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Spherical Projections and Centrally Symmetric Sets

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1. INTRODUCTION

Centrally symmetric convex bodies in d-dimensional Euclidean space $\mathbf{R}^d$ are related to various transforms of functions on the unit sphere S^{d-1} in $\mathbf{R}^d$. In this paper we will investigate how this relationship is affected by projections of the bodies onto lower dimensional subspaces. The results we obtain will also give information about central sections of certain star-shaped sets.

The *cosine transform* T, in $\mathbf{R}^d$, is defined on the Montel space $\mathscr{C}_e^\infty$ of even, infinitely differentiable functions on S^{d-1} by

$$(Tf)(x) = \int_{S^{d-1}} |\langle x, u \rangle| \, f(u) \, \lambda_{d-1}(du), \qquad \text{for} \quad x \in S^{d-1} \quad \text{and} \quad f \in \mathscr{C}_e^\infty,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbf{R}^d$ and λ_j denotes the j-dimensional spherical Lebesgue measure. We will denote by $\mathscr{K}_0$ the class of all convex bodies (non-empty, compact, convex sets) which are centrally symmetric with respect to the origin. In [16] it was shown that, corresponding

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to every $K \in \mathscr{K}_0$ with support function $h(K, \cdot)$, there is a *generating distribution* ρ_K defined by

$$\rho_K(f) = \int_{S^{d-1}} h(K, u)(T^{-1}f)(u)\, \lambda_{d-1}(du), \qquad \text{for} \quad f \in \mathscr{C}_e^\infty.$$

This definition makes use of the fact that T is a bijection on $\mathscr{C}_e^\infty$ (see, for example, [12]). In fact, T can be extended to a bicontinuous bijection on the dual space $\mathscr{D}_e$ of even distributions on S^{d-1}; see [6] for further details. It follows that ρ_K is the inverse $T^{-1}h(K, \cdot)$ where, as usual, even continuous functions (and even signed Borel measures) are thought of as special types of distributions. The body K is said to be a *generalized zonoid* if ρ_K is a signed measure, and a *zonoid* if ρ_K is a positive measure. Throughout the paper, the word positive means non-negative when it is applied to functions, measures, or distributions. Background information on zonoids, generalized zonoids, and centrally symmetric bodies can be found in [14; 7; 13, in particular Section 3.5]).

Orthogonal projections $K|L$ of zonoids and generalized zonoids K onto subspaces L of $\mathbf{R}^d$ where discussed in [17]. There, a projection operator π_L, acting on functions and measures on S^{d-1}, was defined in such a way that the generating measure of $K|L$ (which is again a generalized zonoid) satisfies

$$\rho_{K|L} = \pi_L \rho_K. \tag{1.1}$$

This was used to obtain a characterization of those signed measures which are generating measures of generalized zonoids. Our first goal in this paper is to extend these results to arbitrary centrally symmetric convex bodies and to obtain a characterization of those distributions which are generating distributions of such bodies.

Equation (1.1) reveals a kind of commutativity between the operators π_L and T. For example, for $f \in \mathscr{C}_e^\infty$, we have

$$(Tf)|_L = T_L \pi_L f. \tag{1.2}$$

Here T_L denotes the cosine transform in L and, for functions g defined on S^{d-1}, we denote by $g|_L$ the restriction of g to $S^{k-1}(L)$, the unit sphere of the k-dimensional subspace L. In (1.2) and all that follows, we assume $k \geqslant 2$. Our second goal is to establish a relation similar to (1.2) for another important spherical transform in $\mathbf{R}^d$, the *Radon transform* R. This is the (bicontinuous) bijection on $\mathscr{C}_e^\infty$ defined by

$$(Rf)(x) = \int_{S^{d-2}(x^\perp)} f(v)\, \lambda_{d-2}(dv), \qquad \text{for} \quad x \in S^{d-1} \quad \text{and} \quad f \in \mathscr{C}_e^\infty,$$

where $x^{\perp}$ is the subspace orthogonal to x. We will show that

$$(Rf)|_L = R_L \tau_L f, \tag{1.3}$$

where R_L denotes the spherical Radon transform in the subspace L and τ_L is another projection operator on the sphere. We will use (1.3) in the study of centrally symmetric, star-shaped sets which are *intersection bodies*. Intersection bodies have recently been of interest because of their connection with the Busemann–Petty problem (see [3, 10, 18]). Corresponding to the fact that the projection of a zonoid is a zonoid, we will show, in Section 4, that the central section of an intersection body is an intersection body.

It is known that there is a strong connection between the operators R and T, namely

$$R = \Box T = T\Box, \tag{1.4}$$

where $\Box$ is a second-order differential operator on S^{d-1} (see [6], where a slightly different normalization is used). The connection between (1.2) and (1.3) becomes more apparent from the relationship between the projection operators π_L and τ_L and the *block operators* $\Box$ on S^{d-1} and $\Box_L$ on $S^{k-1}(L)$. These equations could be used to show that

$$\pi_L \Box f = \Box_L \tau_L f, \tag{1.5}$$

for all $f \in \mathscr{C}_e^{\infty}$. In the final section we will use techniques from harmonic analysis to show that (1.5) holds for all $f \in \mathscr{C}^{\infty}$, the space of infinitely differentiable functions on S^{d-1}.

The reader is referred to [8, 15] for background information on functional analysis, in particular on the theory of distributions; to [1, 9, 13] for the geometry of convex bodies, and to [4] for star-shaped bodies, in particular intersection bodies.

2. SPHERES AND SUBSPHERES

We will denote by $\mathscr{C}_e$ the Banach space of even continuous functions on S^{d-1}. Its dual $\mathscr{M}_e$ is the space of finite signed (even) Borel measures on S^{d-1}. The space $\mathscr{M}_e$ will be given the weak* topology whereas $\mathscr{D}_e$ will be given the strong topology, unless otherwise stated. If $f \in \mathscr{C}_e$ we may put

$$\mu_f(A) = \int_A f(u)\, \lambda_{d-1}(du), \qquad \text{for all Borel sets} \quad A \subset S^{d-1}, \tag{2.1}$$

to obtain a measure $\mu_f \in \mathscr{M}_e$ which is absolutely continuous with respect to λ_{d-1} and which has Radon Nikodym derivative $d\mu_f/d\lambda_{d-1} = f$.

