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Harmonic Analysis on $SL(n, \mathbf{C})$*

RONALD L. LIPSMAN

Department of Mathematics, Yale University, New Haven, Connecticut 06520

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1. INTRODUCTION

In this paper we are concerned with a rather unusual result that arises in the study of harmonic analysis on semisimple Lie groups. Suppose G is a locally compact unimodular group. Let $f \in L_2(G)$ and $h \in L_p(G)$, $1 \leqslant p < 2$. By Young's inequality ([4], p. 296) we know that $\int_G |f(g_0 g^{-1}) h(g)| \, dg < \infty$ for almost all $g_0 \in G$, and that if

$$(f * h)(g_0) = \int f(g_0 g^{-1}) \, h(g) \, dg, \tag{1.1}$$

then $f * h \in L_r(G)$, $1/r = 1/p - 1/2$. However if $G = SL(2, \mathbf{R})$ or $G = SL(2, \mathbf{C})$, then the following "L_p convolution theorem" is valid:

THEOREM. *For each p, $1 \leqslant p < 2$, there exists a constant A_p such that $\| f * h \|_2 \leqslant A_p \| f \|_2 \| h \|_p$ for all $f \in L_2(G)$, $h \in L_p(G)$. That is, convolution by an $L_p(G)$ function, $1 \leqslant p < 2$, is a bounded operator on $L_2(G)$.*

The proof for $G = SL(2, \mathbf{R})$ is in [6], for $G = SL(2, \mathbf{C})$ in [9]. Our purpose here is to prove this theorem for $G = SL(n, \mathbf{C})$, $n > 2$.

The idea of the proof is to utilize the "analytic continuation of the principal series" of $SL(n, \mathbf{C})$ constructed by Kunze and Stein in [7]. Let us recall in rather loose terms what this means. The (non-degenerate) principal series of representations of $G = SL(n, \mathbf{C})$ is parameterized (partially) by $n - 1$ purely imaginary numbers $it_1, \ldots, it_{n-1}$; i.e. for each $t_1, \ldots, t_{n-1} \in \mathbf{R}$, we have an irreducible unitary representation $g \to R(g; it_1, \ldots, it_{n-1})$ of G on a (fixed)

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Hilbert space $\mathscr{H}$. It is possible to extend the definition of R from $(i\mathbf{R})^{n-1}$ to an open domain $\mathscr{T}$ in $\mathbf{C}^{n-1}$ so that:

(1) for each $(s_1, \ldots, s_{n-1}) \in \mathscr{T}$, the representation $g \to R(g; s_1, \ldots, s_{n-1})$ is uniformly bounded, i.e.

$$\sup_{g \in G} \| R(g; s_1, \ldots, s_{n-1})\|_\infty = A(s_1, \ldots, s_{n-1}) < \infty;$$

(2) R depends analytically on $(s_1, \ldots, s_{n-1}) \in \mathscr{T}$ in the sense that for each $g \in G$, ξ, $\eta \in \mathscr{H}$, the function

$$(s_1, \ldots, s_{n-1}) \to (R(g; s_1, \ldots, s_{n-1})\, \xi, \eta)$$

is analytic on $\mathscr{T}$.

The uniform bounds $A(s_1, \ldots, s_{n-1})$ exhibit polynomial growth in the imaginary parts of the complex parameters $s_1, \ldots, s_{n-1}$. By utilizing Phragmén–Lindelöf techniques, we shall show that these bounds can be sharpened considerably. We then obtain a kind of Hausdorff–Young theorem for G by applying an operator-valued convexity theorem to the uniform bounds and the Plancherel formula for G. The L_p convolution theorem follows from that result by suitable analyticity arguments.

If one compares the derivation (in [6] or [9]) of the L_p convolution theorem from the analytic continuation of the principal series with the above outline, one sees that the method is basically the same. The main difference (and new difficulty) is the dependence on $n-1$ rather than *one* complex variable. This necessitates the convexity (interpolation) theorems proven in Sections 4 and 5. Theorem 3 in Section 4 is rather general in nature and quite likely of independent interest. Theorem 4 in Section 5, on the other hand, is tailored to fit the situation that arises in the case $G = SL(n, \mathbf{C})$. Both of them are generalizations of the results proven in Sections 3 and 4 of [6].

The author is grateful to Professor E. Stein for several interesting conversations on these and other matters relating to representations of semisimple Lie groups. Some of the results contained herein have been obtained independently by him and Professor R. Kunze. However, as of this writing, none of these have been published.

The paper is arranged as follows. In Section 2 we review the definition of the principal series of representations of $SL(n, \mathbf{C})$. We then describe in detail the analytic continuation of the principal series à la Kunze and Stein [7]. We proceed in Section 3 to strengthen the uniform bounds of the analytic continuation. The main tool that we use (Lemma 3) is a natural generalization of the Phragmén–Lindelöf type result found in ([*9*], Lemma 1). As we mentioned previously, Section 4 and Section 5 contain the interpolation theorems in several variables that we shall need. The origin of these results is of course the Riesz convexity theorem. Generalizations to operator-valued functions depending analytically on a complex parameter are rather standard by now (see for example [*6*], Section 3 and Section 4 or [*10*]). Here we carry these methods one step further—namely, to dependence on several complex parameters. We define the operator-valued Fourier transform and prove a strong version of the Hausdorff–Young theorem in Section 6. We obtain our main result, the L_p convolution theorem for $G = SL(n, \mathbf{C})$, in Section 7. Finally in Section 8, we show that the Fourier transform extends to $L_p(G)$, $1 \leqslant p < 2$, as an analytic operator-valued function on a suitable domain $\mathscr{T}_p$ in $\mathbf{C}^{n-1}$, and we deduce an analog of the Riemann–Lebesgue lemma on G.

Notation. Unless specified otherwise G will denote the $n \times n$ complex unimodular group $SL(n, \mathbf{C})$. The letter A, possibly with subscripts or arguments, shall denote a constant—not necessarily the same from one line to the next.

A *regular measure space* $(X, d\mu)$ shall consist of a locally compact Hausdorff space X and a regular Borel measure μ on X. We denote by $C_0(X)$ and $S_0(X)$ the continuous and simple functions of compact support on X, respectively. By a *bounded* subset of X we mean a measurable subset of a compact set in X. When $X = G$ and $g \in G$ is generic, we will use dg to denote Haar measure.

If $\mathscr{H}$ is a separable Hilbert space, we use $\beta(\mathscr{H})$ for the Banach space of all bounded linear operators on $\mathscr{H}$. In addition, let $\beta_p(\mathscr{H})$, $1 \leqslant p < \infty$, be the Banach spaces of operators with finite pth norms: $\beta_p(\mathscr{H}) = \{T \in \beta(\mathscr{H}) : \| T \|_p{}^p = \text{trace } (T^*T)^{p/2} < \infty\}$. For the elementary properties of β_p, see ([*6*], Section 2). We note here that the norms are non-increasing as $p \to \infty$ and that for $T \in \beta_p$, $S \in \beta_q$, $1/r = 1/p + 1/q \leqslant 1$,

$$\| TS \|_r \leqslant \| T \|_p \| S \|_q . \tag{1.2}$$

Let $\mathscr{F}$ be a function from a measure space $(X, d\mu)$ to the space $\beta(\mathscr{H})$. We say $\mathscr{F}$ is *measurable* if $x \to (\mathscr{F}(x)\, \xi, \eta)$ is μ-measurable on X for all $\xi, \eta \in \mathscr{H}$. If X has an analytic structure, we say $\mathscr{F}$ is *analytic* if $x \to (\mathscr{F}(x)\, \xi, \eta)$ is analytic on X for all $\xi, \eta \in \mathscr{H}$. We shall also denote by $\beta_p(X, \mathscr{H})$, $1 \leqslant p \leqslant \infty$, the Banach spaces of functions $\mathscr{F} : X \to \beta(\mathscr{H})$ such that

$$\| \mathscr{F} \|_p = \left(\int_X \| \mathscr{F}(x) \|_p^p \, d\mu(x) \right)^{1/p} < \infty, 1 \leqslant p < \infty,$$

$$\| \mathscr{F} \|_\infty = \operatorname*{ess\,sup}_X \| \mathscr{F}(x) \|_\infty < \infty.$$

Again, see ([6], Section 2) for the elementary properties of $\beta_p(X, \mathscr{H})$.

Finally we shall use boldface to denote both variables and fixed vectors in r-dimensional complex linear space $\mathbf{C}^r$, $r > 1$. *For example*, we shall consider analytic functions $\varphi(\mathbf{z}) = \varphi(z_1, \ldots, z_r)$ on

$$| \operatorname{Re}(z_j) | < 1, j = 1, \ldots, r$$

and vectors $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_r)$, each $\delta_j = 0$ or 1. At times the vector $\boldsymbol{\delta}$ may occur in the same proof with a real number $\delta > 0$. The appropriate meaning will be clear from the context as well as the notation.

2. The Analytic Continuation

We single out some subgroups of the group $G = SL(n, \mathbf{C})$:

(i) $U = U(n)$, the subgroup of upper triangular unipotent matrices

$$U = \{u \in G: u_{jk} = 0, j > k, u_{jj} = 1\};$$

(ii) $C = C(n)$, the (Cartan) subgroup of diagonal matrices

$$C = \{c \in G: c_{jk} = 0, j \neq k\};$$

(iii) $V = V(n)$, the subgroup of lower triangular unipotent matrices

$$V = \{v \in G: v_{jk} = 0, j < k, v_{jj} = 1\};$$

(iv) $S = S(n)$, the finite subgroup of G generated by the $n - 1$ elements

$$\begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ -1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & & \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & & & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & 1 & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & -1 & 0 \end{bmatrix}.$$

The Weyl group may be indentified with a quotient group of S.

Let λ denote a continuous character of the subgroup C. Then

$$\lambda(c) = \prod_{j=1}^{n} (c_j/|c_j|)^{m_j} |c_j|^{s_j}$$

where $s_1, ..., s_n$ are complex numbers and $m_1, ..., m_n$ are integers. λ is determined *uniquely* by *either* of the following specifications:

(a) $s_1 + s_2 + \cdots + s_n = 0$ and $0 \leqslant m_1 + m_2 + \cdots + m_n < n$

(b) $s_n = 0$ and $m_n = 0$.

Kunze and Stein's construction of the analytic continuation of the principal series [7] assumes the parameterization (a); Gelfand and Naimark's computation of the Plancherel formula [3] assumes (b). Part of our task in this section therefore is the reconciliation of these two choices. Note that λ is unitary precisely when $\mathrm{Re}(s_j) = 0$, $j = 1, ..., n$. We denote the unitary characters by the letter Λ. We also define a special character μ on C by

$$\mu(c) = \prod_{j=1}^{n} |c_j|^{2(n-2j+1)}, \; c \in C.$$

Gelfand and Naimark have shown that (with the exception of a set of measure zero) every $g \in G$ can be uniquely written $g = ucv$, $u \in U$, $c \in C$, $v \in V$. Let $v \in V$, $g \in G$ and suppose vg is decomposable in this manner; say $vg = u(v, g) \cdot c(v, g) \cdot g(v)$, $u(v, g) \in U$, $c(v, g) \in C$, $g(v) \in V$. One sees easily that $dg(v) = \mu(c(v, g))\, dv$. Moreover, to each continuous unitary character λ of C, we can associate an irreducible unitary representation $g \to T(g, \lambda)$ of G on the Hilbert space $\mathscr{H} = L_2(V)$:

$$T(g, \lambda): f(v) \to \lambda(c(v, g))\, \mu^{1/2}(c(v, g)) f(g(v)).$$

These constitute the (nondegenerate) *principal series.*

Before proceeding, we quote one well-known result concerning these representations. The group S acts on Λ (in fact on all continuous characters) by $p\lambda(c) = \lambda(p^{-1}cp)$, $p \in S$. Moreover, for each $p \in S$ and $\lambda \in \Lambda$, there exists a unitary operator $A(p, \lambda)$ such that

$$A(p, \lambda)\, T(g, \lambda) = T(g, p\lambda)\, A(p, \lambda), \; g \in G. \tag{2.1}$$

This expresses the invariance of the principal series under the action of the Weyl group. For proofs of these facts, the reader is referred to [3].

