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GENERALIZED MAYER–VIETORIS SEQUENCES IN ALGEBRAIC K-THEORY

Reinhard C. LAUBENBACHER

Department of Mathematics, New Mexico State University, Las Cruces, NM 88003, U.S.A.

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Long exact sequences of algebraic K-groups for certain kinds of multiple pullback rings are constructed, with special emphasis on Dedekind-like rings. In the case where excision holds they reduce to the usual Mayer–Vietoris sequences. These sequences are then used to obtain information about the K-groups of integral group rings of abelian groups of square-free order; about certain rings of integers in number fields; and about the coordinate ring of n lines in the plane.

Introduction

One frequently encounters the following situation when attempting computations in algebraic K-theory: R is a ring with unit, I_1 and I_2 are two-sided ideals of R such that $I_1 \cap I_2 = 0$. Denote R/I_i by R_i and $R/I_1 + I_2$ by k. Then one obtains a Cartesian square of rings

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & k \end{array} \qquad (0.1)$$

leading to a map of quasi-fibration sequences (that is, sequences which induce long exact sequences of homotopy groups)

$$\begin{array}{ccccc} X & \longrightarrow & BQ\mathcal{P}(R) & \longrightarrow & BQ\mathcal{P}(R_1) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow & & \downarrow \\ Y & \longrightarrow & BQ\mathcal{P}(R_2) & \longrightarrow & BQ\mathcal{P}(k) \end{array} \qquad (0.2)$$

where Q is Quillen's K-theory functor, $\mathcal{P}(A)$ is the exact category of finitely generated projective modules over the ring A, and X and Y are the appropriate homotopy fibres. If the map ϵ induces an isomorphism of homotopy groups, then one can construct a long exact Mayer–Vietoris sequence (MV-sequence)

$$\cdots \to K_n(R) \to K_n(R_1) \oplus K_n(R_2) \to K_n(k) \to K_{n-1}(R) \to \cdots . \quad (0.3)$$

In general, however, such sequences do not exist, a fact which is well known as the failure of excision in algebraic K-theory [21].

A natural way to study the excision problem is to introduce the *birelative* K-groups [5, 6, 8]

$$K_n(R, I_1, I_2) = \pi_{n+1}(\text{homotopy fibre } (\epsilon)), \quad n \geq 1 .$$

When they vanish, sequence (0.3) exists. In [6], $K_i(R, I_1, I_2)$ was determined for $i = 1, 2$; in particular it was shown that $K_2(R, I_1, I_2)$ is nonzero in general. (Birelative K_1 is discussed in greater generality in [5].)

This paper studies the existence of Mayer–Vietoris type sequences for Cartesian squares (0.1) which satisfy the additional hypothesis that

> R_1 and R_2 are Dedekind domains with common residue field k of characteristic $p > 0$, such that $K_n(k)$ is a torsion group for $n > 0$, which contains no p-torsion. (0.4)

One such example is the Rim square [11, p.29]. Under these assumptions we prove (Theorem 2.1) that sequence (0.3) exists, provided we replace the term $K_n(k)$ by $K_n(k) \oplus K_{n-1}(R, I_1, I_2)$. Subsequently, this sequence will be referred to as the 'generalized Mayer–Vietoris sequence' associated to (0.1).

If $\text{char}(k) = p$, let $\mathscr{H}$ be the exact category of finitely generated p-torsion R-modules of homological dimension one. Assume for simplicity that the ideals $p \cdot R_i$ do not split in R_i. Then we show (Theorem 2.4) that with certain assumptions about the rings in (0.1):

$$K_n(\mathscr{H}) \cong K_n(R, I_1, I_2) \oplus K_{n+1}(k) \oplus K_n(k) \oplus K_n(k) .$$

Assuming (0.4), the ring R is an example of a *Dedekind-like* ring. The main reference for these rings is [9]. Roughly speaking, a Dedekind-like ring is constructed as a multiple pullback ring from finitely many Dedekind domains and their residue fields. Examples include integral group rings of cyclic groups of square-free order and many rings of algebraic integers which are not integrally closed in their field of fractions.

We show (Theorem 2.2) that suitably modified generalized MV-sequences exist for all Dedekind-like rings satisfying an analogue of (0.4).

Section 1 of the paper contains generalities about Dedekind-like rings and some technical results about categories like $\mathscr{H}$, which are needed in later sections.

Section 2 contains the main results of the paper.