Throughout this work we will only use Radon Nikodym derivatives in the context of absolutely continuous measures. As usual, measures of the form μ_f defined in (2.1) allow us to view $\mathscr{C}_e$ as a subspace of $\mathscr{M}_e$ which, in turn, is a subspace of $\mathscr{D}_e$. We mentioned in the Introduction that the transforms T and R can be extended to bicontinuous bijections on $\mathscr{D}_e$. We recall that this follows from the self adjointness of both operators. To be precise, if $S = T$ or R, we have

$$\int_{S^{d-1}} f(u)(Sg)(u)\,\lambda_{d-1}(du)$$

$$= \int_{S^{d-1}} (Sf)(u)\, g(u)\,\lambda_{d-1}(du) \qquad \text{for all} \quad f, g \in \mathscr{C}_e^{\infty},$$

see [4], for example. It follows that the extensions to $\mathscr{D}_e$ may be defined by

$$(T\rho)(f) = \rho(Tf) \quad \text{and} \quad (R\rho)(f) = \rho(Rf) \qquad \text{for all} \quad f \in \mathscr{C}_e^{\infty} \quad \text{and} \quad \rho \in \mathscr{D}_e.$$

Consequently, $T\colon \mathscr{M}_e \to \mathscr{M}_e$ and $R\colon \mathscr{M}_e \to \mathscr{M}_e$, and these are continuous transformations. If $\mu \in \mathscr{M}_e$ and $f \in \mathscr{C}_e^{\infty}$, we have

$$(T\mu)(f) = \int_{S^{d-1}} (Tf)(u)\,\mu(du) = \int_{S^{d-1}} \int_{S^{d-1}} |\langle u, x\rangle|\, f(x)\,\lambda_{d-1}(dx)\,\mu(du)$$

and so, by Fubini's Theorem, $T\mu \in \mathscr{C}_e$ is the function given by

$$(T\mu)(u) = \int_{S^{d-1}} |\langle u, x\rangle|\,\mu(dx) \qquad \text{for} \quad u \in S^{d-1}. \tag{2.2}$$

The analogue for the Radon transform is

$$(R\mu)(f) = \int_{S^{d-1}} (Rf)(u)\,\mu(du) = \int_{S^{d-1}} \int_{S^{d-1}(u^{\perp})} f(v)\,\lambda_{d-2}(dv)\,\mu(du).$$

So $R\mu \in \mathscr{M}_e$ is the measure given by

$$(R\mu)(A) = \int_{S^{d-1}} \lambda_{d-2}(A \cap u^{\perp})\,\mu(du) \qquad \text{for all Borel sets} \quad A \subset S^{d-1}.$$

Consequently, $R\mu$ is a mixture of spherical Lebesgue measures λ_{d-2} on the subspheres $S^{d-2}(u^{\perp})$.

It follows from (2.2) that $T(\mathcal{M}_e)$ is actually a subspace of $\mathcal{C}_e$. On the other hand, there are measures whose Radon transforms are not continuous functions. This difference in the behaviour of T and R seems to be at the heart of the differences that arise in the treatment of projection bodies and intersection bodies in Sections 3 and 4.

If L is a subspace of $\mathbf{R}^d$ of dimension $k \geqslant 2$ and $\mathcal{G}$ is a space of functions, measures or distributions on S^{d-1}, we denote by $\mathcal{G}(L)$ the corresponding space on $S^{k-1}(L)$. We note that if $f \in \mathcal{C}_e$, then $f|_L \in \mathcal{C}_e(L)$. On the other hand, the restriction $\mu_f|_L$, defined by restricting A in (2.1) to be a Borel subset of $S^{k-1}(L)$, only yields the trivial measure. So the restriction operator should only be applied to $\mathcal{C}_e$ and not to $\mathcal{M}_e$ or $\mathcal{D}_e$. In the other direction, though, we have the following continuous embeddings

$$\mathcal{M}_e(L) \hookrightarrow \mathcal{M}_e \qquad \text{and} \qquad \mathcal{D}_e(L) \hookrightarrow \mathcal{D}_e.$$

In the case of distributions $\rho \in \mathcal{D}_e(L)$ and functions $f \in \mathcal{C}_e^\infty$, we obtain the latter embedding from the definition $\rho(f) = \rho(f|_L)$. For measures $\mu \in \mathcal{M}_e(L)$ this gives

$$\mu(f) = \mu(f|_L) = \int_{S^{d-1}} (f|_L)(u)\, \mu(du) = \int_{S^{k-1}(L)} f(v)\, \mu(dv),$$

for $f \in \mathcal{C}_e$. It follows that the extension of $\mu \in \mathcal{M}_e(L)$ to $\mathcal{M}_e$ is, as usual, the measure defined by

$$\mu(A) = \mu(A \cap S^{k-1}(L)) \qquad \text{for all Borel sets } A \subset S^{d-1}.$$

These extensions also allow us to apply the transforms T and R to elements of $\mathcal{M}_e(L)$ and $\mathcal{D}_e(L)$. For $\rho \in \mathcal{D}_e(L)$ and $f \in \mathcal{C}_e^\infty$, we have

$$(T\rho)(f) = \rho((Tf)|_L) \qquad \text{and} \qquad (R\rho)(f) = \rho((Rf)|_L).$$

One of the main tools we will use in the following is a disintegration of the spherical Lebesgue measure which appears in [17]. We will state it here for completeness. If L is a subspace of $\mathbf{R}^d$ and $u \in S^{d-1}$, we denote by $u|L$ the orthogonal projection of u onto L. If, moreover, $\dim L = k$ and $v \in S^{k-1}(L)$, we let

$$H^{d-k}(L, v) = \{u \in S^{d-1} : u|L = \alpha v \text{ with } \alpha > 0\}$$

be the open half sphere of dimension $d-k$ generated by v and the orthogonal space $L^\perp$.

LEMMA 2.1. *Let $L \subset \mathbf{R}^d$ be a subspace of dimension k and $A \subset S^{d-1}$ a Borel set. Then we have*

$$\lambda_{d-1}(A) = \int_{S^{k-1}(L)} \int_{H^{d-k}(L,\, v) \cap A} |\langle w, v \rangle|^{k-1} \, \lambda_{d-k}(dw) \, \lambda_{k-1}(dv).$$

3. PROJECTIONS OF GENERATING DISTRIBUTIONS

Our primary objective in this section is the extension of the results of Weil [17] to arbitrary centrally symmetric bodies. This requires us to find an extension of (1.2), namely,

$$(T\rho)|_L = T_L \pi_L \rho, \tag{3.1}$$

which is valid for a class of distributions ρ large enough to include the generating distributions of all centrally symmetric convex bodies. In order for the left side of (3.1) to make sense, we should restrict ρ to be a distribution with $T\rho \in \mathscr{C}_e$. It is easy to see that any such distribution can be extended to a continuous linear functional on $\mathscr{E} = T(\mathscr{M}_e)$ when the latter space carries the final topology for the mapping T on $\mathscr{M}_e$; see [17] for a similar argument. It follows that $T^{-1}(\mathscr{C}_e) \subset \mathscr{E}'$. In fact, we will show that the reverse inclusion also holds. First, we note that the natural embedding of $\mathscr{C}_e^\infty$ in $\mathscr{E}$ is continuous, and therefore $\mathscr{E}' \subset \mathscr{D}_e$. So, if $\rho \in \mathscr{E}'$, we see that $T\rho$ is a distribution which can be extended to a continuous linear functional on $\mathscr{M}_e$ by putting $(T\rho)(\mu) = \rho(T\mu)$. But $\mathscr{C}_e$ is the dual of $\mathscr{M}_e$, therefore $T\rho \in \mathscr{C}_e$ and so we have $\mathscr{E}' = T^{-1}(\mathscr{C}_e)$. We note that if $\rho \in \mathscr{E}'$ with $T\rho = f \in \mathscr{C}_e$, we have

$$\rho(T\mu) = (T\rho)(\mu) = \int_{S^{d-1}} f(u) \, \mu(du) \qquad \text{for all} \quad \mu \in \mathscr{M}_e.$$

We will give $\mathscr{E}'$ the initial topology for the mapping $T\colon \mathscr{E}' \to \mathscr{C}_e$.