Until further notice we assume the parameterization (a). Consider the complex hyperplane

$$\mathscr{P} = \{(s_1, ..., s_n) \in \mathbf{C}^n : s_1 + s_2 + \cdots + s_n = 0\}.$$

Let $0 < d \leqslant 1$ and let $B(d)$ denote the smallest convex set in the real hyperplane $\sigma_1 + \cdots + \sigma_n = 0$ which contains the points $(\sigma, -\sigma, 0, ..., 0)$, $|\sigma| < d$, and all permutations of these points. Next let $\mathscr{T}(d)$ denote the tube in $\mathscr{P}$ whose basis is $B(d)$, that is $\mathscr{T}(d) = \{(s_1, ..., s_n) \in \mathscr{P}, \quad s_j = \sigma_j + it_j, \quad (\sigma_1, ..., \sigma_n) \in B(d),$ $t_1 + \cdots + t_n = 0$, but the t_j's arbitrary otherwise$\}$. $\mathscr{T}(d)$ is an open convex subset of $\mathscr{P}$. Denote $\mathscr{T} = \mathscr{T}(1)$. A function F defined on $\mathscr{T}(d)$ (or on $\mathscr{T}$ or $\mathscr{P}$) is called *analytic* if F is analytic as a function of $s_1, ..., s_{n-1}$. Finally let $\Lambda^* =$ those characters $\lambda = (m_1, ..., m_n; s_1, ..., s_n)$ such that $0 \leqslant \sum m_j < n$ and $(s_1, ..., s_n) \in \mathscr{T}$. A function F on Λ^* is called *analytic* if, for each fixed $(m_1, ..., m_n)$, F is analytic on $\mathscr{T}$.

We now summarize the results of ([7]; Theorem 4, the corollary on page 172, and formula (9.9)) in the following

THEOREM 1. *There exist representations $g \to R(g, \lambda)$ of G on $\mathscr{H} = L_2(V)$ such that:*

(1) *$g \to R(g, \lambda)$ is a continuous representation of G on $\mathscr{H}$ for each $\lambda \in \Lambda^*$;*

(2) *when $\lambda \in \Lambda$, $g \to R(g, \lambda)$ is unitarily equivalent to the element of the principal series $g \to T(g, \lambda)$;*

(3) *for each fixed $g \in G$, $\xi, \eta \in \mathscr{H}$, the function $\lambda \to (R(g, \lambda)\, \xi, \eta)$ is analytic on Λ^*;*

(4) *there exists a function K_d, $0 \leqslant d < 1$, bounded on any interval of the form $0 \leqslant d \leqslant d_0 < 1$ such that:*

$$\sup_{g \in G} \| R(g, \lambda) \|_\infty \leqslant K_d \, | H(\lambda) |,$$

whenever $\lambda = (m_1, ..., m_n; \; s_1, ..., s_n) \in \Lambda^*$ satisfies $|\operatorname{Re}(s_j)| < d$, $1 \leqslant j \leqslant n$. Here $H(\lambda)$ is the analytic function

$$H(\lambda) = \Big(1 + \sum_{j=1}^{n} |m_j|\Big)^3 \prod_{j=1}^{n} (3 - s_j)^6;$$

(5) *$R(g, \lambda) = R(g, p\lambda)$ for all $g \in G$, $\lambda \in \Lambda^*$, $p \in S$;*

(6) *if λ' denotes the contragredient character $\lambda'(c) = \overline{\lambda(c^{-1})}$, $\lambda \in \Lambda^*$, then the representations $R(g, \lambda)$ and $R(g, \lambda)$ are contragredient on $\mathscr{H}$;*

(7) *if* $\lambda' = p\lambda$ *for some nontrivial* $p \in S$, *then* $g \to R(g, \lambda)$ *is unitarily equivalent to a member of the complementary series.*

Remarks. (1) Let us recall very briefly how the proof goes. First we specify the subgroup $G_0 = \{g \in G : g_{jn} = 0,\ 1 \leqslant j \leqslant n - 1\}$. Then for $\lambda \in \Lambda$ we define the *residue* of λ by $\text{res}\,\lambda = m_1 + \cdots + m_n$, $0 \leqslant \text{res}\,\lambda < n$. The following two points are crucial:

(i) if $\lambda_1, \lambda_2 \in \Lambda$ and $\text{res}\,\lambda_1 = \text{res}\,\lambda_2$, then the elements of the principal series $T(g, \lambda_1)$ and $T(g, \lambda_2)$ are unitarily equivalent when restricted to G_0;

(ii) the representations $g \to T(g, \lambda)$, $\lambda \in \Lambda$, are irreducible on G_0.

It follows that for each $\lambda \in \Lambda$ there exists a unique unitary operator $W(\lambda)$ on $\mathscr{H}$ such that: if $R(g, \lambda) = W(\lambda)\, T(g, \lambda)\, W(\lambda)^{-1}$, then $R(g, \lambda)$ depends only on $\text{res}\,\lambda$ when $g \in G_0$. These representations constitute the so-called *normalized principal series*. One can then construct an analytic continuation of the $R(g, \lambda)$. This procedure is rather involved—we mention only that it is facilitated by studying the intertwining operators $A(p, \lambda)$ (see (2.1)). We should indicate that Kunze and Stein have investigated the operators $A(p, \lambda)$ on arbitrary complex semisimple Lie groups [*8*].

(2) The author has also reconstructed the analytic continuation of the principal series according to the general scheme of [*6*]. One obtains stronger uniform bounds by that method. Nevertheless, Theorem 1 part (4) and Lemma 3 (Section 3) will suffice for the results we desire in harmonic analysis. In addition, that method is deficient for another reason. It *employs* (in the proof) the precise form of the complementary series, an object unlikely to be explicity available on more general groups. On the other hand, by the method of [*7*] one *obtains* the complementary series from the analytic continuation. This has led to the conjecture that *all* irreducible unitary representations of complex semisimple Lie groups will arise from suitable analytic continuations of the (degenerate and nondegenerate) principal series ([*11*], p. 580).

Consider the character $\lambda = (m_1, \ldots, m_n; s_1, \ldots, s_n) \in \Lambda^*$. How do we translate from parameterization (a) to parameterization (b)? Clearly, as a character, λ also equals

$$(m_1 - m_n, \ldots, m_{n-1} - m_n, 0; s_1 - s_n, \ldots, s_{n-1} - s_n, 0).$$

Therefore consider the map

$$P : (m_1, \ldots, m_n; s_1, \ldots, s_n) \to [m_1 - m_n, \ldots, m_{n-1} - m_n; s_1 - s_n, \ldots, s_{n-1} - s_n] \tag{2.2}$$

(In this section we use curved parenthesis to indicate n variables and square ones to indicate $r = n - 1$ variables). For $0 < d \leqslant 1$, let

$$\mathscr{D}(d) = \{[s_1, ..., s_r] : s_j = \sigma_j + it_j, |\sigma_j| < d, t_j \in \mathbf{R}, 1 \leqslant j \leqslant r\}.$$

LEMMA 1. *P is a $1 - 1$ mapping of Λ^* onto $\mathbf{Z}^r \times \mathscr{U}(n)$, where $\mathscr{U}(n)$ is some open set in $\mathbf{C}^r$ containing $\mathscr{D}(n/(n-1)^2)$. If F is an analytic function on Λ^*, then $F \circ P^{-1}$ is analytic on $\Omega = \mathbf{Z}^r \times \mathscr{D}(n/(n-1)^2)$.*

Proof. P is well-defined on all $(m_1, ..., m_n; s_1, ..., s_n)$, $0 \leqslant \sum m_j < n$, $(s_1, ..., s_n) \in \mathscr{P}$. It is easy to check that the map defined on $\mathbf{Z}^r \times \mathbf{C}^r$ by

$$[q_1, ..., q_r; z_1, ..., z_r] \to (q_1 - q, ..., q_r - q, -q; z_1 - z/n, ..., z_r - z/n, -z/n)$$

$$\sum_{j=1}^{r} z_j = z, \sum_{j=1}^{r} q_j = nq + p, 0 \leqslant p < n,$$

is an inverse for P. Therefore P is $1 - 1$ onto $\mathbf{Z}^r \times \mathbf{C}^r$, and P is clearly a homeomorphism. The integer part obviously covers $\mathbf{Z}^r$, and $P(\mathscr{T})$ covers $\mathscr{D}(n/(n-1)^2)$ because: $P^{-1}[\sigma_1, ..., \sigma_r]$ can be written as a sum of $(n-1)^2$ terms of the form $(\sigma_j/n, -\sigma_j/n, 0, ..., 0)$ and permutations of these. To illustrate, when $n = 3$,

$$\begin{aligned}(\sigma_1 - \sigma/3, \sigma_2 - \sigma/3, -\sigma/3) &= \tfrac{1}{4}[(4\sigma_1/3, -4\sigma_1/3, 0) + (4\sigma_1/3, 0, -4\sigma_1/3) + (-4\sigma_2/3, 4\sigma_2/3, 0) \\ &\quad + (0, 4\sigma_2/3, -4\sigma_2/3)].\end{aligned}$$

The formula in the general case is analogous. Therefore

$$P^{-1}(\mathscr{D}(n/(n-1)^2)) \subsetneq \mathscr{T} \Rightarrow \mathscr{D}(n/(n-1)^2) \subsetneq P(\mathscr{T}).$$

The last statement of the lemma follows trivially from the fact that P^{-1} is an analytic mapping of $\mathbf{C}^r$ to $\mathscr{P}$.

Remarks. (1) If $n \geqslant 3$, $n/(n-1)^2 < 1$, while $2/(2-1)^2 = 2$. Therefore we shall assume $n \geqslant 3$ in what follows. In fact, by altering P slightly, we can include the case $n = 2$. We find it more convenient to work with (2.2); and, since $n = 2$ is already handled in [*9*], we shall leave (2.2) as it is.

(2) The set $\mathscr{D}(n/(n-1)^2)$ is clearly not the largest possible domain of definition for the analytic continuation; but it will suffice for our purposes.

Henceforth parameterization (b) will always be assumed. By using

Lemma 1, we can translate Theorem 1 into this new setting. Since we need only conditions (1)–(4) in this paper, we shall not bother with the last three.

THEOREM 1a. *There exist representations* $g \to R(g, \lambda)$ *of* G *on* $\mathscr{H}$ *such that:*

(1a) $g \to R(g, \lambda)$ *is a continuous representation of* G *on* $\mathscr{H}$ *for each* $\lambda \in \Omega$;

(2a) *when* $\lambda \in \Lambda \simeq \mathbf{Z}^r \times (i\mathbf{R})^r$, $g \to R(g, \lambda)$ *is unitarily equivalent to a member of the principal series;*

(3a) *for each fixed* $g \in G$, $\xi, \eta \in \mathscr{H}$, *the function* $\lambda \to (R(g, \lambda)\, \xi, \eta)$ *is analytic on* Ω;

$$\text{(4a)} \qquad \sup_{g \in G} \| R(g, \lambda)\|_\infty \leqslant A(n, d) \prod_{j=1}^{r} (1 + | m_j |)^3 \prod_{j=1}^{r} (1 + | t_j |)^{6n}, \qquad (2.3)$$

whenever $\lambda = [m_1, ..., m_r\,;\, s_1, ..., s_r] \in \Omega$ *satisfies*

$$| \operatorname{Re}(s_j)| < d < n/(n-1)^2, \; 1 \leqslant j \leqslant r.$$

Moreover, for fixed n, $A(n, d)$ *is bounded on intervals of the form* $0 \leqslant d \leqslant d_0 < n/(n-1)^2$.