Section 3 is devoted to applications. We derive a comparison theorem (Theorem 3.1) for $K_*(R)$ and $G_*(R)$ of a Dedekind-like ring R, and show that $G_*(R)$ injects into the K-theory of the total ring of fractions of R, if the same is true for the Dedekind domains used to construct R (Theorem 3.2).

In [20] Stein proved a number of results concerning the surjectivity of the map $K_3(R) \to K_3(R/\mathfrak{m})$ for a commutative ring R and a maximal ideal $\mathfrak{m}$ of R, under the assumption that the map $K_3(R) \to K_3(R_{\mathfrak{m}})$ is surjective. If R is a Dedekind-like group ring, we describe all maximal ideals $\mathfrak{m}$ for which this assumption holds (Theorem 3.3).

Computations include an upper bound for $K_2(\mathbb{Z}C_{30})$, C_{30} a cyclic group of order 30 (Theorem 3.4), and a determination of $K_2(\mathbb{Z}(p))$, where

$$\mathbb{Z}(p) = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m \equiv n \pmod{p}\},\ p \text{ prime}.$$

Furthermore, we show (Theorem 3.6) that for $n > 1$ square-free, $n \equiv 1 \pmod 8$ there is an exact sequence

$$\mathbb{Z}/2 \to K_2(\mathbb{Z}[\sqrt{n}]) \to K_2(\mathbb{Z}[\tfrac{1}{2}(1+\sqrt{n})]) \to 0.$$

In Theorem 3.7 the results of Section 2 are applied to the coordinate ring of n lines in the plane, generalizing a K_2-result of Dayton and Roberts.

1. Dedekind-like rings

Definition 1.1. Let $\{R_\beta\}_\beta$ be a finite collection of Dedekind domains and $\{k_\alpha \mid \alpha = 1, \ldots, m\}$ a finite collection of fields. For each α let a pair of surjective ring homomorphisms be given, as shown below (and subsequently referred to as *generalized pullback diagrams*):

$$\begin{array}{ccc} R_{i(\alpha)} & & \\ & \overset{f_\alpha}{\searrow} & \\ & & k_\alpha \\ & \underset{g_\alpha}{\nearrow} & \\ R_{j(\alpha)} & & \end{array} \qquad (1.1)$$

where $R_{i(\alpha)}$ and $R_{j(\alpha)}$ are in $\{R_\beta\}$. The case $i(\alpha) = j(\alpha)$ is allowed. These homomorphisms are required to satisfy the following *independence condition*:

> Whenever two of the homomorphisms $f_1, g_1, f_2, \ldots$ are defined on the same ring R_β, then they have distinct kernels.

Define the *Dedekind-like* ring R determined by these data to be

$$R = \left\{ (r_1, r_2, \ldots) \in \prod_\beta R_\beta \mid f_\alpha(r_{i(\alpha)}) = g_\alpha(r_{j(\alpha)}) \text{ for all } \alpha \right\}.$$

Examples 1.2. (i) Dedekind rings ($\{k_\alpha\}_\alpha = \emptyset$).

(ii) The integral group ring $\mathbb{Z}C_n$ of a cyclic group C_n of square-free order n [10]. As it is needed later we will briefly recall the construction. Choose $\{R_\beta\} = \{\mathbb{Z}[\zeta_d] \mid d|n\}$ (including $d = 1$ and n), where ζ_d is a primitive dth root of unity. For each d and each prime p such that $dp|n$, define ring homomorphisms

$$\begin{array}{ccc} \mathbb{Z}[\zeta_d] & & \\ & \searrow & \\ & & \mathbb{Z}[\zeta_d]/(p) \\ & \nearrow & \\ \mathbb{Z}[\zeta_{dp}] & & \end{array} \qquad \begin{array}{ccc} \zeta_d & & \\ & \searrow & \\ & & \zeta_d + (p) \\ & \nearrow & \\ \zeta_{dp} & & \end{array} \tag{1.2}$$

Here (p) denotes the ideal $p \cdot \mathbb{Z}[\zeta_d]$. As p does not divide d (since n is square-free) the ring $\mathbb{Z}[\zeta_d]/(p)$ is a direct sum of (finite) fields. Then choose $\{k_\alpha\}$ to contain all these summands for all pairs (d, p) such that $dp|n$. The Dedekind-like ring determined by these data is isomorphic to $\mathbb{Z}C_n$.