Our objective now is to show that (3.1) holds for all distributions $\rho \in \mathscr{E}'$. The spherical projection operator π_L is defined for $f \in \mathscr{C}_e$ and subspaces L of dimension k by

$$(\pi_L f)(v) = \int_{H^{d-k}(L,\, v)} |\langle w, v \rangle|^k \, f(w) \, \lambda_{d-k}(dw), \qquad \text{for} \quad v \in S^{k-1}(L), \tag{3.2}$$

see [17]. In order to extend this to $\mathscr{E}'$, it is convenient to introduce the adjoint operator π_L^* which lifts functions in $\mathscr{C}_e(L)$ to functions in $\mathscr{C}_e$. For

$f \in \mathscr{C}_e(L)$ we denote by $\hat{f}$ the positive homogeneous extension of f to the whole of L. Then π_L^* is defined by

$$(\pi_L^* f)(x) = \hat{f}(x \mid L) \qquad \text{for} \quad f \in \mathscr{C}_e(L) \quad \text{and} \quad x \in S^{d-1}. \tag{3.3}$$

We note that for $\mu \in \mathscr{M}_e(L)$, we have $T_L \mu \in \mathscr{E}(L)$ and

$$(\pi_L^* T_L \mu)(x) = \int_{S^{k-1}(L)} |\langle x \mid L, v \rangle| \, \mu(dv)$$

$$= \int_{S^{d-1}} |\langle x, u \rangle| \, \mu(du) = (T\mu)(x), \tag{3.4}$$

for all $x \in S^{d-1}$. Consequently $\pi_L^* \colon \mathscr{E}(L) \to \mathscr{E}$ and this mapping is continuous.

The following lemma shows that π_L^* is indeed an adjoint of π_L and helps us to see how the latter should be extended to $\mathscr{E}'$.

LEMMA 3.1. *If $f \in \mathscr{C}_e$ and $g \in \mathscr{C}_e(L)$, we have*

$$\int_{S^{k-1}(L)} (\pi_L f)(v) \, g(v) \, \lambda_{k-1}(dv) = \int_{S^{d-1}} f(u)(\pi_L^* g)(u) \, \lambda_{d-1}(du).$$

Proof. We see from Lemma 2.1, (3.2), and (3.3) that

$$\int_{S^{k-1}(L)} (\pi_L f)(v) \, g(v) \, \lambda_{k-1}(dv)$$

$$= \int_{S^{k-1}(L)} g(v) \int_{H^{d-k}(L,v)} |\langle w, v \rangle|^k \, f(w) \, \lambda_{d-k}(dw) \, \lambda_{k-1}(dv)$$

$$= \int_{S^{k-1}(L)} \int_{H^{d-k}(L,v)} \hat{g}(w \mid L) \, f(w) \, |\langle w, v \rangle|^{k-1} \, \lambda_{d-k}(dw) \, \lambda_{k-1}(dv)$$

$$= \int_{S^{d-1}} f(u)(\pi_L^* g)(u) \, \lambda_{d-1}(du),$$

as required.

It follows from Lemma 3.1 that, for $\rho \in \mathscr{E}'$, it is appropriate to define $\pi_L \rho$ by

$$(\pi_L \rho)(f) = \rho(\pi_L^* f) \qquad \text{for} \quad f \in \mathscr{E}(L). \tag{3.5}$$

We now formulate the main result of this section.

THEOREM 3.2. (a) *Let* $L \subset \mathbf{R}^d$ *be a subspace, then* π_L *is a continuous mapping from* $\mathscr{E}'$ *to* $\mathscr{E}'(L)$. *If* $\rho \in \mathscr{E}'$, *we have*

$$(T\rho)|_L = T_L \pi_L \rho. \tag{3.6}$$

Moreover, if $\rho = \rho_K$ *is the generating distribution of a (centrally symmetric) convex body* K, *then* $\pi_L \rho$ *is the generating distribution of* $K|L$.

(b) *A distribution* $\rho \in \mathscr{E}'$ *is the generating distribution of a (centrally symmetric) convex body, if and only if* $\pi_L \rho \geqslant 0$ *for all planes (two-dimensional subspaces)* L.

Proof. (a) First we assume that $\rho \in \mathscr{E}'$ and that $\mu \in \mathscr{M}_e(L)$. Then, from (3.4) and (3.5) we have

$$(\pi_L \rho)(T_L \mu) = \rho(\pi_L^* T_L \mu) = \rho(T\mu).$$

It immediately follows that $\pi_L \rho \in \mathscr{E}'(L)$. So we know that both sides of (3.6) are elements of $\mathscr{C}_e(L)$. If we put $T\rho = f \in \mathscr{C}_e$ and $T_L \pi_L \rho = f_L \in \mathscr{C}_e(L)$, then (3.6) is equivalent to $f|_L = f_L$. To prove the latter, we choose $g \in \mathscr{C}_e(L)$ and let μ_g be the measure whose Radon Nikodym derivative with respect to λ_{k-1} on $S^{k-1}(L)$ is g. Then we see from (3.4) and (3.5) that

$$\begin{aligned}\int_{S^{k-1}(L)} (f|_L)(v)\, g(v)\, \lambda_{k-1}(dv) &= \int_{S^{d-1}} f(u)\, \mu_g(du) = \mu_g(T\rho) = \rho(T\mu_g)\\ &= \rho(\pi_L^* T_L \mu_g) = (\pi_L \rho)(T_L \mu_g) = (T_L \pi_L \rho)(\mu_g)\\ &= \int_{S^{k-1}(L)} f_L(v)\, g(v)\, \lambda_{k-1}(dv)\end{aligned}$$

and so $f|_L = f_L$, as required. The continuity of $\pi_L : \mathscr{E}' \to \mathscr{E}'(L)$ follows immediately from (3.6). For the final statement in (a), we use the fact that the support function of $K|L$ is the restriction of the support function of K to $S^{k-1}(L)$.

(b) If $\rho \in \mathscr{E}'$ is the generating distribution of a convex body K and if L is a two-dimensional subspace, then (from (a)) $\pi_L \rho$ is the generating distribution of $K|L$. Since any two-dimensional centrally symmetric body is a zonoid, $K|L$ is a zonoid, and hence $\rho_{K|L} \geqslant 0$.

Conversely, if $\pi_L \rho \geqslant 0$ for all two-dimensional subspaces L then, for each such L, we see that $T_L \pi_L \rho$ is the support function of a zonoid in L. By (a), $T_L \pi_L \rho = (T\rho)|_L$ and so the function $T\rho$ is convex on all two-dimensional subspaces. It follows that $T\rho$ is the support function of a centrally symmetric convex body $K \subset \mathbf{R}^d$ and therefore $\rho = \rho_K$.