Proof. (1a)–(3a) are obvious from Theorem 1 and Lemma 1. We verify (4a). Let $\lambda = [m_1, ..., m_r\,;\, s_1, ..., s_r]$, $m_j \in \mathbf{Z}$, $s_j = \sigma_j + it_j$, $| \sigma_j | < n/(n-1)^2$. Then

$$P^{-1}\lambda = (m_1 - q, ..., m_r - q, -q; s_1 - s/n, ..., s_r - s/n, -s/n),$$

$$\sum_{j=1}^{r} s_j = s, \; \sum_{j=1}^{r} m_j = nq + p, \; 0 \leqslant p < n.$$

The following are easily checked:

(i) $| -\sigma/n | < \max | \sigma_j |, \; | \sigma_j - \sigma/n | < \max | \sigma_j | \cdot 2(n-1)/n,$

(ii) $| -q | \leqslant 1 + \sum | m_j |, \; | m_j - q | \leqslant 1 + \sum | m_j |,$

(iii) $| 3 + s/n | \leqslant 4 + \sum | t_j | \leqslant 4 \prod (1 + | t_j |),$
$| 3 - s_j + s/n | \leqslant 5 + \sum | t_j | \leqslant 5 \prod (1 + | t_j |).$

Putting these together with part (4) of Theorem 1 we obtain

$$\sup_g \| R(g, \lambda)\|_\infty \leqslant A(n, d) \Big(1 + \sum_{j=1}^r | m_j |\Big)^3 \prod_{k=1}^n \Big(\prod_{j=1}^r (1 + | t_j |)^6\Big)$$

$$\leqslant A(n, d) \prod_{j=1}^r (1 + | m_j |)^3 \prod_{j=1}^r (1 + | t_j |)^{6n}.$$

The last statement of (4a) follows from $A(n, d) = A(n)\, K_{2(n-1)d/n}$.

We close this section with the Plancherel formula for G ([3], p. 159). Let $f \in (L_1 \cap L_2)(G)$ and $R(f, \lambda) = \int R(g, \lambda) f(g)\, dg$, $\lambda \in \Lambda$. Then

$$\|f\|_2^2 = C_n \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_r=-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \| R(f, \lambda)\|_2^2 \prod_{1 \leqslant j < k \leqslant n} \omega_{kj}(\lambda)\, dt_1, \ldots, dt_r \tag{2.4}$$

where $C_n = 2^{n(n-1)/2}/n!\,(2\pi)^{(n-1)(n+2)}$ and

$$\omega_{kj}(\lambda) = [(m_k - m_j)^2 + (t_k - t_j)^2], \tag{2.5}$$

$m_n = t_n = 0$.

3. Improving the Uniform Bounds

In this section we employ suitable analyticity arguments to sharpen the bounds in (2.3).

Lemma 2. *Let $\varphi(s)$ be an analytic function of one complex variable on the closed strip $|\operatorname{Re}(s)| \leqslant \tau < 1$. Suppose that*

$$| \varphi(it)| \leqslant k^\alpha(1 + | t |)^{c\alpha},\ t \in \mathbf{R},$$

$k \geqslant 1, 0 \leqslant \alpha \leqslant 1, c \geqslant 0$, *and that*

$$\sup_{0 \leqslant |\sigma| \leqslant \tau} | \varphi(\sigma + it)| \leqslant k(1 + | t |)^c, \qquad t \in \mathbf{R}.$$

Then there exists a constant A, depending only on c, such that

$$| \varphi(\sigma + it)| \leqslant Ak^{\alpha(\sigma)}(1 + | t |)^{c\alpha(\sigma)}, \qquad 0 \leqslant | \sigma | \leqslant \tau, \qquad t \in \mathbf{R},$$

and $\alpha(\sigma) = (| \sigma |/\tau)(1 - \alpha) + \alpha$.

Proof. The case $\alpha = 0$ is precisely ([9], Lemma 1). Recall that the proof is a slight modification of the classical Phragmén–Lindelöf theorem in a strip; and that $A(c) \leqslant A(c')$ if $c \leqslant c'$.

Assume $0 < \alpha \leqslant 1$. Define $\Phi(s) = 2^{-c\alpha/2}\varphi(s)/k^{\alpha}(2+s)^{c\alpha}$, an analytic function on $|\operatorname{Re}(s)| \leqslant \tau$. Moreover $|\Phi(it)| \leqslant 1$ and

$$\sup_{0\leqslant|\sigma|\leqslant\tau} |\Phi(\sigma+it)| \leqslant k^{1-\alpha}(1+|t|)^{c(1-\alpha)}, \qquad t\in\mathbf{R}.$$

Applying the case $\alpha = 0$ to the function Φ, we obtain

$$|\Phi(\sigma+it)| \leqslant A(c)\,k^{(1-\alpha)|\sigma|/\tau}(1+|t|)^{c(1-\alpha)|\sigma|/\tau}, \qquad 0\leqslant|\sigma|\leqslant\tau, \qquad t\in\mathbf{R}.$$

Therefore

$$|\varphi(\sigma+it)| \leqslant A(c)\,k^{\alpha(\sigma)}(1+|t|)^{c\alpha(\sigma)}, \qquad 0\leqslant|\sigma|\leqslant\tau, \qquad t\in\mathbf{R}.$$

With this lemma we can now use an inductive technique to obtain the necessary result in several variables.

LEMMA 3. *Let $\varphi(\mathbf{s}) = \varphi(s_1,\dots,s_r)$ be an analytic function on $|\operatorname{Re}(s_j)| \leqslant \tau < 1$, $j = 1,\dots,r$. Suppose that $|\varphi(i\mathbf{t})| \leqslant 1$, $\mathbf{t}\in\mathbf{R}^r$ and*

$$\sup_{0\leqslant|\sigma_j|\leqslant\tau} |\varphi(\boldsymbol{\sigma}+i\mathbf{t})| \leqslant K\prod_{j=1}^{r}(1+|t_j|)^c, \qquad \mathbf{t}=(t_1,\dots,t_r)\in\mathbf{R}^r, \tag{3.1}$$

$K \geqslant 1$, $c \geqslant 0$. Then there exists a constant A, depending only on c and r, such that

$$|\varphi(\mathbf{s})| \leqslant AK^{rd/\tau}\prod_{j=1}^{r}(1+|t_j|)^{crd/\tau}, \qquad 0\leqslant|\sigma_j|\leqslant\tau, \qquad \mathbf{t}\in\mathbf{R}^r$$

and $d = \max_{1\leqslant j\leqslant r}|\sigma_j|$.

Proof. Let $s_j = it_j$, $j = 2,\dots,r$ be arbitrary but fixed. Define $\psi_1(s_1) = \varphi(s_1, it_2,\dots,it_r)$, an analytic function on $0\leqslant|\sigma_1|\leqslant\tau$. Moreover $|\psi_1(it_1)| \leqslant 1$, and by (3.1)

$$|\psi_1(\sigma_1+it_1)| \leqslant K\prod_{j=1}^{r}(1+|t_j|)^c.$$

By Lemma 2, with $\alpha = 0$, $k = K\prod_{j=2}^{r}(1+|t_j|)^c$, we get

$$|\psi_1(\sigma_1+it_1)| \leqslant A_1(c)\,K^{|\sigma_1|/\tau}\prod_{j=1}^{r}(1+|t_j|)^{c|\sigma_1|/\tau}.$$

That is

$$|\varphi(s_1, it_2,\dots,it_r)| \leqslant A_1(c)\,K^{|\sigma_1|/\tau}\prod_{j=1}^{r}(1+|t_j|)^{c|\sigma_1|/\tau}, \quad 0\leqslant|\sigma_1|\leqslant\tau. \tag{3.2}$$

We may assume $A_1(c) \geqslant 1$.

Now let $s_1 = \sigma_1 + it_1$, $|\sigma_1| \leqslant \tau$, and $s_j = it_j$, $j = 3,..., r$ be arbitrary but fixed. Define $\psi_2(s_2) = \varphi(s_1, s_2, it_3, ..., it_r)/A_1(c)$, analytic on $0 \leqslant |\sigma_2| \leqslant \tau$. By (3.2)

$$|\psi_2(it_2)| \leqslant K^{|\sigma_1|/\tau} \prod_{j=1}^{r} (1 + |t_j|)^{c|\sigma_1|/\tau},$$

and by the assumption (3.1)

$$|\psi_2(s_2)| \leqslant K \prod_{j=1}^{r} (1 + |t_j|)^c, \qquad 0 \leqslant |\sigma_2| \leqslant \tau.$$

We apply Lemma 1 with $\alpha = |\sigma_1|/\tau$, $k = K \prod_{j\neq 2} (1 + |t_j|)^c$. The result is:

$$|\psi_2(s_2)| \leqslant A(c)\, K^{\alpha(\sigma_2)} \prod_{j=1}^{r} (1 + |t_j|)^{c\alpha(\sigma_2)}$$

where $\alpha(\sigma_2) = (|\sigma_2|/\tau)(1 - |\sigma_1|/\tau) + |\sigma_1|/\tau$. Hence

$$|\varphi(s_1, s_2, it_3, ..., it_r)| \leqslant A_2(c)\, K^{\alpha(\sigma_2)} \prod_{j=1}^{r} (1 + |t_j|)^{c\alpha(\sigma_2)},$$
$$0 \leqslant |\sigma_1|, |\sigma_2| \leqslant \tau.$$

Continuing in this manner we get

$$|\varphi(s_1, ..., s_r)| \leqslant A_r(c)\, K^{\alpha(\sigma_r)} \prod_{j=1}^{r} (1 + |t_j|)^{c\alpha(\sigma_r)}, \qquad 0 \leqslant |\sigma_j| \leqslant \tau,$$

where $\alpha(\sigma_r) = (|\sigma_r|/\tau)(1 - \alpha'(\sigma_{r-1})) + \alpha'(\sigma_{r-1})$ and $\alpha'(\sigma_{r-1})$ is the exponent obtained at the $(r - 1)$st stage. A simple induction argument shows $\alpha(\sigma_r) \leqslant (|\sigma_1| + \cdots + |\sigma_r|)/\tau$; and the proof is completed once we note $|\sigma_1| + \cdots + |\sigma_r| \leqslant rd$, $d = \max |\sigma_j|$.

THEOREM 2. *The uniform bound of Theorem* 1a *part* (4a) *may be strengthened as follows: for every* $\epsilon > 0$,

$$\sup_g \| R(g, \lambda)\|_\infty \leqslant A(n, d, \epsilon) \prod_{j=1}^{r} (1 + |m_j|)^{3n^2 d(1+\epsilon)} \prod_{j=1}^{r} (1 + |t_j|)^{6n^3 d(1+\epsilon)}, \tag{3.3}$$

$\lambda = (m_1, ..., m_r; s_1, ..., s_r) \in \Omega$, $d = \max |\sigma_j| < n/(n - 1)^2$.

Proof. Let $\xi, \eta \in \mathscr{H}$ be arbitrary such that $\| \xi \| \leqslant 1$, $\| \eta \| \leqslant 1$, and let $(m_1, ..., m_r) \in \mathbf{Z}^r$. Define

$$\varphi(\mathbf{s}) = (R(g, \lambda)\, \xi, \eta), \qquad \lambda = (m_1, ..., m_r; s_1, ..., s_r),$$

an analytic function on $\mathscr{D}(n/(n-1)^2)$. Then $|\varphi(i\mathbf{t})| \leqslant 1$ and by (2.3):

$$\sup_{0 \leqslant |\sigma_j| \leqslant \tau} |\varphi(\boldsymbol{\sigma} + i\mathbf{t})| \leqslant A(n, \tau_1) \prod_{j=1}^{r} (1 + |m_j|)^3 \prod_{j=1}^{r} (1 + |t_j|)^{6n}$$

whenever $\tau < \tau_1 < n/(n-1)^2$.

Let $\mathbf{s} \in \mathscr{D}(n/(n-1)^2)$ and $\epsilon > 0$. Choose τ and τ_1 so that $\max |\sigma_j| < \tau < \tau_1 < n/(n-1)^2$ and $1/\tau \leqslant (1 + \epsilon)(n-1)^2/n$. Applying Lemma 3 with $c = 6n$, we obtain

$$|\varphi(\mathbf{s})| \leqslant A_n \left(A(n, \tau_1) \prod_{j=1}^{r} (1 + |m_j|)^3\right)^{rd/\tau} \prod_{j=1}^{r} (1 + |t_j|)^{6nrd/\tau}$$

$$\leqslant A(n, d, \epsilon) \prod_{j=1}^{r} (1 + |m_j|)^{3n^2 d(1+\epsilon)} \prod_{j=1}^{r} (1 + |t_j|)^{6n^3 d(1+\epsilon)}.$$

Taking the supremum over all ξ, η such that $\|\xi\| \leqslant 1$, $\|\eta\| \leqslant 1$, we get the result (3.3).