(iii) Let $n > 1$ be a square-free integer such that $n \equiv 1 \pmod 8$, and let $\omega = \frac{1}{2}(1 + \sqrt{n})$. Choosing $\{R_\beta\} = \{\mathbb{Z}[\omega]\}$ and $\{k_\alpha\} = \{\mathbb{F}_2\}$ together with the generalized pullback diagram

$$\mathbb{Z}[\omega] \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{F}_2$$

where $f(a + b\omega) = \bar{a} + \bar{b}$ and $g(a + b\omega) = \bar{a}$, it is easily seen that the resulting Dedekind-like ring is isomorphic to $\mathbb{Z}[\sqrt{n}]$.

(iv) There are infinitely many Dedekind-like subrings of $\mathbb{Z} \times \cdots \times \mathbb{Z}$. One example is

$$\mathbb{Z}(p) = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n \equiv m \pmod p\}, \quad p \text{ prime}.$$

(v) The coordinate ring of n lines in the plane. This is a special case of the rings constructed in [14]. Let k be a field, and $f_1, \ldots, f_n \in k[X, Y]$ be equations of n distinct straight lines no three of which pass through one point. Let $m \geq 1$ be the number of intersection points. Then the ring $A = k[X, Y]/f_1 \cdot \ldots \cdot f_n$ can be constructed as follows: Let $R_i = k[u_i] \cong A/f_i$. Then A is obtained from $\prod_i R_i$ via the m pullback diagrams (where the maps are the obvious ones)

$$\begin{array}{ccc} k[u_i] & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ k[u_j] & & \end{array}$$

one for each intersection point.

Remarks 1.3. It is easily seen that Dedekind-like rings are noetherian [9, Lemma 1.1(i)] and have Krull dimension one. Furthermore, they are *not* regular; in fact, each of the component rings R_β has infinite homological dimension as R-module [9, Lemma 12.6]. They are, however, seminormal [22, Corollary 3.3]. From [9] it follows easily that $\operatorname{Spec}(R)$ can be obtained as the pushout of the diagram

$$\{\operatorname{Spec}(k_\alpha)\} \to \{\operatorname{Spec}(R_\beta)\}\ .$$

The maximal ideals $\mathfrak{m} = \ker(R \to k_\alpha)$ for all α play a special role throughout the theory of Dedekind-like rings. (In fact, these are the maximal ideals at which R is not integrally closed in its total ring of fractions [9, Corollary 6.5].) Of particular interest for our purposes is the behavior of R under localization and completion at these ideals.

Theorem 1.4. *Let R be a Dedekind-like ring, $\mathfrak{m}$ a maximal ideal of R.*

(i) *If $\mathfrak{m} = \ker(R \to k_\alpha)$ for some α, then there is a Cartesian square of rings*

$$\begin{array}{ccc} \hat{R}_{\mathfrak{m}} & \longrightarrow & (R_{i(\alpha)})\hat{}_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ (R_{j(\alpha)})\hat{}_{\mathfrak{m}} & \longrightarrow & k_\alpha \end{array} \tag{1.3}$$

(ii) *If $\mathfrak{m}$ is any other maximal ideal, then there is some β and a maximal ideal $\mathfrak{n}$ of R_β such that $\hat{R}_{\mathfrak{m}} \cong (R_\beta)\hat{}_{\mathfrak{n}}$.*

Proof. The assertions follow easily from results in [9]; Proposition 6.2 and Lemma 6.3 describe $R_{\mathfrak{m}}$, and Proposition 6.9 and Remarks 6.10 describe what happens under completion. □

To each Dedekind-like ring R is associated a filtration of the ring $R^0 = \prod_\beta R_\beta$ by Dedekind-like subrings ending at R as follows: Define inductively

$$R^\alpha = \{(r_1, r_2, \ldots) \in R^{\alpha-1} \mid f_\alpha(r_{i(\alpha)}) = g_\alpha(r_{j(\alpha)})\}\ .$$

It is clear that each R^α is Dedekind-like and that $R^m = R$, that is, one has a chain

$$R = R^m \subset R^{m-1} \subset \cdots \subset R^1 \subset R^0 = \prod_\beta R_\beta\ . \tag{1.4}$$

For later use (namely, to prove Theorem 2.7) we will now derive some information about certain functors induced by the above ring inclusions. As all rings in (1.4) are Dedekind-like, it is sufficient to study the last inclusion

$R \subset R^{m-1}$ which shall be denoted by $R \subset S$ for simplicity.