We note that $\mathcal{M}_e \subset \mathcal{E}'$ and that we could use (3.5) to show that if $\mu \in \mathcal{M}_e$ then $\pi_L \mu \in \mathcal{M}_e(L)$. In fact, we have a rather stronger result. If $\mu \in \mathcal{M}_e$ then the measure $\tilde{\mu} \in \mathcal{M}_e$ is defined in [17] by

$$\tilde{\mu}(A) = \int_A \|u \mid L\| \, \mu(du), \qquad \text{for all Borel sets} \quad A \subset S^{d-1}.$$

LEMMA 3.3. *Let $\mu \in \mathcal{M}_e \subset \mathcal{E}'$ and let $L \subset \mathbf{R}^d$ be a subspace, then $\pi_L \mu \in \mathcal{M}_e(L)$. In fact, $\pi_L \mu$ is the image measure of $\tilde{\mu}$ under the ($\tilde{\mu}$-almost everywhere defined) measurable mapping $x \mapsto (x \mid L)/\|x \mid L\|$, for $x \in S^{d-1}$.*

Proof. Let $\hat{\mu}$ be the image measure mentioned in the statement of the lemma and let $f \in \mathcal{C}_e^\infty(L)$. The transformation theorem for measures, together with (3.3) and (3.5), shows that $\hat{\mu}$ satisfies

$$\int_{S^{k-1}(L)} f(v) \, \hat{\mu}(dv) = \int_{S^{d-1}} f\left(\frac{u \mid L}{\|u \mid L\|}\right) \|u \mid L\| \, \mu(du) = \int_{S^{d-1}} \hat{f}(u \mid L) \, \mu(du)$$

$$= \int_{S^{d-1}} (\pi_L^* f)(u) \, \mu(du) = (\pi_L \mu)(f),$$

and so $\hat{\mu} = \pi_L \mu$, as required.

This result could be used, together with Theorem 3.2, to show that the projection of a (generalized) zonoid is again a (generalized) zonoid. Of course, the fact that zonoids are limits of finite vector sums of line segments provides a simple proof of this fact.

We conclude this section with two observations concerning the operator π_L^*. First, if $h \in \mathcal{C}_e(L)$ is the support function of $K \subset L$ and if $u \in S^{d-1}$, we have

$$\max_{x \in K} \langle x, u \rangle = \max_{x \in K} \langle x, u \mid L \rangle = \hat{h}(u \mid L) = (\pi_L^* h)(u).$$

Consequently, π_L^* lifts support functions of bodies in L to support functions of the same (lower dimensional) bodies in $\mathbf{R}^d$. Second, (3.4) shows that π_L^* can be extended to the whole of $\mathcal{D}_e(L)$ using the definition

$$\pi_L^* \rho = T T_L^{-1} \rho \qquad \text{for} \quad \rho \in \mathcal{D}_e(L).$$

In this setting, the continuity of the embedding $\mathcal{D}_e(L) \hookrightarrow \mathcal{D}_e$ gives the continuity of $\pi_L^*\colon \mathcal{D}_e(L) \to \mathcal{D}_e$.

4. THE SPHERICAL RADON TRANSFORM AND INTERSECTION BODIES

In this section we consider the class $\mathscr{S}_0$ of star bodies. These are compact star-shaped sets, centrally symmetric with respect to the origin, and such that the radial function $r(K, \cdot)$ of $K \in \mathscr{S}_0$ is continuous. In this context, the intersection bodies and generalized intersection bodies introduced by Lutwak [10] (see also [5]) play a role in $\mathscr{S}_0$ analogous to that of the zonoids and generalized zonoids in $\mathscr{K}_0$. A body $K \in \mathscr{S}_0$ is said to be a *generalized intersection body*, if $r(K, \cdot)$ is the Radon transform of an even signed measure μ, and an *intersection body*, if this measure μ is nonnegative. Since the Radon transform R is also a bicontinuous bijection on $\mathscr{D}_e$, generalized intersection bodies (respectively, intersection bodies) K are characterized by the fact that $R^{-1} r(K, \cdot)$ is an element of $\mathscr{M}_e$ (respectively, a nonnegative element).

Our next objective is to find an analogue of Theorem 3.2 for the Radon transform. To this end, we first define the spherical projection operator τ_L. For $f \in \mathscr{C}_e$ and for subspaces L with $\dim L = k$, we put

$$(\tau_L f)(v) = \int_{H^{d-k}(L,\, v)} |\langle w, v \rangle|^{k-2} f(w)\, \lambda_{d-k}(dw) \qquad \text{for } f \in \mathscr{C}_e \text{ and } v \in S^{k-1}(L). \tag{4.1}$$

We define a corresponding lift operator τ_L^* by

$$(\tau_L^* f)(x) = \|x \,|\, L\|^{-2} \hat{f}(x \,|\, L) = \|x \,|\, L\|^{-1} f\left(\frac{x \,|\, L}{\|x \,|\, L\|}\right), \tag{4.2}$$

for $f \in \mathscr{C}_e$ and almost all $x \in S^{d-1}$. Lemma 2.1 shows that

$$\int_{S^{d-1}} \|x \,|\, L\|^{-1} \lambda_{d-1}(dx) = \int_{S^{k-1}(L)} \int_{H^{d-k}(L,\, v)} |\langle w, v \rangle|^{k-2} \lambda_{d-k}(dw)\, \lambda_{k-1}(dv),$$

which is finite, for $k \geqslant 2$. It follows that τ_L^* maps $\mathscr{C}_e(L)$ into the integrable functions on S^{d-1}.

We will be particularly interested in distributions ρ for which $R\rho \in \mathscr{C}_e$. Just as in the previous section, this class is the dual of the space $\mathscr{F} = R(\mathscr{M}_e)$ endowed with the final topology for the mapping R on $\mathscr{M}_e$. In this setting, if $\rho \in \mathscr{F}'$ with $R\rho = f \in \mathscr{C}_e$, then

$$\rho(R\mu) = (R\rho)(\mu) = \int_{S^{d-1}} f(u)\, \mu(du) \qquad \text{for all } \mu \in \mathscr{M}_e.$$

We will give $\mathscr{F}' = R^{-1}(\mathscr{C}_e)$ the initial topology for the mapping $R: \mathscr{F}' \to \mathscr{C}_e$.