The important advantage of (3.3) over (2.3) is that both exponents $\to 0$ as $d = \max |\sigma_j| \to 0$.

4. A General Interpolation Theorem

In order to carry out the necessary Fourier analysis on $SL(n, \mathbf{C})$, we have to generalize the results of ([*6*], Sections 3 and 4). First we recall some of this material.

Let $z = x + iy$ denote one complex variable. If $\mathscr{D} = \mathscr{D}(\alpha, \beta) = \{z \in \mathbf{C} : \alpha \leqslant \operatorname{Re}(z) \leqslant \beta\}$, $\alpha < \beta$, we shall call a function φ *analytic* on $\mathscr{D}$ if φ is analytic (in the usual sense) in $\alpha < \operatorname{Re}(z) < \beta$ and continuous on $\alpha \leqslant \operatorname{Re}(z) \leqslant \beta$. We say φ is *admissible* on $\mathscr{D}$ if φ is analytic on $\mathscr{D}$ and

$$\sup_{\alpha \leqslant x \leqslant \beta} \log |\varphi(x + iy)| = O(e^{a|y|})$$

where $a < \pi/(\beta - \alpha)$ and a in general depends on φ. Now let $(M, d\mu)$ and $(N, d\nu)$ be regular measure spaces. Suppose that for each $z \in \mathscr{D}(\alpha, \beta)$, B_z is a bilinear form on $S_0(M) \times S_0(N)$. We say the collection $\{B_z : z \in \mathscr{D}\}$ is *admissible* if for each $\varphi \in S_0(M)$, $\psi \in S_0(N)$, the function $\theta(z) = B_z(\varphi, \psi)$ is admissible on $\mathscr{D}$.

Let us fix some notation. Let $0 < \tau < 1$ and let p_0, p_1, q_0, q_1 be indices, $1 \leqslant p_i, q_i \leqslant \infty$. Denote by q_i' the conjugate indices of the q_i. Let $\gamma = (1 - \tau)\alpha + \tau\beta$ and

$$1/p = (1 - \tau) \cdot 1/p_0 + \tau \cdot 1/p_1, \qquad 1/q = (1 - \tau) \cdot 1/q_0 + \tau \cdot 1/q_1.$$

Also let A_0 and A_1 be non-negative continuous functions of $y \in \mathbf{R}$ with

$$\log A_i(y) = O(e^{a|y|}), \qquad a < \pi/(\beta - \alpha).$$

LEMMA 4. *Assume $\{B_z\}$ is admissible on $\mathscr{D}$ and that*

$$| B_{\alpha+iy}(\varphi, \psi)| \leqslant A_0(y) \| \varphi \|_{p_0} \| \psi \|_{q_0'},$$
$$| B_{\beta+iy}(\varphi, \psi)| \leqslant A_1(y) \| \varphi \|_{p_1} \| \psi \|_{q_1'},$$

for all $\varphi \in S_0(M)$, $\psi \in S_0(N)$. Then

$$| B_\gamma(\varphi, \psi)| \leqslant A_\tau \| \varphi \|_p \| \psi \|_{q'}$$

for all $\varphi \in S_0(M)$, $\psi \in S_0(N)$ and the following estimate holds for A_τ:

$$\log A_\tau = \int \omega(1 - \tau, y) \log A_0[(\beta - \alpha) y]\, dy + \int \omega(\tau, y) \log A_1[(\beta - \alpha) y]\, dy,$$

where

$$\omega(\tau, y) = \frac{1/2 \tan(\pi\tau/2) \operatorname{sech}^2(\pi y/2)}{\tan^2(\pi\tau/2) + \tanh^2(\pi y/2)}, \qquad 0 \leqslant \tau \leqslant 1, \qquad y \in \mathbf{R}.$$

This result is stated in ([*6*], Lemma 8) and proven in ([*10*], Theorem 1). We wish to generalize this lemma to bilinear forms that are indexed by several complex variables.

Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ denote r complex variables $\mathbf{z} = (z_1, ..., z_r)$ (see Section 1 for the notation convention). Suppose $\varphi(\mathbf{z}) = \varphi(z_1, ..., z_r)$ is analytic in $\alpha_j < \operatorname{Re}(z_j) < \beta_j$, $1 \leqslant j \leqslant r$, and continuous on $\alpha_j \leqslant \operatorname{Re}(z_j) \leqslant \beta_j$, $1 \leqslant j \leqslant r$. Such a function φ will be called *analytic* on $\mathscr{D} = \mathscr{D}_r(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \{z \in \mathbf{C}^r : \alpha_j \leqslant \operatorname{Re}(z_j) \leqslant \beta_j\}$, $\alpha_j < \beta_j$.

DEFINITION. φ is called admissible on $\mathscr{D}$ if φ is analytic on $\mathscr{D}$ and

$$\sup_{\alpha_j \leqslant x_j \leqslant \beta_j} \log | \varphi(\mathbf{x} + i\mathbf{y})| = O\left(\exp\left(\sum_{j=1}^{r} a_j \left| y_j \right|\right)\right), \; a_j < \pi/(\beta_j - \alpha_j). \tag{4.1}$$

Again, let $(M, d\mu)$ and $(N, d\nu)$ be regular measure spaces and suppose that for each $\mathbf{z} \in \mathscr{D}$, $B_\mathbf{z}$ is a bilinear form on $S_0(M) \times S_0(N)$. We call the collection $\{B_{\mathbf{z}} : \mathbf{z} \in \mathscr{D}\}$ *admissible* if $\theta(\mathbf{z}) = B_{\mathbf{z}}(\varphi, \psi)$ is admissible for each $\varphi \in S_0(M)$, $\psi \in S_0(N)$.

We now fix some notation for the general case. Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_r)$, $0 < \tau_j < 1$ and let $\boldsymbol{\gamma} = (\mathbf{1} - \boldsymbol{\tau})\,\boldsymbol{\alpha} + \boldsymbol{\tau}\boldsymbol{\beta}$, i.e. $\gamma_j = (1 - \tau_j)\,\alpha_j + \tau_j\beta_j$, $j = 1,..., r$. Denote by $\mathfrak{A} = \mathfrak{A}^r$ the set of 2^r elements consisting of $\{\boldsymbol{\delta} = (\delta_1, ..., \delta_r)$, each $\delta_j = 0$ or $1\}$. If we let $\boldsymbol{\epsilon} = (\mathbf{1} - \boldsymbol{\delta})\,\boldsymbol{\alpha} + \boldsymbol{\delta}\boldsymbol{\beta}$, then $\epsilon_j = \alpha_j$ or β_j according as $\delta_j = 0$ or 1. Let $p(\boldsymbol{\delta})$, $q(\boldsymbol{\delta})$ be indices such that $1 \leqslant p(\boldsymbol{\delta}), q(\boldsymbol{\delta}) \leqslant \infty$, and define

$$1/p = \sum_{\boldsymbol{\delta}\in\mathfrak{A}} 1/p(\boldsymbol{\delta}) \prod_{j=1}^{r} [(1 - \tau_j)(1 - \delta_j) + \delta_j\tau_j] \tag{4.2}$$

$$1/q = \sum_{\boldsymbol{\delta}\in\mathfrak{A}} 1/q(\boldsymbol{\delta}) \prod_{j=1}^{r} [(1 - \tau_j)(1 - \delta_j) + \delta_j\tau_j]. \tag{4.3}$$

Note that

$$\sum_{\boldsymbol{\delta}\in\mathfrak{A}} \prod_{j=1}^{r} [(1 - \zeta_j)(1 - \delta_j) + \delta_j\zeta_j] = 1, \tag{4.4}$$

for any $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_r) \in \mathbf{C}^r$. Finally let $A(\boldsymbol{\delta}, \mathbf{y})$ be 2^r nonnegative continuous functions of $\mathbf{y} \in \mathbf{R}^r$ such that

$$\log A(\boldsymbol{\delta}, \mathbf{y}) = O\left(\exp\left(\sum a_j \left| y_j \right|\right)\right), \qquad a_j < \pi/(\beta_j - \alpha_j).$$

LEMMA 5. *Let $\{B_{\mathbf{z}}\}$ be admissible on $\mathscr{D}$. Suppose that for each $\boldsymbol{\delta} \in \mathfrak{A}$*

$$| B_{\boldsymbol{\epsilon}+i\mathbf{y}}(\varphi, \psi)| \leqslant A(\boldsymbol{\delta}, \mathbf{y}) \| \varphi \|_{p(\boldsymbol{\delta})} \| \psi \|_{q(\boldsymbol{\delta})'}$$

for all $\varphi \in S_0(M)$, $\psi \in S_0(N)$. Then

$$| B_{\boldsymbol{\gamma}}(\varphi, \psi)| \leqslant A_{\boldsymbol{\tau}} \| \varphi \|_p \| \psi \|_{q'}$$

for all $\varphi \in S_0(M)$, $\psi \in S_0(N)$ and the following estimate holds for $A_{\boldsymbol{\tau}}$:

$$\log A_{\boldsymbol{\tau}} = \sum_{\boldsymbol{\delta}\in\mathfrak{A}} \int_{R^r} \prod_{j=1}^{r} \omega[(1 - \tau_j)(1 - \delta_j) + \delta_j\tau_j, y_j] \log A(\boldsymbol{\delta}, (\boldsymbol{\beta} - \boldsymbol{\alpha})\,\mathbf{y})\, d\mathbf{y}. \tag{4.5}$$

Proof. The proof is a straightforward induction argument using Lemma 4 and (4.1)–(4.4). We omit the details.

Now for each $\mathbf{z} \in \mathscr{D} = \mathscr{D}_r(\boldsymbol{\alpha}, \boldsymbol{\beta})$, let $T_{\mathbf{z}}$ be a linear transformation $T_{\mathbf{z}} : S_0(M) \to \beta_\infty(N, \mathscr{H})$, $\mathscr{H}$ a separable Hilbert space. So for each $f \in S_0(M)$, $T_{\mathbf{z}}(f) = F_{\mathbf{z}}$ is a measurable operator-valued function

$F_{\mathbf{z}} : N \to \beta(\mathscr{H})$. We call $\{T_{\mathbf{z}}\}$ *admissible* if: (1) for each $f \in S_0(M)$, ξ, $\eta \in \mathscr{H}$, the function $t \to (F_{\mathbf{z}}(t)\, \xi, \eta)$ is locally integrable on N; and (2) for each bounded subset K of N, the function $\theta(\mathbf{z}) = \int_K (F_{\mathbf{z}}(t)\, \xi, \eta)\, d\nu(t)$ is admissible on $\mathscr{D}$.

We remark that the derivation of ([*6*], Theorem 5) from ([*6*], Lemma 8) is independent of how many complex parameters are present. Indeed that proof (with trivial modifications) applied to Lemma 5 yields the following

THEOREM 3. *Let* $T_{\mathbf{z}} : S_0(M) \to \beta_\infty(N, \mathscr{H})$ *be linear and admissible on* $\mathscr{D}$. *Suppose*

$$\| T_{\boldsymbol{\epsilon}+i\mathbf{y}}(f)\|_{q(\boldsymbol{\delta})} \leqslant A(\boldsymbol{\delta}, \mathbf{y})\| f \|_{p(\boldsymbol{\delta})}$$

for all $\boldsymbol{\delta} \in \mathfrak{A}$, $f \in S_0(M)$, $\mathbf{y} \in \mathbf{R}^r$. *Then, for every* $f \in S_0(M)$

$$\| T_{\boldsymbol{\gamma}}(f)\|_q \leqslant A_{\boldsymbol{\tau}}\| f \|_p ,$$

with $A_{\boldsymbol{\tau}}$ *again given by* (4.5).

5. A Special Interpolation Theorem

Suppose $\mathscr{D} = \mathscr{D}_r(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathscr{F}$ is an analytic operator-valued function on $\mathscr{D}$.