Lemma 1.5. *Let* $\mathfrak{m}$ *be a maximal ideal of* R. *There are ring isomorphisms*
(i) $S_{\mathfrak{m}} \cong (\prod_\beta R_\beta)_{\mathfrak{m}}$ *if* $\mathfrak{m} = \ker(R \to k_m)$,
(ii) $S_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ *if* $\mathfrak{m}$ *is any other maximal ideal.*

Proof. First observe that $R_{\mathfrak{m}} \subset S_{\mathfrak{m}} \subset (\prod_\beta R_\beta)_{\mathfrak{m}}$. Then (ii) follows easily from [9, Proposition 6.2].

(i) From the independence condition in Definition 1.1 it follows that $\mathfrak{m} \neq \ker(R \to k_\alpha)$ for all $\alpha \neq m$. Therefore, the ideal $\mathfrak{p} = \bigcap_{\alpha \neq m} \ker(R \to k_\alpha)$ is not contained in $\mathfrak{m}$. Let $r \in \mathfrak{p} \setminus \mathfrak{m}$. Then r becomes a unit in $R_{\mathfrak{m}}$, and for every $\tilde{r} \in \prod R_\beta$ one has that $r \cdot \tilde{r} \in S$ (by definition of S, since $r\tilde{r}$ maps to zero in every k_α for $\alpha \neq m$). But then $\tilde{r}/1 = \tilde{r}r/r \in S_{\mathfrak{m}}$, hence $S_{\mathfrak{m}} = (\prod R_\beta)_{\mathfrak{m}}$. This proves (i). □

Using the independence condition in Definition 1.1 and the Chinese Remainder Theorem one can find an element $t_\alpha \in \mathfrak{m}_\alpha = \ker(R \to k_\alpha)$ for each α, which has only nonzero coordinates. Since the R_β's are domains, t_α is a nonzero divisor in R and in $\prod R_\beta$. Let T denote the multiplicative subset of $A = R$, S and $\prod R_\beta$ generated by all the t_α's. Let $\mathcal{H}_T(A)$ denote the exact category of finitely generated T-torsion A-modules of homological dimension one.

Lemma 1.6. *The functor* $- \otimes_R S : \mathcal{H}_T(R) \to \mathcal{H}_T(S)$ *is well defined and exact.*

Proof. It is sufficient to show that $\mathrm{Tor}_1^R(M, S) = 0$ for all M in $\mathcal{H}_T(R)$. Let

$$0 \to P_1 \to P_0 \to M \to 0$$

be a projective R-resolution of M. One obtains an exact sequence of S-modules

$$0 \to \mathrm{Tor}_1^R(M, S) \to P_1 \otimes_R S \to P_0 \otimes_R S \to M \otimes_R S \to 0 .$$

Since M is T-torsion, so is $\mathrm{Tor}_1^R(M, S)$. As T consists of nonzero divisors, $P_1 \otimes_R S$ has no T-torsion, therefore $\mathrm{Tor}_1^R(M, S) = 0$.

Lemma 1.7. *Let* $\mathfrak{m}_\alpha = \ker(R \to k_\alpha)$. *There are category equivalences*
(i) $\mathcal{H}_T(R) \cong \coprod_{\mathfrak{p}} \mathcal{H}_T(\hat{R}_{\mathfrak{p}}) = \coprod_\alpha \mathcal{H}_T(\hat{R}_{\mathfrak{m}_\alpha}) \times \mathcal{H} = \mathcal{H}_T(\hat{R}_{\mathfrak{m}_m}) \times \mathcal{K}$
(ii) $\mathcal{H}_T(S) \cong \coprod_{\mathfrak{p}} \mathcal{H}_T(\hat{S}_{\mathfrak{p}}) \cong \mathcal{H}_T(\hat{S}_{\mathfrak{m}_m}) \times \mathcal{K}$, *where* $\mathfrak{p}$ *runs over all maximal ideals of* R, *resp.* S, $\mathcal{H} = \coprod_{\mathfrak{p} \neq \mathfrak{m}_\alpha} \mathcal{H}_T(\hat{R}_{\mathfrak{p}})$, *and* $\mathcal{K} = \coprod_{\mathfrak{p} \neq \mathfrak{m}_m} \mathcal{H}_T(\hat{R}_{\mathfrak{p}})$,
(iii) $\mathcal{H}_T(\prod_\beta R_\beta) \cong \coprod_\alpha \mathcal{H}_T((\prod_\beta R_\beta)\hat{}_{\mathfrak{m}_\alpha}) \times \mathcal{H}$.
Furthermore, the functor $- \otimes_R S$ *(resp.* $- \otimes_R (\prod R_\beta)$*) induces an equivalence on* $\mathcal{K}$ *(resp.* $\mathcal{H}$*).*