We now define τ_L^* on $\mathscr{F}(L)$ by

$$\tau_L^* R_L \mu = R\mu \qquad \text{for} \quad \mu \in \mathscr{M}_e(L). \tag{4.3}$$

To see that (4.3) is consistent with (4.2) we assume $R_L \mu = f \in \mathscr{C}_e(L)$ and let $g \in \mathscr{C}_e$. Then we have, from Lemma 2.1,

$$\begin{aligned}
&\int_{S^{d-1}} (\tau_L^* f)(u)\, g(u)\, \lambda_{d-1}(du) \\
&= \int_{S^{k-1}(L)} \int_{H^{d-k}(L, v)} |\langle w, v \rangle|^{k-1}\, (\tau_L^* f)(w)\, g(w)\, \lambda_{d-k}(dw)\, \lambda_{k-1}(dv) \\
&= \int_{S^{k-1}(L)} \int_{H^{d-k}(L, v)} |\langle w, v \rangle|^{k-2}\, g(w)\, f(v)\, \lambda_{d-k}(dw)\, \lambda_{k-1}(dv) \\
&= \int_{S^{k-1}(L)} \int_{H^{d-k}(L, v)} |\langle w, v \rangle|^{k-2}\, g(w)\, \lambda_{d-k}(dw)(R_L \mu)(dv) \\
&= \int_{S^{k-1}(L)} \left[R_L \int_{H^{d-k}(L, \cdot)} |\langle w, \cdot \rangle|^{k-2}\, g(w)\, \lambda_{d-k}(dw) \right] (v)\, \mu(dv) \\
&= \int_{S^{k-1}(L)} \int_{S^{k-2}(L \cap v^{\perp})} \int_{H^{d-k}(L, z)} |\langle w, z \rangle|^{k-2}\, g(w) \\
&\qquad \lambda_{d-k}(dw)\, \lambda_{k-2}(dz)\, \mu(dv) \\
&= \int_{S^{k-1}(L)} \int_{S^{d-1}(v^{\perp})} g(w)\, \rho_{v^{\perp}}(dw)\, \mu(dv) \\
&= \int_{S^{d-1}} \int_{S^{d-2}(x^{\perp})} g(w)\, \rho_{x^{\perp}}(dw)\, \mu(dx),
\end{aligned}$$

where $\rho_{x^{\perp}}$ is the measure on $S^{d-2}(x^{\perp})$ given by

$$\rho_{x^{\perp}}(A) = \int_{S^{k-1}(L \cap x^{\perp})} \int_{H^{d-k}(L, z) \cap A} |\langle w, z \rangle|^{k-2}\, \lambda_{d-k}(dw)\, \lambda_{k-2}(dz)$$

for Borel subsets A of $S^{d-2}(x^{\perp})$. If $x \in L$ and $z \in L \cap x^{\perp}$, we have

$$H^{d-k}(L, z) \subset x^{\perp},$$

and so we may apply Lemma 2.1 in $x^\perp$ to see that $\rho_{x^\perp}$ is Lebesgue measure λ_{d-2} on $S^{d-2}(x^\perp)$. Hence

$$\begin{aligned}\int_{S^{d-1}} (\tau_L^* f)(u)\, g(u)\, \lambda_{d-1}(du) &= \int_{S^{d-1}} \int_{S^{d-2}(x^\perp)} g(w)\, \lambda_{d-2}(dw)\, \mu(dx)\\ &= \int_{S^{d-1}} (Rg)(x)\, \mu(dx)\\ &= \int_{S^{d-1}} g(u)(R\mu)(du).\end{aligned}$$

It follows that (4.3) is consistent with (4.2). We also see, from (4.3) and the continuity of the embedding $\mathscr{M}_e(L) \hookrightarrow \mathscr{M}_e$, that $\tau_L^*: \mathscr{F}(L) \to \mathscr{F}$ is continuous. The next result shows that τ_L^* is the adjoint of τ_L.

LEMMA 4.1. (a) *If $f \in \mathscr{C}_e$ and $g \in \mathscr{C}_e(L)$, we have*

$$\int_{S^{k-1}(L)} (\tau_L f)(v)\, g(v)\, \lambda_{k-1}(dv) = \int_{S^{d-1}} f(u)(\tau_L^* g)(u)\, \lambda_{d-1}(du).$$

(b) *If $f \in \mathscr{C}_e$ and $\mu \in \mathscr{F}(L)$, we have*

$$\int_{S^{k-1}(L)} (\tau_L f)(v)\, \mu(dv) = \int_{S^{d-1}} f(u)(\tau_L^* \mu)(du). \tag{4.4}$$

Proof. (a) Again the main tool in the proof will be Lemma 2.1. We see that

$$\begin{aligned}&\int_{S^{k-1}(L)} (\tau_L f)(v)\, g(v)\, \lambda_{k-1}(dv)\\ &\quad= \int_{S^{k-1}(L)} \int_{H^{d-k}(L,\, v)} |\langle w, v\rangle|^{k-2} f(w)\, g(v)\, \lambda_{d-k}(dw)\, \lambda_{k-1}(dv)\\ &\quad= \int_{S^{d-1}} f(u)\, \|u \,|\, L\|^{-2}\, \hat{g}(u \,|\, L)\, \lambda_{d-1}(du)\\ &\quad= \int_{S^{d-1}} f(u)(\tau_L^* g)(u)\, \lambda_{d-1}(du),\end{aligned}$$

as required.

(b) We note that (a) and (b) are the same if the measure μ has Radon Nikodym derivative g with respect to λ_{k-1}. Since R_L maps $\mathscr{C}_e^\infty(L)$ into $\mathscr{C}_e^\infty(L)$, the denseness of $\mathscr{C}_e^\infty(L)$ in $\mathscr{M}_e(L)$ shows that $\mathscr{C}_e^\infty(L)$ is also dense in $\mathscr{F}(L)$. It follows that (b) may be proved by showing that both sides of (4.4) are continuous as functions of $\mu \in \mathscr{F}(L)$. For the left side, this is a consequence of the continuity of $R_L: \mathscr{M}_e(L) \to \mathscr{M}_e(L)$, and for the right side, we use the continuity of $R: \mathscr{M}_e \to \mathscr{M}_e$.

We use Lemma 4.1 to extend the definition of τ_L from $\mathscr{C}_e$ to $\mathscr{F}'$. For $\rho \in \mathscr{F}'$ and $\mu \in \mathscr{F}(L)$, we put

$$(\tau_L \rho)(\mu) = \rho(\tau_L^* \mu). \tag{4.5}$$

THEOREM 4.2. *Let $L \subset \mathbf{R}^d$ be a subspace, then τ_L is a continuous mapping from $\mathscr{F}'$ to $\mathscr{F}'(L)$. If $\rho \in \mathscr{F}'$, we have*

$$(R\rho)|_L = R_L \tau_L \rho. \tag{4.6}$$

Moreover, if $r(K, \cdot) = R\rho$ is the radial function of a star body $K \in \mathscr{S}_0$, then $R_L \tau_L \rho$ is the radial function of the central section $K \cap L$ of K.

Proof. The proof is completely analogous to that of Theorem 3.2(a). The final step uses the fact that the restriction of the radial function of a star body is the radial function of its corresponding section.

We could obtain an analogue of Theorem 3.2(b) as a consequence of (4.6). This would state that, for $\rho \in \mathscr{F}'$, the function $R\rho$ is positive if and only if $\tau_L \rho$ is positive for all two-dimensional subspaces L. But this result is trivial since, for $\dim L = 2$, we see that $\tau_L \rho$ and $R\rho$ are just rotations of one another. A different analogue of Theorem 3.2(b) would provide a characterization of $\rho \in \mathscr{F}'$ for which $R\rho$ is the radial function of a convex body. Here we can use (4.6) and Gardner's positive solution of the three-dimensional Busemann–Petty problem [3] to obtain a partial result. We see from (4.6) that $R\rho$ is the radial function of a convex body if and only if, for all L of a fixed dimension, $R_L \tau_L \rho$ is the radial function of a convex body. In the case $\dim L = 3$, Gardner's result shows that this implies that $\tau_L \rho$ is positive. So we conclude that if $R\rho$ is the radial function of a convex body then $\tau_L \rho$ is positive for all subspaces L of dimension three.