DEFINITION. $\mathscr{F}$ is called admissible if

$$\sup_{\alpha_j \leqslant x_j \leqslant \beta_j} \log \| \mathscr{F}(\mathbf{x} + i\mathbf{y})\|_\infty = O\left(\exp\left(\sum a_j \left| y_j \right|\right)\right), \qquad a_j < \pi/(\beta_j - \alpha_j). \tag{5.1}$$

We now make an additional assumption on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$; namely, we require that $\beta_k - \alpha_k \neq \beta_j - \alpha_j$ for $j \neq k$.

THEOREM 4. *Let* $(M, d\mu)$ *be a regular measure space and* T *a linear map from* $S_0(M)$ *to analytic operator-valued functions on* $\mathscr{D}$. *In addition, suppose* $\mathscr{F} = Tf$ *is admissible on* $\mathscr{D}$ *for each* $f \in S_0(M)$. *Let* $\boldsymbol{\delta}^0 = (\delta_1{}^0, \ldots, \delta_r{}^0) \in \mathfrak{A}$, $t_n = 0$, $r = n - 1$ *as usual, and suppose that for every* $f \in S_0(M)$:

$$\sup_{\mathbf{t} \in \mathbf{R}^r} \| \mathscr{F}(\boldsymbol{\epsilon} + i\mathbf{t})\|_\infty \prod_{1 \leqslant j < k \leqslant n} (1 + | t_k - t_j |)^{c_{kj}} \leqslant A_{\boldsymbol{\delta}}\| f \|_1 , \tag{5.2}$$

for all $\boldsymbol{\delta} \in \mathfrak{A}$, $\boldsymbol{\delta} \neq \boldsymbol{\delta}^0$, *and*

$$\left(\int \| \mathscr{F}(\boldsymbol{\epsilon}^0 + i\mathbf{t})\|_2^2 \prod_{1 \leqslant j < k \leqslant n} | t_k - t_j |^{2a_{kj}}\, d\mathbf{t}\right)^{1/2} \leqslant A_0\| f \|_2 . \tag{5.3}$$

Here we require a_{kj} to be a nonnegative integer for $1 \leqslant j < k < n$, $a_j \equiv a_{nj} \geqslant 0$, and $c_{kj} \leqslant 0$, $1 \leqslant j < k \leqslant n$. Then we can conclude:

$$\left(\int \| \mathscr{F}(\gamma + i\mathbf{t})\|_q^q \prod_{1 \leqslant j < k \leqslant n} (1 + | t_k - t_j |)^{q d_{kj}}\, d\mathbf{t}\right)^{1/q} \leqslant A \|f\|_p, \quad f \in S_0(M), \tag{5.4}$$

where $1 < p < 2$, $1/p + 1/q = 1$, $\gamma = (\mathbf{1} - \boldsymbol{\tau})\, \boldsymbol{\alpha} + \boldsymbol{\tau}\boldsymbol{\beta}$ *and* $d_{kj} = (1 - \tau_0{}^r)\, c_{kj} + \tau_0{}^r a_{kj}$. *The parameters* $\tau_0 \in \mathbf{R}$, $\boldsymbol{\tau} \in \mathbf{R}^r$ *are determined by* $\tau_0{}^r/2 = 1/q$, $0 < \tau_0 < 1$, *and* $\boldsymbol{\tau} = (1 - \tau_0)(\mathbf{1} - \boldsymbol{\delta}^0) + \boldsymbol{\delta}^0 \tau_0$ *so that*

$$\tau_j = \begin{cases} 1 - \tau_0\,, & \delta_j{}^0 = 0 \\ \tau_0\,, & \delta_j{}^0 = 1 \end{cases}, \qquad j = 1, \ldots, r.$$

Proof. We set up a situation in which we can employ Theorem 3. Let $N = \mathbf{R}^r$ and $d\nu(\mathbf{t}) = \prod_{j<k} (1 + | t_k - t_j |)^{2(a_{kj} - c_{kj})}\, d\mathbf{t}$, $t_n \equiv 0$. For $f \in S_0(M)$ and $\mathscr{F} = Tf$, set

$$F_{\mathbf{z}}(\mathbf{t}) = \mathscr{F}(\mathbf{z} + i\mathbf{t}) \prod_{j<k} (1 + | t_k - t_j |)^{c_{kj} - a_{kj}}$$
$$\times [(z_k - \epsilon_k{}^0) - (z_j - \epsilon_j{}^0) + i(t_k - t_j)]^{a_{kj}}, \tag{5.5}$$

$\mathbf{z} = (z_1, \ldots, z_r) \in \mathscr{D}$, $z_n = \epsilon_n{}^0 = 0$.

Then $T_{\mathbf{z}} : f \to F_{\mathbf{z}}$ is a linear map from $S_0(M)$ to $\beta_\infty(N, \mathscr{H})$. Since $a_{kj} \in \mathbf{Z}^+$, $k \neq n$, the factor $\prod_{1 \leqslant j < k < n}$ in (5.5) is an analytic function of $\mathbf{z} \in \mathbf{C}^r$. In addition, $a_j \geqslant 0$ and $x_j - \epsilon_j{}^0 \neq 0$ for $\alpha_j < x_j < \beta_j \Rightarrow$ we may choose a single-valued analytic branch of $[-(z_j - \epsilon_j{}^0) - it_j]^{a_j}$, $1 \leqslant j \leqslant r$. Therefore, $F_{\mathbf{z}}(\mathbf{t})$ is analytic on $\mathscr{D}$.

Now we estimate the growth of $\| F_{\mathbf{z}}(\mathbf{t})\|_\infty$. Indeed for $\mathbf{z} \in \mathscr{D}$, $\mathbf{t} \in \mathbf{R}^r$, we have

$$\| F_{\mathbf{z}}(\mathbf{t})\|_\infty = \| \mathscr{F}(\mathbf{x} + i(\mathbf{y} + \mathbf{t}))\|_\infty \prod_{j<k} (1 + | t_k - t_j |)^{c_{kj} - a_{kj}}$$
$$\times |(x_k - \epsilon_k{}^0) - (x_j - \epsilon_j{}^0) + i(y_k - y_j + t_k - t_j)|^{a_{kj}}$$
$$\leqslant A(\boldsymbol{\alpha}, \boldsymbol{\beta}) \| \mathscr{F}(\mathbf{x} + i(\mathbf{y} + \mathbf{t}))\|_\infty$$
$$\times \prod_{j<k} (1 + | t_k - t_j |)^{c_{kj}} (1 + | y_k - y_j |)^{a_{kj}}. \tag{5.6}$$

Hence, using (5.1),

$$\log \| F_{\mathbf{z}}(\mathbf{t})\|_\infty \leqslant \log A(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \sum_{j<k} [c_{kj} \log(1 + | t_k - t_j |)$$
$$+ a_{kj} \log(1 + | y_k - y_j |)] + O\left(\exp\left(\sum a_j | y_j + t_j |\right)\right). \tag{5.7}$$

It follows from (5.5) and (5.7) that $\{T_z\}$ is an admissible family. Furthermore, substituting $\mathbf{x} = \boldsymbol{\epsilon}$ in (5.6) and using (5.2), we obtain

$$\|F_{\boldsymbol{\epsilon}+i\mathbf{y}}(\mathbf{t})\|_\infty \leqslant A(\boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{j<k} (1 + |y_k - y_j + t_k - t_j|)^{-c_{kj}} (1 + |t_k - t_j|)^{c_{kj}}$$

$$\times (1 + |y_k - y_j|)^{a_{kj}} A_{\boldsymbol{\delta}} \|f\|_1$$

$$\leqslant A(\boldsymbol{\alpha}, \boldsymbol{\beta}) A_{\boldsymbol{\delta}} \prod_{j<k} (1 + |y_k - y_j|)^{a_{kj} - c_{kj}} \|f\|_1,$$

since $-c_{kj} \geqslant 0$. Therefore

$$\|F_{\boldsymbol{\epsilon}+i\mathbf{y}}\|_\infty \leqslant A(\boldsymbol{\delta}, \mathbf{y}) \|f\|_1$$

with $A(\boldsymbol{\delta}, \mathbf{y}) = A(\boldsymbol{\alpha}, \boldsymbol{\beta}) A_{\boldsymbol{\delta}} \prod_{j<k} (1 + |y_k - y_j|)^{a_{kj} - c_{kj}}$, $\boldsymbol{\delta} \in \mathfrak{A}$, $\boldsymbol{\delta} \neq \boldsymbol{\delta}^0$.
We also get from (5.3) that

$$\|F_{\boldsymbol{\epsilon}^0+i\mathbf{y}}\|_2^2 = \int \|\mathscr{F}(\boldsymbol{\epsilon}^0 + i(\mathbf{y} + \mathbf{t}))\|_2^2 \prod_{j<k} (1 + |t_k - t_j|)^{2(c_{kj} - a_{kj})}$$

$$\times |y_k - y_j + t_k - t_j|^{2a_{kj}} \, d\nu(\mathbf{t})$$

$$= \int \|\mathscr{F}(\boldsymbol{\epsilon}^0 + i(\mathbf{y} + \mathbf{t}))\|_2^2 \prod_{j<k} |(y_k + t_k) - (y_j + t_j)|^{2a_{kj}} \, d\mathbf{t}$$

$$\leqslant A_0^2 \|f\|_2^2.$$

But $q(\boldsymbol{\delta}) = \infty$, $\boldsymbol{\delta} \neq \boldsymbol{\delta}^0$ and $q(\boldsymbol{\delta}^0) = 2$. Therefore the right side of (4.3) is

$$1/2 \prod_{j=1}^{r} [(1 - \tau_j)(1 - \delta_j^0) + \delta_j^0 \tau_j] = 1/2\tau_0^r = 1/q.$$

Since $1/p(\boldsymbol{\delta}) + 1/q(\boldsymbol{\delta}) = 1$, all $\boldsymbol{\delta} \in \mathfrak{A}$, the right side of (4.2) is $1/p$. We can now apply Theorem 3 to conclude: $\|F_{\boldsymbol{\gamma}}\|_q \leqslant A_{\boldsymbol{\tau}} \|f\|_p$, $f \in S_0(M)$, with $A_{\boldsymbol{\tau}}$ determined by (4.5). That is

$$\Big(\int \|\mathscr{F}(\boldsymbol{\gamma} + i\mathbf{t})\|_q^q \prod_{j<k} (1 + |t_k - t_j|)^{q(c_{kj} - a_{kj})}$$

$$\cdot |(\gamma_k - \epsilon_k^0) - (\gamma_j - \epsilon_j^0) + i(t_k - t_j)|^{qa_{kj}} \, d\nu(\mathbf{t})\Big)^{1/q} \leqslant A_{\boldsymbol{\tau}} \|f\|_p, \qquad (5.8)$$

$\gamma_n = \epsilon_n^0 = 0$. But $\gamma_j - \epsilon_j^0 = \pm(1 - \tau_0)(\beta_j - \alpha_j)$, $1 \leqslant j \leqslant r$. Since

$\beta_k - \alpha_k \neq \beta_j - \alpha_j$, $j \neq k$, the real part of the term inside the absolute value is nonzero. Hence we infer from (5.8) that:

$$\Big(\int \| \mathscr{F}(\boldsymbol{\gamma} + i\mathbf{t})\|_q^q \prod_{j<k} (1 + | t_k - t_j |)^{qc_{kj}+2(a_{kj}-c_{kj})} \, d\mathbf{t}\Big)^{1/q}$$

$$\leqslant A(p, \boldsymbol{\alpha}, \boldsymbol{\beta}) \, A_{\boldsymbol{\tau}} \, \|f\|_p = A \, \|f\|_p \, ,$$

$f \in S_0(M)$. It suffices to note finally that $qc_{kj} + 2(a_{kj} - c_{kj}) = qc_{kj} + q\tau_0{}^r(a_{kj} - c_{kj}) = qd_{kj}$. Q.E.D.