Proof. It is sufficient to show that the modules M in $\mathcal{H}_T(R)$ and $\mathcal{H}_T(S)$ have finite length, in light of the fact that then $M \cong \bigoplus_{\mathfrak{p}} \hat{M}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all maximal ideals of R, resp. S [1, Chapter 4, Section 2.5, Proposition 8]. But this follows by an easy induction argument from the fact that if A is a one-dimensional noetherian ring and $t \in A$ is a nonzero divisor, then $A/(t)$ has finite length [1, Chapter 4, Section 2.5, Proposition 9]. The decomposition of the categories and functors under consideration is then straightforward, using Theorem 1.4, Lemma 1.5 and the fact that $\hat{S}_{\mathfrak{m}_m} \cong \hat{S}_{\mathfrak{n}_i} \times \hat{S}_{\mathfrak{n}_j}$, where $\mathfrak{n}_i = \ker(S \to R_{i(m)} \xrightarrow{f_m} k)$ and $\mathfrak{n}_j = \ker(S \to R_{j(m)} \xrightarrow{g_m} k)$.

2. Generalized Mayer–Vietoris sequences

The main purpose of this section is to prove, for Dedekind-like rings, an analogue of the following:

Theorem 2.1. *Let $f : A \to B$ be a ring homomorphism, $I \subset A$ a two-sided ideal such that $f|_I$ is an isomorphism. Suppose that B/I has characteristic p and $K_n(B/I)$ is a torsion group for $n > 0$ and contains no p-torsion, for some prime p. Let $K_*(A, B, I)$ denote the birelative K-groups associated to f [5]. Then there is a long exact sequence*

$$\cdots \to K_n(A) \to K_n(B) \oplus K_n(A/I) \to K_n(B/I) \oplus K_{n-1}(A, B, I) \to \cdots .$$

Proof. Consider the commutative diagram of spaces

$$\begin{array}{ccccc}
F & \longrightarrow & U & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & BQ\mathcal{P}(A) & \longrightarrow & BQ\mathcal{P}(A/I) \\
\downarrow & & \downarrow{\scriptstyle BQ(f \times \mathrm{pr})} & & \downarrow{\scriptstyle BQ(\bar{f} \times \mathrm{id})} \\
Y & \longrightarrow & BQ\mathcal{P}(B \times A/I) & \underset{BQ(\mathrm{pr} \times \mathrm{id})}{\longrightarrow} & BQ\mathcal{P}(B/I \times A/I)
\end{array}$$

in which all rows and columns are quasi-fibration sequences. By definition, $\pi_{n+1}(F) \cong K_n(A, B, I)$, and, clearly, $\pi_{n+1}(Z) \cong K_{n+1}(B/I)$. The ideal I is a $\{p\}$-excision ideal, hence the groups $K_n(A, B, I)$ are p-primary [3]. Therefore, the long exact homotopy sequence associated to the top row of the diagram breaks up into split short exact sequences and we obtain

$$\pi_{n+1}(U) \cong K_{n+1}(B/I) \oplus K_n(A, B, I) . \qquad \square$$

Remark. If B/I is a perfect ring of positive characteristic, whose K-groups in positive dimensions are torsion, then the hypotheses of Theorem 2.1 are satisfied [7, Theorem 5.4].

For the rest of the section we will restrict our attention to Dedekind-like rings satisfying the following condition:

$$\text{All the fields } k_\alpha \text{ have positive characteristic } p_\alpha, \text{ and } K_n(k_\alpha) \text{ is a torsion group with no } p_\alpha\text{-torsion for } n>0. \tag{2.1}$$

Theorem 2.2. *Let R be a Dedekind-like ring satisfying condition* (2.1). *For every α, let $B_*^{(\alpha)}$ denote the birelative K-groups associated to square* (1.3) *for all α. Then there is a long exact sequence for $n \geq 1$,*

$$\cdots \to K_n(R) \to \bigoplus_\beta K_n(R_\beta) \to \bigoplus_\alpha (K_n(k_\alpha) \oplus B_{n-1}^{(\alpha)}) \xrightarrow{d} K_{n-1}(R) \to \cdots .$$