We will now consider the case where $\mu \in \mathscr{F}'$ is a positive measure. In particular, we want to show that, in this case, $\tau_L \mu$ is also a positive measure.

LEMMA 4.3. *If $\mu \in \mathscr{F}'$ is a positive measure, then the set function $\bar{\mu}$, defined by*

$$\bar{\mu}(A) = \int_A \|u \,|\, L\|^{-1} \mu(du), \qquad \textit{for Borel sets} \quad A \subset S^{d-1},$$

is a finite positive measure and $\tau_L\mu$ *is the image measure of* $\bar{\mu}$ *under the* ($\bar{\mu}$*-almost everywhere defined*) *measurable mapping* $x \mapsto (x|L)/\|x|L\|$, *for* $x \in S^{d-1}$.

Proof. First we show that the positive measure $\bar{\mu}$ is finite. If $R\mu = f \in \mathscr{C}_e$ and $g_0 \in \mathscr{C}_e(L)$ is identically one on $S^{k-1}(L)$, it follows from (4.2), (4.5), and (4.6) that

$$\begin{aligned}\bar{\mu}(S^{d-1}) &= \int_{S^{d-1}} \|u|L\|^{-1}\,\mu(du) = \mu(\tau_L^* g_0) = (\tau_L\mu)(g_0)\\ &= c_k(\tau_L\mu)(R_L g_0) = c_k(R_L\tau_L\mu)(g_0) = c_k(R\mu)|_L\,(g_0)\\ &= c_k\int_{S^{k-1}(L)} f(v)\,\lambda_{k-1}(dv),\end{aligned}$$

for some constant c_k, dependent only on k. So $\bar{\mu}$ is finite, as claimed. Moreover, if $g \in \mathscr{C}_e^\infty(L)$, we also have

$$(\tau_L\mu)(g) = \int_{S^{d-1}} (\tau_L^* g)(u)\,\mu(du) = \int_{S^{d-1}} \|u|L\|^{-1}\, g\left(\frac{u|L}{\|u|L\|}\right)\mu(du),$$

which completes the proof.

COROLLARY 4.4. *Let* $K \in \mathscr{S}_0$ *be an intersection body and* $L \subset \mathbf{R}^d$ *a subspace. Then* $K \cap L$ *is an intersection body in* L.

Proof. For an intersection body $K \in \mathscr{S}_0$, there is a positive measure $\mu \in \mathscr{M}_e$ such that $r(K,\cdot) = R\mu$. If r is the radial function of $K \cap L$ in the subspace L, then $r = r(K,\cdot)|_L$, and Theorem 4.2 implies that

$$r = (R\mu)|_L = R_L\tau_L\mu.$$

It follows from Lemma 4.3 that $\tau_L\mu$ is a positive measure and therefore $K \cap L$ is an intersection body.

We note that we have been unable to show that $\tau_L\colon \mathscr{M}_e \cap \mathscr{F}' \to \mathscr{M}_e(L) \cap \mathscr{F}'(L)$. For signed measures μ, it is not clear that $\bar{\mu}$ is well-defined. Consequently, we cannot prove that sections of generalized intersection bodies are necessarily generalized intersection bodies. Nor is it clear that the radial function of a generalized intersection body is the difference of radial functions of intersection bodies. The difficulty arises from the fact that if $\mu \in \mathscr{F}'$, it might not be true that the positive and negative parts of μ are in $\mathscr{F}'$.

As in the case of π_L^* we can extend τ_L^* so that it is a continuous transformation from $\mathscr{D}_e(L)$ to $\mathscr{D}_e$ by putting

$$\tau_L^* \rho = R R_L^{-1} \rho \qquad \text{for} \quad \rho \in \mathscr{D}_e(L). \tag{4.7}$$

If K is a convex body in L, we will denote its first surface area measure (calculated in L) by $S_1'(K, \cdot)$, and denote by $S_1(K, \cdot)$ the first surface area measure of the same body now considered as a subset of $\mathbf{R}^d$. We recall from Goodey and Weil [6] that, for $K \in \mathscr{K}_0$, we have $S_1(K, \cdot) = R\rho_K$. It follows from (4.7) that, for all centrally symmetric convex bodies K in L,

$$\tau_L^* S_1'(K, \cdot) = \tau_L^* R_L \rho_K = R\rho_K = S_1(K, \cdot).$$

So τ_L^* has an effect on first surface area measures similar to that of π_L^* on support functions. There is, of course, a simple connection between support functions and first surface area measures, namely, $\square h(K, \cdot) = S_1(K, \cdot)$, where $\square$ is the differential operator relating R and T in (1.4). It is reasonable to expect that many of the analogies between Sections 3 and 4 may be explained by this relationship. This expectation is reinforced by the observation that radial functions of intersection bodies are Radon Nikodym derivatives of first surface area measures of zonoids. We will investigate these relationships further in the final section.

5. PROJECTIONS AND THE LAPLACE–BELTRAMI OPERATOR

Our objective in this section is to investigate the connections between the results of Sections 3 and 4. In particular, we will show that (3.6) and (4.6) imply Eq. (1.5) for all $f \in \mathscr{C}_e^\infty$. However, we will give an independent proof that (1.5) holds for all $f \in \mathscr{C}^\infty$. This will be achieved by studying the effects of the various operators on spherical harmonics. For background information on the latter, the reader is referred to Müller [11].

We recall that, for $n = 0, 1, \ldots$, the spherical harmonics of degree n on S^{d-1} are the eigenvectors of the Laplace–Beltrami operator Δ with eigenvalue $-n(d+n-2)$. They span a finite-dimensional subspace of the Hilbert space $L^2(S^{d-1})$ and we will denote its dimension by $N(d, n)$. In fact, these spaces of spherical harmonics are the irreducible invariant (with respect to the rotation group) subspaces of $L^2(S^{d-1})$. Each of the operators T, R, and $\square = (\Delta + d - 1)/2$ acts as a multiple of the identity on the spherical harmonics of fixed degree. For the block operator $\square$ and for the spherical harmonic of degree n on S^{d-1}, we will denote this multiple by $b_{n,d}$. It follows that

$$b_{n,d} = (-n(d+n-2) + d - 1)/2 = -(n-1)(d+n-1)/2. \tag{5.1}$$

The corresponding multiples for T and R will be denoted by $t_{n,d}$ and $r_{n,d}$ respectively. It is clear that they are both zero for odd values of n. Our next objective is to give explicit formulas for them in the case of even values of n. To do this, we need only apply the appropriate operator to one spherical harmonic of degree n, and a natural choice for the latter is one which has rotational symmetry. In fact, there is precisely one (up to scalar multiples) such choice and this is obtained from the Legendre polynomials. In view of our later calculations we will describe them in terms of the Gegenbauer polynomials C_n^{ν}. These can be defined by $C_0^{\nu}(t)=1$, $C_1^{\nu}(t)=2\nu t$, and the recursion formula

$$(n+1)\, C_{n+1}^{\nu}(t)=2(n+\nu)\, tC_n^{\nu}(t)-(n+2\nu-1)\, C_{n-1}^{\nu}(t); \tag{5.2}$$

see, for example Erdély *et al.* [2]. It follows that, for even values of n,

$$C_n^{\nu}(0)=(-1)^{n/2}\,\frac{\Gamma(\nu+(n/2))}{\Gamma((n/2)+1)\,\Gamma(\nu)}. \tag{5.3}$$