Remark. From the proof we see that the constant A in (5.4) has the following form: $A = A(p, \boldsymbol{\alpha}, \boldsymbol{\beta})A_{\boldsymbol{\tau}}$ where $A_{\boldsymbol{\tau}}$ is given by (4.5) and

$$A(\boldsymbol{\delta}, \mathbf{y}) = A(\boldsymbol{\alpha}, \boldsymbol{\beta}) \, A_{\boldsymbol{\delta}} \prod_{j<k} (1 + | y_k - y_j |)^{a_{kj}-c_{kj}}, \qquad \boldsymbol{\delta} \in \mathfrak{A}, \qquad \boldsymbol{\delta} \neq \boldsymbol{\delta}^0$$

$$A(\boldsymbol{\delta}^0, \mathbf{y}) = A_0 \, .$$

6. A Hausdorff–Young Theorem for $SL(n, \mathbf{C})$

Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G = SL(n, \mathbf{C})$. The Fourier transform of a function $f \in L_1(G)$ is usually defined to be the operator-valued function $\mathscr{F}(\lambda) = \int_G \lambda_g f(g) \, dg$, $\lambda \in \hat{G}$. But it follows from the Plancherel formula (2.4) that only the nondegenerate principal series occurs with nonzero measure. In particular, the degenerate principal series, complementary series, degenerate complementary series, and any other exotic representations of G lurking in the wings awaiting discovery[1] do not appear in the support of the Plancherel measure. Therefore we shall consider the Fourier transform of $f \in L_1(G)$ to be the operator-valued function

$$\mathscr{F}(\lambda) = \int_G R(g, \lambda) f(g) \, dg, \qquad \lambda \in \Lambda.$$

By the uniform boundedness of $R(g, \lambda)$, $\lambda \in \Omega$, we can extend the definition of $\mathscr{F}$ to $\lambda \in \Omega$. Claim: $\mathscr{F}$ is actually an analytic operator-valued function of $\lambda \in \Omega$. It suffices to show that $\lambda \to (\mathscr{F}(\lambda) \, \xi, \eta)$, ξ, $\eta \in \mathscr{H}$, is analytic locally, say on polydiscs. But for fixed $s_1, \ldots, s_{j-1}$, $s_{j+1}, \ldots, s_r$, we can use Morera's theorem to show that $\mathscr{F}$ is analytic as a function of s_j (see [9], Section 3). The analyticity in all variables then follows by Hartog's theorem ([5], p. 28).

[1] *Added in Proof.* Stein, *Ann. of Math.*, **86**, pp. 461–490, has found some.

THEOREM 5. *Let* $1 < p < 2$, $1/p + 1/q = 1$, *and* $\tau_0 = (2/q)^{1/r}$, $0 < \tau_0 < 1$. *Let* $\lambda = (m_1, ..., m_r; s_1, ..., s_r) \in \Omega$, $m_n \equiv t_n \equiv 0$, *and* $\sigma_0 = (1 - (2/q)^{1/r})\, n/(n-1)^2$. *Suppose* $|\sigma_j| < \sigma_0$, $j = 1, ..., r$ *and that* $|\sigma_j| \neq |\sigma_k|$, $1 \leqslant j < k \leqslant n$. *Then for every* $\delta > 0$, *there exist constants* $A(n, p, \boldsymbol{\sigma}, \delta)$ *such that*

$$\left(\int_{\mathbf{R}^r} \|\mathscr{F}(\lambda)\|_q^q \prod_{j=1}^{r} (1 + |t_j|)^{1-q\nu-\delta}\, dt\right)^{1/q} \leqslant A(n, p, \boldsymbol{\sigma}, \delta) \prod_{j=1}^{r} (1 + |m_j|)^{\mu} \|f\|_p \tag{6.1}$$

for all $f \in S_0(G)$. *Here* $r = n - 1$

$$\mu = 6n^2 d[1 - (1-\tau_0)^r]/(1-\tau_0) - 1/2(1-\tau_0)^r \tag{6.2}$$

$$\nu = 6n^3 d(1 - \tau_0{}^r)/(1 - \tau_0), \tag{6.3}$$

and $d = \max_{1 \leqslant j \leqslant r} |\sigma_j| < \sigma_0$.

Proof. We set up a situation to which Theorem 4 is applicable. We are given p, $1 < p < 2$, so τ_0 is determined by $1/p = 1 - \tau_0{}^r/2$ or $1/q = \tau_0{}^r/2$, $0 < \tau_0 < 1$. We are also given $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_r)$ satisfying $d = \max |\sigma_j| < \sigma_0$ and $|\sigma_k| \neq |\sigma_j|$, $1 \leqslant j < k \leqslant n$. Let $\boldsymbol{\rho} = \boldsymbol{\sigma}/(1 - \tau_0)$. Then $\max |\rho_j| = d/(1 - \sigma_0)$, $|\rho_j| \neq |\rho_k|$, $1 \leqslant j < k \leqslant n$, and

$$|\rho_j| = |\sigma_j|/(1 - \tau_0) < n/(n-1)^2, \qquad j = 1, ..., r.$$

Next define

$$(\alpha_j, \beta_j) = \begin{cases} (\rho_j, 0), & \sigma_j < 0 \\ (0, \rho_j), & \sigma_j > 0 \end{cases}, \qquad j = 1, ..., r.$$

Clearly $\beta_j - \alpha_j = |\rho_j| \neq |\rho_k| = \beta_k - \alpha_k$, $j \neq k$. We also choose

$$\delta_j{}^0 = \begin{cases} 1, & \sigma_j < 0 \\ 0, & \sigma_j > 0 \end{cases}$$

so that $\epsilon_j{}^0 = (1 - \delta_j{}^0)\, \alpha_j + \delta_j{}^0 \beta_j = 0$ for all j.

Now for any ϵ, $0 < \epsilon < 1$, let $c = 6n^3 d(1 + \epsilon)/(1 - \tau_0)$ and define

$$c_{kj} = \begin{cases} 0, & 1 \leqslant j < k < n \\ -c, & 1 \leqslant j < k = n. \end{cases}$$

It follows from (3.3), using $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mu_0 = 3n^2d/(1-\tau_0)$, that

$$\sup_{\mathbf{t}} \| \mathscr{F}(\mathbf{m}, \boldsymbol{\epsilon} + i\mathbf{t})\|_\infty \prod_{1 \leqslant j < k \leqslant n} (1 + | t_k - t_j |)^{c_{kj}}$$

$$\leqslant A(n, d/(1-\tau_0), \epsilon) \prod_{j=1}^{r} (1 + | m_j |)^{\mu_0(1+\epsilon)} \| f \|_1 \tag{6.4}$$

for $f \in S_0(G)$ and $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}^0$.

Next we obtain an estimate for $\boldsymbol{\epsilon}^0 = \mathbf{0}$ from the Plancherel formula (2.4). The following inequalities are obvious from (2.5):

$$\omega_{kj}(\lambda) \geqslant \begin{cases} | t_k - t_j |^2, & 1 \leqslant j < k < n \\ | t_j |^2, & 1 \leqslant j < k = n, \quad m_j = 0 \\ (1 + | m_j |) | t_j |, & 1 \leqslant j < k = n, \quad m_j \neq 0. \end{cases}$$

Therefore if we let

$$a_{kj} = \begin{cases} 1, & 1 \leqslant j < k < n \\ 1, & 1 \leqslant j < k = n, \quad m_j = 0 \\ 1/2, & 1 \leqslant j < k = n, \quad m_j \neq 0, \end{cases}$$

we obtain from (2.4):

$$\left(\int \| \mathscr{F}(\mathbf{m}, \boldsymbol{\epsilon}^0 + i\mathbf{t})\|_2^2 \prod_{j<k} | t_k - t_j |^{2a_{kj}} \, d\mathbf{t}\right)^{1/2} \leqslant A(n) \prod_{j=1}^{r} (1 + | m_j |)^{-1/2} \| f \|_2 . \tag{6.5}$$

We apply Theorem 4 to (6.4) and (6.5), taking $M = G$, $d\mu =$ Haar measure on G. The result is: for $f \in S_0(G)$

$$\left(\int \| \mathscr{F}(\boldsymbol{\gamma} + i\mathbf{t})\|_q^q \prod_{j<k} (1 + | t_k - t_j |)^{qd_{kj}} \, d\mathbf{t}\right)^{1/q} \leqslant A \| f \|_p , \tag{6.6}$$

where $d_{kj} = (1 - \tau_0{}^r) c_{kj} + \tau_0{}^r a_{kj}$, $\boldsymbol{\gamma} = (1 - \boldsymbol{\tau}) \boldsymbol{\alpha} + \boldsymbol{\tau\beta}$, $\boldsymbol{\tau} = (1 - \tau_0)(1 - \boldsymbol{\delta}^0) + \boldsymbol{\delta}^0 \tau_0$, and the constant A remains to be estimated.

(1) $$\gamma_j = (1 - \tau_j) \alpha_j + \tau_j \beta_j = \begin{cases} (1 - \tau_j) \rho_j , & \sigma_j < 0 \\ \tau_j \rho_j , & \sigma_j > 0 \end{cases} =$$

$(1 - \tau_0) \rho_j = \sigma_j$, that is $\boldsymbol{\gamma} = \boldsymbol{\sigma}$.

(2) $$d_{kj} \geqslant \begin{cases} \tau_0{}^r, & 1 \leqslant j < k < n \\ \tfrac{1}{2}\tau_0{}^r - (1 - \tau_0{}^r) c, & 1 \leqslant j < k = n. \end{cases}$$

Hence $qd_{kj} \geqslant 0$, $1 \leqslant j < k < n$ and

$$qd_j = qd_{nj} = 1 - q6n^3\, d(1 - \tau_0{}^r)(1 + \epsilon)/(1 - \tau_0)$$
$$\geqslant 1 - q\nu - \delta$$

so long as $\epsilon \leqslant \delta/q\nu$.

(3) By the remark following Theorem 4, $A = A(p, \boldsymbol{\alpha}, \boldsymbol{\beta})\, A_\tau$; here $A(p, \boldsymbol{\alpha}, \boldsymbol{\beta})$ depends on n, p, and $\boldsymbol{\sigma}$, and A_τ is given by (4.5) with

$$A(\boldsymbol{\delta}, \mathbf{y}) = A(\boldsymbol{\alpha}, \boldsymbol{\beta})\, A_{\boldsymbol{\delta}} \prod_{j<k} (1 + | y_k - y_j |)^{a_{kj} - c_{kj}},$$

$$A_{\boldsymbol{\delta}} = A(n, p, \boldsymbol{\sigma}, \epsilon) \prod_{j=1}^{r} (1 + | m_j |)^{\mu_0(1+\epsilon)} \qquad \boldsymbol{\delta} \neq \boldsymbol{\delta}^0$$

$$A_0 = A(n) \prod_{j=1}^{r} (1 + | m_j |)^{-1/2}.$$

It follows that

$$\log A_\tau = O(1) + \log \prod_{j=1}^{r} (1 + | m_j |)^{\mu_0(1+\epsilon)}$$
$$\times \sum_{\boldsymbol{\delta} \neq \boldsymbol{\delta}^0} \int \prod_{j=1}^{r} \omega[(1 - \tau_j)(1 - \delta_j) + \delta_j \tau_j , y_j]\, d\mathbf{y}$$
$$+ \log \prod_{j=1}^{r} (1 + | m_j |)^{-1/2} \int \prod_{j=1}^{r} \omega[(1 - \tau_j)(1 - \delta_j{}^0) + \delta_j{}^0 \tau_j , y_j]\, d\mathbf{y}$$
$$= O(1) + A_1 \log \prod_{j=1}^{r} (1 + | m_j |)^{\mu_0(1+\epsilon)} + A_2 \log \prod_{j=1}^{r} (1 + | m_j |)^{-1/2},$$

where the constant $O(1)$ depends on n, p, $\boldsymbol{\sigma}$, and δ, but not on $\mathbf{m}$ or f. But $(1 - \tau_j)(1 - \delta_j{}^0) + \delta_j{}^0 \tau_j = \tau_0$ and $\int \omega(\tau_0 , y_j)\, dy_j = 1 - \tau_0$, $j = 1,..., r$; therefore $A_2 = (1 - \tau_0)^r$. Since $A_1 + A_2 = 1$, we must have $A_1 = 1 - (1 - \tau_0)^r$ and

$$A = A(n, p, \boldsymbol{\sigma}, \delta) \prod_{j=1}^{r} (1 + | m_j |)^{\mu_0(1+\epsilon)[1-(1-\tau_0)^r] - (1-\tau_0)^r/2}$$
$$\leqslant A(n, p, \boldsymbol{\sigma}, \delta) \prod_{j=1}^{r} (1 + | m_j |)^{\mu}.$$

From (6.6) and (1)–(3) above, we get the result (6.1).

