \rtimes \Gamma & \to 0 \\ & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & \\ 0 \to & (C_{n+1} \otimes C_c^\infty(F)) \rtimes \Gamma_2 & \to & (E \otimes C_c^\infty(F)) \rtimes \Gamma_2 & \to & C_c^\infty(S^*\widetilde{B} \times F) \rtimes \Gamma_2 & \to 0 \end{array}$$

where the vertical arrows are defined by the diagonal morphism $\Gamma \to \Gamma_2 = \Gamma \times \Gamma$ and the bottom line is obtained from the exact sequence of $E(\widetilde{B}) \rtimes \Gamma$ by tensoring with $C_c^\infty(F) \rtimes \Gamma$.

The map $(S^*\widetilde{B} \times F)//\Gamma \to (S^*\widetilde{B} \times F)//\Gamma_2$ is $(\pi_0, f_0)\colon S^*\mathscr{F} \to S^*B \times (F//\Gamma)$ which shows that the morphism β^* satisfies the relation

$$\beta^*(\Phi(\xi_0) \otimes \Phi(\xi_1)) = \Phi(\pi_0^*(\xi_0)\, f_0^*(\xi_1)) \in HC_{\mathrm{per}}^*(C_c^\infty(S^*\widetilde{B} \times F) \rtimes \Gamma)$$

where $\xi_0 \in H^{*+1}(S^*B)$ and $\xi_1 \in H^{*+q}(F//\Gamma, \mathcal{O}(F))$. This follows from the functoriality and multiplicativity of the geometric morphism Φ.

We have, using the multiplicativity of the boundary morphism and Lemma 4,

$$\begin{aligned} \partial_{\mathscr{F}}(\Phi_0(\xi)) &= \partial_{\mathscr{F}}(Tr \otimes \Phi(\xi)) \\ &= \partial_{\mathscr{F}}(\alpha_0^*(Tr \otimes 1 \otimes \Phi(\xi))) \\ &= \beta^*(\partial_{E \rtimes \Gamma}(Tr \otimes 1) \otimes \Phi(\xi)) \\ &= (-1)^n\, \beta^*(\Phi(\mathscr{I}(B)\, g^*(1)) \otimes \Phi(\xi)) \\ &= (-1)^n\, \Phi(\pi_0^*(\mathscr{I}(B))\, f_0^*(\xi)) \\ &= (-1)^n\, \Phi(\mathscr{I}(\mathscr{F})\, f_0^*(\xi)). \end{aligned}$$

(here $g\colon S^*B \to B\Gamma$ classifies the universal covering of S^*B as in Lemma 4).

We obtain, using the compatibility between the index map in K-theory and the boundary map in periodic cyclic cohomology,

$$\begin{aligned} \langle \Phi_0(\xi), \mathrm{Ind}[u] \rangle &= \langle \partial_{\mathscr{F}}(\Phi_0(\xi)), \mathrm{Ch}[u] \rangle \\ &= (-1)^n \langle \Phi(\mathscr{I}(\mathscr{F})\, f_0^*(\xi)), \mathrm{Ch}[u] \rangle \\ &= (-1)^n \langle \mathrm{Ch}[u]\, \mathscr{I}(\mathscr{F})\, f_0^*(\xi), [S^*\mathscr{F}] \rangle. \end{aligned}$$

This proves the theorem. ∎

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