The Legendre polynomial $P_n(d,\cdot)$ of degree n in dimension d can be defined by

$$P_n(d,\cdot)=\frac{\Gamma(d-2)\,\Gamma(n+1)}{\Gamma(d+n-2)}\,C_n^{(d-2)/2}. \tag{5.4}$$

If u_0 is the north pole of S^{d-1} then $P_n(d,\langle u_0,\cdot\rangle)$ is the unique spherical harmonic of degree n which has value 1 at u_0 and is invariant under rotations which leave u_0 fixed. We put

$$\omega_i=\lambda_i(S^i)=\frac{2\pi^{(i+1)/2}}{\Gamma((i+1)/2)}\qquad \text{for}\quad i=0,1,\ldots.$$

For even values of n, we use (5.3) and (5.4) to see that

$$\begin{aligned} r_{n,d} &= (RP_n(d,\langle u_0,\cdot\rangle))(u_0)=\omega_{d-2}P_n(d,0)\\ &=(-1)^{n/2}\,\frac{\Gamma(d-2)\,\Gamma(n+1)\,\Gamma((d+n-2)/2)}{\Gamma(d+n-2)\,\Gamma((n+2)/2)\,\Gamma((d-2)/2)}\,\omega_{d-2}\\ &=(-1)^{n/2}\,\frac{\pi^{(d-1)/2}\Gamma(n)}{2^{n-2}\Gamma((n/2))\,\Gamma((d+n-1)/2)}. \end{aligned} \tag{5.5}$$

The corresponding calculation for $t_{n,d}$ can be found in Schneider [13, Eq. (3.5.7)] where it is shown that, for even values of n,

$$t_{n,d}=(-1)^{(n-2)/2}\,\frac{\pi^{(d-1)/2}\Gamma(n-1)}{2^{n-2}\Gamma((n/2))\,\Gamma((d+n+1)/2)}. \tag{5.6}$$

It follows immediately from (5.1), (5.5), and (5.6) that $r_{n,d} = b_{n,d} t_{n,d}$ and so we have a proof of the (well-known) relation

$$R = \Box T = T \Box, \tag{5.7}$$

which was mentioned in the Introduction.

It follows from (5.7) that $\rho \in \mathscr{F}'$ if and only if $\Box \rho \in \mathscr{E}'$ and so (3.6) gives

$$(R\rho)|_L = (T\Box\rho)|_L = T_L \pi_L \Box \rho.$$

On the other hand, (4.6) gives

$$(R\rho)|_L = R_L \tau_L \rho = T_L \Box_L \tau_L \rho.$$

Using the injectivity of T_L, we deduce that

$$\pi_L \Box \rho = \Box_L \tau_L \rho, \tag{5.8}$$

for all $\rho \in \mathscr{F}'$.

We already showed that $\pi_L: \mathscr{E}' \to \mathscr{E}'(L)$ and $\tau_L: \mathscr{F}' \to \mathscr{F}'(L)$ are continuous. It follows immediately from (5.7) that $\Box: \mathscr{F}' \to \mathscr{E}'$ is continuous. So both sides of (5.8) are continuous as functions of $\rho \in \mathscr{F}'$. Since $\mathscr{C}_e^\infty$ is clearly dense in $\mathscr{F}'$, we see that (5.8) is equivalent to the validity of (1.5) for all $f \in \mathscr{C}_e^\infty$. We now come to the main result of this section which shows that (1.5) actually holds for all $f \in \mathscr{C}^\infty$.

Theorem 5.1. *If $L \subset \mathbf{R}^d$ is a subspace and $f \in \mathscr{C}^\infty$ we have*

$$\pi_L \Box f = \Box_L \tau_L f. \tag{5.9}$$

First we will prove a lemma which allows us to assume that $\dim L = d - 1$.

If L is a subspace of $\mathbf{R}^d$ of dimension $k \geqslant 2$, we assume that $\pi_L: \mathscr{C}^\infty \to \mathscr{C}^\infty(L)$ and $\tau_L: \mathscr{C}^\infty \to \mathscr{C}^\infty(L)$ are defined by (3.2) and (4.1). Furthermore, if L and M are subspaces with $M \subset L$, we use the following notation for the corresponding operators in L:

$$\pi_M^L: \mathscr{C}^\infty(L) \to \mathscr{C}^\infty(M) \qquad \text{and} \qquad \tau_M^L: \mathscr{C}^\infty(L) \to \mathscr{C}^\infty(M).$$

Lemma 5.2. *If L and M are subspaces of $\mathbf{R}^d$ with $M \subset L$, we have*

$$\pi_M^L \pi_L f = \pi_M f \qquad \textit{and} \qquad \tau_M^L \tau_L f = \tau_M f \qquad \textit{for} \quad f \in \mathscr{C}^\infty.$$

Proof. We just give the proof of the first statement since the second is entirely analogous. We assume that $\dim M = m$ and $\dim L = k$. For $z \in S^{m-1}(M)$ and $f \in \mathscr{C}^\infty$, we have

$$(\pi_M f)(z) = \int_{H^{d-m}(M,\, z)} |\langle v, z\rangle|^m\, f(v)\, \lambda_{d-m}(dv).$$

We note that, for $v \in H^{d-m}(M, z)$, we have

$$(v \,|\, L) \,|\, M = v \,|\, M = \alpha z$$

for some $\alpha > 0$. So if y is the unit vector in the direction of $v \,|\, L$, then $y \in H^{k-m}(M, z) \subset L$, and $v \in H^{d-k}(L, y)$ satisfies $\langle v, z\rangle = \langle v, y\rangle \langle y, z\rangle$. Lemma 2.1 therefore gives

$$\begin{aligned}(\pi_M f)(z) &= \int_{H^{k-m}(M,\, z)} |\langle y, z\rangle|^m \int_{H^{d-k}(L,\, y)} |\langle w, y\rangle|^k\, \lambda_{d-k}(dw)\, \lambda_{k-m}(dy)\\ &= (\pi_M^L \pi_L f)(z),\end{aligned}$$

as required.