COROLLARY. *Let* $1 < p < 2$, $1/p + 1/q = 1$. *Then there exists* $\sigma_0' > 0$ *such that*

$$\left(\int_{\mathbf{R}^r} \| \mathscr{F}(\mathbf{m}, \mathbf{s}) \|_q^q \, dt\right)^{1/q} \leqslant A(n, p, \boldsymbol{\sigma}) \| f \|_p, \qquad f \in S_0(G)$$

whenever $0 < |\sigma_j| < \sigma_0$, $j = 1,..., r$ *and* $|\sigma_j| \neq \sigma_k$, $j \neq k$.

Proof. From (6.1) we see that σ_0' must satisfy three requirements. First σ_0' must be small enough ($\sigma_0' < \sigma_0$) so that Theorem 5 is valid. Next σ_0' must satisfy $1 - q6n^3\sigma_0'(1 - \tau_0{}^r)/(1 - \tau_0) \geqslant 0$. Then for any $\boldsymbol{\sigma} = (\sigma_1,..., \sigma_r)$ such that $0 < |\sigma_j| < \sigma_0'$, $1 \leqslant j \leqslant r$, we can select $\delta > 0$ so that $1 - q\nu - \delta > 0$ (see (6.3)). Finally we require that $6n^2\sigma_0'[1 - (1 - \tau_0)^r]/(1 - \tau_0) \leqslant 1/2(1 - \tau_0)^r$. Then for $\boldsymbol{\sigma} = (\sigma_1,..., \sigma_r)$, $0 < |\sigma_j| < \sigma_0'$, $1 \leqslant j \leqslant r$, we have $\mu < 0$, μ given by (6.2). Q.E.D.

7. THE L_p CONVOLUTION THEOREM

We now prove our main result: convolution by an $L_p(G)$ function, $1 \leqslant p < 2$, is a bounded operator on $L_2(G)$. Recall that the analyticity result ([*6*], Lemma 26) or ([*9*], Lemma 3) plays a crucial role in the proof of the L_p convolution theorem for $SL(2, \mathbf{R})$ and $SL(2, \mathbf{C})$. We first generalize that result.

LEMMA 6. *Let* $\varphi(\mathbf{z}) = \varphi(z_1,..., z_r)$ *be an analytic function in an open region containing* $\mathscr{D} = \mathscr{D}_r(\boldsymbol{\alpha}, \boldsymbol{\beta})$. *Suppose that for some* $K \geqslant 1$ *and* $c \geqslant 0$,

$$\sup_{\alpha_j \leqslant x_j \leqslant \beta_j} |\varphi(\mathbf{x} + i\mathbf{y})| \leqslant K \prod_{j=1}^{r} (1 + |y_j|)^c, \qquad \mathbf{y} \in \mathbf{R}^r.$$

Let $\boldsymbol{\epsilon} = (\mathbf{1} - \boldsymbol{\delta})\, \boldsymbol{\alpha} + \boldsymbol{\delta}\boldsymbol{\beta}$, $\boldsymbol{\delta} \in \mathfrak{A}^r$ *and suppose that for some* $q > 1$,

$$\int_{\mathbf{R}^r} |\varphi(\boldsymbol{\epsilon} + i\mathbf{y})|^q \, dy \leqslant 1, \qquad \text{all} \quad \boldsymbol{\delta} \in \mathfrak{A}.$$

Then for $\alpha_j < \gamma_j < \beta_j$, $j = 1,..., r$, *we have*

$$\sup_{\mathbf{y}} |\varphi(\boldsymbol{\gamma} + i\mathbf{y})| \leqslant \prod_{j=1}^{r} A_j$$

where $A_j = c_1[(\gamma_j - \alpha_j)^{-1/q} + (\beta_j - \gamma_j)^{-1/q}]$, c_1 *an absolute constant* (i.e. *independent of* φ, K *or* c).

Proof. For $r = 1$, this follows from ([6], Lemma 26). Suppose $r > 1$; we give a proof by induction. Assume the result for all dimensions $< r$. Choose $\delta_r = 0$ and let $p = q'$, i.e. $1/p + 1/q = 1$, $1 < p < \infty$. Let $\theta \in C_0(\mathbf{R})$ be arbitrary such that $\int_{\mathbf{R}} | \theta |^p \leqslant 1$. Define

$$\Phi(\mathbf{z}') = \Phi(z_1, \ldots, z_{r-1}) = \int \varphi(z_1, \ldots, z_{r-1}, \alpha_r + iy_r)\, \theta(y_r)\, dy_r .$$

(In this proof we use primes, e.g. $\mathbf{z}' = \mathbf{x}' + i\mathbf{y}'$, to denote $r - 1$ variables.) Then Φ is analytic in an open region containing $\alpha_j \leqslant x_j \leqslant \beta_j$, $j = 1, \ldots, r - 1$ and

$$\sup_{\substack{\alpha_j \leqslant x_j \leqslant \beta_j \\ 1 \leqslant j \leqslant r-1}} | \Phi(\mathbf{x}' + i\mathbf{y}')| \leqslant K_1 \prod_{j=1}^{r-1} (1 + | y_j |)^c,$$

since θ has compact support.

Next let $\psi \in C_0(\mathbf{R}^{r-1})$ be arbitrary such that $\int_{\mathbf{R}^{r-1}} | \psi |^p \leqslant 1$. We compute

$$\begin{aligned} \left| \int \Phi(\boldsymbol{\epsilon}' + i\mathbf{y}')\, \psi(\mathbf{y}')\, d\mathbf{y}' \right| &= \left| \iint \varphi(\boldsymbol{\epsilon}' + i\mathbf{y}', \alpha_r + iy_r)\, \theta(y_r)\, \psi(\mathbf{y}')\, dy_r\, d\mathbf{y}' \right| \\ &\leqslant \left(\iint | \varphi(\boldsymbol{\epsilon}' + i\mathbf{y}', \alpha_r + iy_r)|^q\, dy_r\, d\mathbf{y}' \right)^{1/q} \\ &\quad \times \left(\int | \theta(y_r)|^p\, dy_r \right)^{1/p} \left(\int | \psi(\mathbf{y}')|^p\, d\mathbf{y}' \right)^{1/p} \\ &\leqslant 1, \qquad \text{for any} \quad \boldsymbol{\delta}' \in \mathfrak{A}^{r-1}. \end{aligned}$$

Since ψ is arbitrary $\int | \Phi(\boldsymbol{\epsilon}' + i\mathbf{y}')|^q\, d\mathbf{y}' \leqslant 1$. By the induction hypothesis

$$\sup_{\mathbf{y}'} | \Phi(\boldsymbol{\gamma}' + i\mathbf{y}')| \leqslant \prod_{j=1}^{r-1} A_j = A(r-1);$$

that is

$$\sup_{\mathbf{y}'} \left| \int \varphi(\boldsymbol{\gamma}' + i\mathbf{y}', \alpha_r + iy_r)\, \theta(y_r)\, dy_r \right| \leqslant A(r-1).$$

Since θ is arbitrary, we conclude

$$\sup_{\mathbf{y}'} \int | \varphi(\boldsymbol{\gamma}' + i\mathbf{y}', \alpha_r + iy_r)|^q\, dy_r \leqslant A(r-1)^q. \tag{7.1}$$

By a similar argument with $\delta_r = 1$, we get

$$\sup_{\mathbf{y}'} \int |\varphi(\boldsymbol{\gamma}' + i\mathbf{y}', \beta_r + iy_r)|^q \, dy_r \leqslant A(r-1)^q. \tag{7.2}$$

Finally let $\mathbf{y}' \in R^{r-1}$ be arbitrary but fixed. Define $\Psi(z_r) = (1/A(r-1))\, \varphi(\boldsymbol{\gamma}' + i\mathbf{y}', z_r)$. Ψ is analytic in an open region containing $\alpha_r \leqslant \operatorname{Re}(z_r) \leqslant \beta_r$, and

$$\sup_{\alpha_r \leqslant x_r \leqslant \beta_r} |\Psi(x_r + iy_r)| \leqslant K_2(1 + |y_r|)^c,$$

since $\mathbf{y}'$ is fixed. By (7.1) and (7.2)

$$\int |\Psi(\alpha_r + iy_r)|^q \, dy_r \leqslant 1; \qquad \int |\Psi(\beta_r + iy_r)|^q \, dy_r \leqslant 1.$$

Applying the one-dimensional case, we obtain

$$\sup_{y_r} |\Psi(\gamma_r + iy_r)| \leqslant A_r .$$

But A_r is independent of $\mathbf{y}'$, therefore

$$\sup_{\mathbf{y} \in \mathbf{R}^r} |\varphi(\boldsymbol{\gamma} + i\mathbf{y})| \leqslant \prod_{j=1}^{r} A_j . \qquad \text{Q.E.D.}$$

LEMMA 7. *Let* $1 \leqslant p < 2$. *Then for every* $f \in S_0(G)$

$$\|\mathscr{F}(\lambda)\|_\infty \leqslant A(n, p) \|f\|_p , \qquad \lambda \in \Lambda.$$

$A(n, p)$ depends only *on the parameter p and the dimension n of the complex semisimple Lie group $G = SL(n, \mathbf{C})$.*

Proof. The case $p = 1$ is trivial (since $R(g, \lambda)$, $\lambda \in \Lambda$, is unitary). Suppose $1 < p < 2$, $1/p + 1/q = 1$. Normalize by assuming $\|f\|_p = 1$. Consider σ_0' of the corollary in Section 6. Choose $\rho_1, \ldots, \rho_r$ such that $0 < \rho_1 < \rho_2 < \cdots < \rho_r < \sigma_0'$. For $\boldsymbol{\delta} \in \mathfrak{A} = \mathfrak{A}^r$, denote $\boldsymbol{\rho}(\boldsymbol{\delta}) = (\pm\rho_1, \ldots, \pm\rho_r)$, where we choose $+\rho_j$ if $\delta_j = 0$, $-\rho_j$ if $\delta_j = 1$. By that corollary

$$\int \|\mathscr{F}(\mathbf{m}, \boldsymbol{\rho}(\boldsymbol{\delta}) + i\mathbf{t})\|_q^q \, d\mathbf{t} \leqslant A(n, p, \boldsymbol{\rho}(\boldsymbol{\delta}))^q \leqslant A(n, p)^q, \qquad \boldsymbol{\delta} \in \mathfrak{A}.$$

There are only 2^r possibilities for $\boldsymbol{\delta}$ and we consider $\rho_1, \ldots, \rho_r$ fixed throughout the remainder of the proof.

Let $\xi, \eta \in \mathscr{H}$, $\|\xi\| \leqslant 1$, $\|\eta\| \leqslant 1$. If we define $\varphi(\mathbf{s}) = (\mathscr{F}(\mathbf{m}, \mathbf{s})\, \xi, \eta)/A(n, p)$, then

$$|\varphi(\mathbf{s})| \leqslant \|\mathscr{F}(\mathbf{m}, \mathbf{s})\|_\infty / A(n, p) \leqslant \|\mathscr{F}(\mathbf{m}, \mathbf{s})\|_q / A(n, p).$$

Hence

$$\int |\varphi(\rho(\delta) + i\mathbf{t})|^q\, d\mathbf{t} \leqslant 1, \qquad \delta \in \mathfrak{A}.$$

Moreover φ is analytic in an open region containing $-\rho_j \leqslant \mathrm{Re}(s_j) \leqslant \rho_j$, $j = 1,\ldots, r$; and by (2.3)

$$\sup_{|\sigma_j| \leqslant |\rho_j|} |\varphi(\mathbf{s})| \leqslant K \prod_{j=1}^{r} (1 + |t_j|)^c,$$

$K = A(n, \sigma_0') \prod_{j=1}^r (1 + |m_j|)^3 \|f\|_1$, $c = 6n$. Note K depends on $\mathbf{m}$ and f, but by Lemma 6, that is immaterial. Indeed Lemma 6 $\Rightarrow$

$$\sup_{\mathbf{t} \in \mathbf{R}^r} |\varphi(i\mathbf{t})| \leqslant \prod_{j=1}^{r} A_j,$$

$A_j = A(-\rho_j, 0, \rho_j, q) = A(q)$. Taking the supremum over all ξ, η of norm $\leqslant 1$, we obtain

$$\sup_{\mathbf{t}} \|\mathscr{F}(\mathbf{m}, i\mathbf{t})\|_\infty \leqslant A(n, p)\, A(q) = A(n, p). \tag{7.3}$$

Dropping the normalization $\|f\|_p = 1$ and noting that the right side of (7.3) is independent of $\mathbf{m}$, we get

$$\|\mathscr{F}(\lambda)\|_\infty \leqslant A(n, p) \|f\|_p, \qquad \lambda \in \Lambda, \qquad f \in S_0(G).$$

The L_p convolution theorem for $G = SL(n, \mathbf{C})$ follows easily.