Proof of Theorem 5.1. As noted above, we may assume that $\dim L = d-1$. It clearly suffices to prove (5.9) for functions f which are spherical harmonics. We will choose the coordinates so that $L = u_0^\perp$ and put $S^{d-2} = L \cap S^{d-1}$. We will use cylindrical coordinates to identify each $u \in S^{d-1}$ with the pair (t, y) where $t = \langle u, u_0\rangle$ and $y \in S^{d-2}$. The spherical harmonics of degree n on S^{d-1} are spanned by the functions of the form

$$f(u) = A_{n,\, j}(d, t)\, S_{j,\, k}(d-1, y) \qquad 0 \leqslant j \leqslant n, \quad 1 \leqslant k \leqslant N(d-1, j), \tag{5.10}$$

see [11], for example. For each j with $0 \leqslant j \leqslant n$, the functions $S_{j,\, k}(d-1, \cdot)$ with $1 \leqslant k \leqslant N(d-1, j)$ span the space of spherical harmonics of degree j on S^{d-2} and $A_{n,\, j}(d, \cdot)$ is the d-dimensional associated Legendre function of degree n of order j. The latter are defined by

$$A_{n,\, j}(d, t) = \alpha_{d,\, n,\, j} (1-t^2)^{j/2}\, P_{n-j}(d+2j, t),$$

where

$$\alpha_{d,\, n,\, j} = \sqrt{\frac{N(d+2j, n-j)\, \omega_{d+2j-2}}{\omega_{d+2j-1}}}.$$

So it will suffice to prove (5.9) for functions f of the form given in (5.10). If $p=1, 2, ...,$ we let π^p be the projection operator defined on $\mathscr{C}^\infty$ by

$$(\pi^p f)(y) = \int_{H^1(L,\, y)} |\langle v, y\rangle|^p\, f(v)\, \lambda_1(dv) \qquad \text{for} \quad y \in S^{d-2}(L).$$

We note that if f is of the form in (5.10), then

$$(\pi^p f)(y) = q_{d,\, n,\, j,\, p} S_{j,\, k}(d-1, y) \qquad \text{for} \quad y \in S^{d-2}(L),$$

where

$$q_{d,\, n,\, j,\, p} = \int_{-1}^{1} (1-t^2)^{(p-1)/2}\, A_{n,\, j}(d, t)\, dt$$

$$= \alpha_{d,\, n,\, j} \int_{-1}^{1} (1-t^2)^{(p+j-1)/2}\, P_{n-j}(d+2j, t)\, dt. \qquad (5.11)$$

It follows that (5.9) is equivalent to

$$b_{n,\, d}\, q_{d,\, n,\, j,\, d-1} = b_{j,\, d-1}\, q_{d,\, n,\, j,\, d-3} \qquad \text{for} \quad 0 \leqslant j \leqslant n$$

for all n and for all d. We may use (5.1), (5.4), and (5.11) to see that this, in turn, is equivalent to

$$(n-1)(n+d-1) \int_{-1}^{1} (1-t^2)^{(d+j-2)/2}\, C_{n-j}^{(d+2j-2)/2}(t)\, dt$$

$$= (j-1)(d+j-2) \int_{-1}^{1} (1-t^2)^{(d+j-4)/2}\, C_{n-j}^{(d+2j-2)/2}(t)\, dt, \qquad (5.12)$$

for all $0 \leqslant j \leqslant n$. Equation (5.12) is clearly true if $n-j$ is odd since, in that case, both sides of the equation are zero. So we will concentrate on the case where $n-j$ is even. It is convenient to reformulate (5.12) in terms of $k = n-j$ and to put $v = (d+2n-2k-2)/2$. Our objective then is to show that

$$(n-1)(n+d-1) \int_{-1}^{1} (1-t^2)^{(d+n-k-2)/2}\, C_k^{v}(t)\, dt$$

$$= (n-k-1)(d+n-k-2) \int_{-1}^{1} (1-t^2)^{(d+n-k-4)/2}\, C_k^{v}(t)\, dt, \qquad (5.13)$$

for all even $k \leqslant n$ and for all n. We note that this is trivially true in the case $n=1$, since then we have $k=0$. For $n \neq 1$ we put

$$I(d, n, k)=\int_{-1}^{1} (1-t^2)^{(d+n-k-2)/2}\, C_k^{\nu}(t)\, dt, \tag{5.14}$$

and prove that

$$I(d, n, k)=\left(\frac{-1}{2}\right)^{k/2} C_k^{\nu}(0)\, \frac{\Gamma(1/2)\, \Gamma((d+n-k)/2)}{(n-1)\, \Gamma((d+n+1)/2)} \times (n-1)(n-3)(n-5)\cdots(n-k-1) \tag{5.15}$$

for all even $k \leqslant n$. We note that (5.13) follows from (5.15) because

$$(n-1)(n+d-1)\, I(d, n, k)=(n-k-1)(d+n-k-2)\, I(d-4, n+2, k).$$

We will prove (5.15) by induction on k. The case $k=0$ is straightforward since

$$I(d, n, 0)=\int_{-1}^{1} (1-t^2)^{(d+n-2)/2}\, dt=\frac{\Gamma(1/2)\, \Gamma((d+n)/2)}{\Gamma((d+n+1)/2)}$$

for all d and for all $n \geqslant 0$, as required. The inductive step will make use of (5.2) and the recursion formula

$$2\nu(1-t^2)\, C_{n-1}^{\nu+1}(t)=(2\nu+n-1)\, C_{n-1}^{\nu}(t)-ntC_n^{\nu}(t) \tag{5.16}$$

which can also be found in [2]. The combination of (5.2) and (5.16) together with our choice of ν gives

$$C_{k+2}^{\nu-2}(t)=\frac{(d+2n-k-6)(d+2n-k-5)}{(k+1)(k+2)}\, C_k^{\nu-2}(t) - \frac{(d+2n-2k-6)(d+2n-4)}{(k+1)(k+2)}\, (1-t^2)\, C_k^{\nu-1}(t). \tag{5.17}$$

Furthermore, (5.2) and (5.16) show that

$$C_{k+2}^{\nu-2}(0)=-\frac{d+2n-k-6}{k+2}\, C_k^{\nu-2}(0)=-\frac{d+2n-2k-6}{k+2}\, C_k^{\nu-1}(0). \tag{5.18}$$

It now follows from (5.14), (5.17), and (5.18) together with the inductive assumption that, for $n \geqslant k+2$, we have

$$I(d, n, k+2) = \int_{-1}^{1} (1-t^2)^{(d+n-k-4)/2} C_{k+2}^{\nu-2}(t)\, dt$$

$$= \frac{(d+2n-k-6)(d+2n-k-5)}{(k+1)(k+2)}$$

$$\times \int_{-1}^{1} (1-t^2)^{(d+n-k-4)/2} C_{k}^{\nu-2}(t)\, dt$$

$$- \frac{(d+2n-2k-6)(d+2n-4)}{(k+1)(k+2)}$$

$$\times \int_{-1}^{1} (1-t^2)^{(d+n-k-2)/2} C_{k}^{\nu-1}(t)\, dt$$

$$= \frac{(d+2n-k-6)(d+2n-k-5)}{(k+1)(k+2)} I(d, n-2, k)$$

$$- \frac{(d+2n-2k-6)(d+2n-4)}{(k+1)(k+2)} I(d+2, n-2, k)$$

$$= \left(\frac{-1}{2}\right)^{(k+2)/2} C_{k+2}^{\nu-2}(0) \frac{\Gamma(1/2)\, \Gamma((d+n-k-2)/2)}{\Gamma((d+n+1)/2)}$$

$$\times (n-3)(n-5) \cdots (n-k-3),$$

as required.

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