THEOREM 6. *For each p, $1 \leqslant p < 2$, there exists a constant $A(n, p)$ such that*

$$\|f * h\|_2 \leqslant A(n, p) \|f\|_2 \|h\|_p \tag{7.4}$$

for all $f \in L_2(G)$, $h \in L_p(G)$.

Proof. Of course it suffices to prove (7.4) for a dense class of functions, so assume $f, h \in S_0(G)$. Denote $R(f, \lambda) = \int R(g, \lambda) f(g)\, dg$ and $R(h, \lambda) = \int R(g, \lambda)\, h(g)\, dg$, $\lambda \in \Lambda$. If $k = f * h$ is defined by (1.1), then

$$R(k, \lambda) = \int R(g, \lambda)\, k(g)\, dg = R(f, \lambda)\, R(h, \lambda), \qquad \lambda \in \Lambda.$$

By (1.2) and Lemma 7

$$\| R(k, \lambda)\|_2 \leqslant \| R(f, \lambda)\|_2 \| R(h, \lambda)\|_\infty$$
$$\leqslant A(n, p)\| R(f, \lambda)\|_2 \| h \|_p , \qquad 1 \leqslant p < 2.$$

Computing the L_2 norms of k and f via the Plancherel formula (2.4), we obtain

$$\| k \|_2^2 = C_n \sum_{\mathbf{Z}^r} \int_{\mathbf{R}^r} \| R(k, \lambda)\|_2^2 \prod_{j<k} \omega_{kj}(\lambda)\, dt$$
$$= \int \| R(k, \lambda)\|_2^2 \, d\hat{\mu}(\lambda)$$
$$\leqslant \int A(n, p)^2 \| h \|_p^2 \| R(f, \lambda)\|_2^2 \, d\hat{\mu}(\lambda)$$
$$= A(n, p)^2 \| h \|_p^2 \| f \|_2^2 . \qquad \text{Q.E.D.}$$

8. A Riemann–Lebesgue Lemma for $SL(n, \mathbf{C})$

In this section we obtain the analog of ([9], Theorem 6)—namely, the Fourier transform of $f \in L_p$, $1 \leqslant p < 2$, vanishes at infinity on Λ. It has already been pointed out ([9], Section 5) that this is not true in general for locally compact abelian groups.

Lemma 8. *Let* $1 < p < 2$ *and* $\lambda = (\mathbf{m}, \mathbf{s}) \in \Omega$. *Assume that* $| \sigma_j | < \sigma_0$, $1 \leqslant j \leqslant r$. *Then there exists a constant* $A(n, p, \lambda)$ *such that*

$$\| \mathscr{F}(\lambda)\|_\infty \leqslant A(n, p, \lambda)\| f \|_p , \tag{8.1}$$

for all $f \in S_0(G)$.

Proof. Recall that σ_0 was defined in Theorem 5,

$$\sigma_0 = (1 - (2/q)^{1/r})\, n/(n-1)^2, \qquad 1/p + 1/q = 1.$$

Normalize by assuming $\| f \|_p = 1$. Choose $\rho_1, ..., \rho_r$ so that $0 < \rho_1 < \rho_2 < \cdots < \rho_r < \sigma_0$. As usual for $\boldsymbol{\delta} \in \mathfrak{A}^r$, let $\boldsymbol{\rho}(\boldsymbol{\delta}) = (\pm\rho_1, ..., \pm\rho_r)$, choosing $+\rho_j$ if $\delta_j = 0$, $-\rho_j$ if $\delta_j = 1$.

Now let $\delta > 0$, $\nu = 6n^3\rho_r(1 - \tau_0{}^r)/(1 - \tau_0)$ (see (6.3)), $\theta = 1 - q\nu - \delta$, and $\mu = 6n^2\rho_r[1 - (1 - \tau_0)^r]/(1 - \tau_0) - \frac{1}{2}(1 - \tau_0)^r$ (see (6.2)). It follows from (6.1) that:

$$\int \| \mathscr{F}(\mathbf{m}, \boldsymbol{\rho}(\boldsymbol{\delta}) + i\mathbf{t})\|_q^q \prod_{j=1}^r (1 + | t_j |)^\theta \, dt \leqslant A^q$$

where

$$A = A(n, p, \rho_r, \delta) \prod_{j=1}^{r} (1 + | m_j |)^{\mu}. \tag{8.2}$$

Let $\xi, \eta \in \mathscr{H}$, $\| \xi \| \leqslant 1$, $\| \eta \| \leqslant 1$. Define $\psi(\mathbf{s}) = (\mathscr{F}(\mathbf{m}, \mathbf{s}) \xi, \eta)$. Clearly ψ is analytic in an open region containing $-\rho_j \leqslant \operatorname{Re}(s_j) \leqslant \rho_j$, $1 \leqslant j \leqslant r$, and

$$\int | \psi(\boldsymbol{\rho}(\boldsymbol{\delta}) + i\mathbf{t})|^q \prod_{j=1}^{r} (1 + | t_j |)^{\theta} \, d\mathbf{t} \leqslant A^q.$$

We also define

$$\varphi(\mathbf{s}) = c_2 A^{-1} \prod_{j=1}^{r} (1 + s_j)^{\theta/q} \psi(\mathbf{s}).$$

If we choose the constant c_2 appropriately (depending on n, p, and ρ_r), then φ is an analytic function on an open region containing $-\rho_j \leqslant \operatorname{Re}(s_j) \leqslant \rho_j$, $1 \leqslant j \leqslant r$, and

$$\int | \varphi(\boldsymbol{\rho}(\boldsymbol{\delta}) + i\mathbf{t})|^q \, d\mathbf{t} \leqslant 1, \qquad \boldsymbol{\delta} \in \mathfrak{A}.$$

The growth condition of Lemma 6 is satisfied for φ with any exponent $> 6n + \theta/q$. Therefore by Lemma 6

$$\sup_{\mathbf{t}} | \varphi(\boldsymbol{\sigma} + i\mathbf{t})| \leqslant \prod_{j=1}^{r} c_1[| \sigma_j - \rho_j |^{-1/q} + | \sigma_j + \rho_j |^{-1/q}],$$

$| \sigma_j | < \rho_j$, $1 \leqslant j \leqslant r$. That is

$$\prod_{j=1}^{r} | 1 + s_j |^{\theta/q} |(\mathscr{F}(\mathbf{m}, \mathbf{s}) \xi, \eta)|$$

$$\leqslant A(n, p, \rho_r) \prod_{j=1}^{r} [| \sigma_j - \rho_j |^{-1/q} + | \sigma_j + \rho_j |^{-1/q}].$$

Taking the supremum over all ξ, η of norm $\leqslant 1$, and dropping the restriction on f, we obtain:

$$\prod_{j=1}^{r} | 1 + s_j |^{\theta/q} \| \mathscr{F}(\mathbf{m}, \mathbf{s})\|_{\infty} \leqslant A \prod_{j=1}^{r} [| \sigma_j - \rho_j |^{-1/q} + | \sigma_j + \rho_j |^{-1/q}] \| f \|_p \tag{8.3}$$

and the constant A is given by (8.2).

Finally we note that (8.3) holds for every $\mathbf{s}$ in the open domain $|\sigma_j| < \sigma_0$, $1 \leqslant j \leqslant r$. Indeed for such an $\mathbf{s}$ we need only choose $\boldsymbol{\rho}$ at the beginning so that $\max |\sigma_j| < \rho_1 < \cdots < \rho_r < \sigma_0$. This completes the proof.

LEMMA 9. *The Fourier transform, initially defined for* $f \in L_1 \cap L_p$, $1 \leqslant p < 2$, *has a unique bounded extension to all of* $L_p(G)$. *For each* $f \in L_p$, $1 \leqslant p < 2$, $\mathscr{F}(\mathbf{m}, \mathbf{s})$ *is analytic in*

$$\Omega_p = \{(\mathbf{m}, \mathbf{s}) \in \mathbf{Z}^r \times \mathbf{C}^r : |\operatorname{Re}(s_j)| < (1 - (2/q)^{1/r})\, n/(n-1)^2\}$$

and satisfies (8.3) *when* $1 < p < 2$.

Proof. Lemma 9 follows from Lemma 8 by the exact same proof used to derive ([9], Theorem 6) from ([9], Lemma 4). The only point we have to make is that the constants $A(n, p, \lambda)$ of (8.1) are uniformly bounded on compact subsets of Ω_p.

We conclude with our version of the Riemann–Lebesgue lemma on G.

THEOREM 7. *Let* $1 \leqslant p < 2$ *and* $f \in L_p(G)$. *Suppose* $\mathscr{F} = \mathscr{F}(\lambda)$ *is the Fourier transform of* f. *Then the function* $\lambda \to \|\mathscr{F}(\lambda)\|_\infty$ *vanishes at infinity on* $\Lambda = \mathbf{Z}^r \times (i\mathbf{R})^r$.

Proof. $p = 1$. Consider the set $\hat{G}$ of equivalence classes of irreducible unitary representations of $G = SL(n, \mathbf{C})$. $\hat{G}$ possesses a certain natural—although non-Hausdorff—topology called the *hull-kernel* topology. For $f \in L_1(G)$ and $\epsilon > 0$, the set $\{\lambda \in \hat{G} : \|\mathscr{F}(\lambda)\| \geqslant \epsilon\}$ is quasi-compact in this topology ([1], p. 317). Since the support of the Plancherel measure, i.e. Λ, is closed in $\hat{G}$ ([1], Sections 18.3.1, 18.3.2, 18.8.4), the set $\{\lambda \in \Lambda : \|\mathscr{F}(\lambda)\| \geqslant \epsilon\}$ is also quasi-compact. But the hull-kernel topology restricted to Λ is actually Hausdorff and coincides with the ordinary topology ([2], Section 17). Therefore $\|\mathscr{F}(\lambda)\|_\infty$ vanishes at infinity on Λ.

The interesting part of the theorem is the case $1 < p < 2$. From Lemma 8, we know $\|\mathscr{F}(\lambda)\|_\infty \leqslant A(n, p, \lambda)\, \|f\|_p$, $\lambda \in \Omega_p$, $f \in S_0(G)$. Here $A(n, p, \lambda)$ is specified by formulas (8.2) and (8.3); and, by Lemma 9, the inequality holds for *all* $f \in L_p(G)$. In the proof of Lemma 8, choose $\rho_r > 0$ and $\delta > 0$ so small that $\theta > 0$ in (8.3) and $\mu < 0$ in (8.2). Upon setting $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) = \mathbf{0}$ in (8.3), we obtain

$$\|\mathscr{F}(\lambda)\|_\infty = \|\mathscr{F}(\mathbf{m}, i\mathbf{t})\|_\infty \leqslant A(n, p, \rho_r, \delta) \prod_{j=1}^{r} (1 + |m_j|)^\mu (1 + t_j^2)^{-\theta/2q} \|f\|_p. \tag{8.4}$$

It is now clear from (8.4) that $\| \mathscr{F}(\lambda)\|_\infty$ vanishes at infinity on $\Lambda = \mathbf{Z}^r \times (i\mathbf{R})^r$.

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