Proof. We shall reduce to the situation of Theorem 2.1. Without loss of generality we may assume that R is connected (that is, contains no nontrivial idempotents). Let T be the multiplicative subset of R and $\prod_\beta R_\beta$ as in Section 1. Then it is easily seen that $T^{-1}R$ and $T^{-1}(\prod_\beta R_\beta)$ are ring-isomorphic. Let $\mathscr{C}(R)$ denote the exact category of those finitely generated R-modules M of homological dimension one for which $\mathrm{Tor}_1^R(M, \prod_\beta R_\beta) = 0$. Observe that by [12, Corollary 3, p. 27], the inclusion $\mathscr{P}(R) \to \mathscr{C}(R)$ induces an isomorphism of K-groups.

Let $\mathscr{H}_T(R)$ and $\mathscr{H}_T(\prod_\beta R_\beta)$ be as in Lemma 1.7. Finally, for any ring A, let $\mathscr{P}^1(A)$ denote the category of finitely generated A-modules of homological dimension less than or equal to one.

By virtue of Lemma 1.6 there is a commutative diagram of spaces

$$\begin{array}{ccccc} BQ\mathscr{H}_T(R) & \longrightarrow & BQ\mathscr{C}(R) & \longrightarrow & BQ\mathscr{P}^1(T^{-1}R) \\ \downarrow & & \downarrow & & \downarrow \\ BQ\mathscr{H}_T(\prod R_\beta) & \longrightarrow & BQ\mathscr{P}^1(\prod R_\beta) & \longrightarrow & BQ\mathscr{P}^1(T^{-1}(\prod R_\beta)) \end{array} \tag{2.2}$$

The middle vertical map will induce the desired long exact sequence. The rows are quasi-fibration sequences by Quillen's Localization Theorems, and the right vertical map is a homeomorphism since $\mathscr{P}^1(T^{-1}R) \cong \mathscr{P}^1(T^{-1}(\prod R_\beta))$. Consequently, the homotopy fibres of the left and middle vertical maps are weakly equivalent. Lemma 1.7 then immediately implies that we need only study the homotopy fibres of the maps $BQ\mathscr{H}_T(\hat{R}_{\mathfrak{m}_\alpha}) \to BQ\mathscr{H}_T((\prod R_\beta)\hat{}_{\mathfrak{m}_\alpha})$ for the various α, to know the homotopy fibre of the left (and, therefore, the middle) map in (2.2). Hence, we may assume that R is complete local, that is, we have a Cartesian square

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & k \end{array} \tag{2.3}$$

where R_1 and R_2 are complete DVR's with common residue field k. The theorem now follows from Theorem 2.1. □

Now let R be defined by (2.3) with R_1 and R_2 not necessarily complete local such that k satisfies (2.1). We will describe the K-theory of the category $\mathcal{H}_T(R)$ of all finitely generated T-torsion R-modules of homological dimension one. Let $\{\mathfrak{p}_i\}$ be the set of all maximal ideals of R_1 and R_2 other than $\mathfrak{m}_i = \ker(R_i \to k)$ $(i = 1, 2)$, which lie over the image in R_i of the generator $t \in \mathfrak{m} = \ker(R \to k)$ of T, and let $\{F_i\}$ be the set of residue fields associated to these ideals.

Theorem 2.3. *If R_1 and R_2 are k-algebras or k is algebraic over a finite field, then there is an isomorphism for $n \geq 0$:*

$$K_n(\mathcal{H}_T(R)) \cong K_{n+1}(k) \oplus B_n \oplus K_n(k) \oplus K_n(k) \oplus \Big(\bigoplus_i K_n(F_i)\Big) .$$

Proof. By Lemma 1.7 we know that $K_n(\mathcal{H}_T(R)) \cong K_n(\mathcal{H}_T(\hat{R}_\mathfrak{m})) \oplus K_n(\mathcal{K})$. A Devissage argument combined with Theorem 1.4(ii) shows that $K_n(\mathcal{K}) \cong \bigoplus_i K_n(F_i)$.

To compute $K_n(\mathcal{H}_T(\hat{R}_\mathfrak{m}))$ assume that R is complete local with residue field k. Consider the map of localization sequences

$$\begin{array}{ccccc} \cdots \to K_{n+1}(R[T^{-1}]) & \longrightarrow & K_n(\mathcal{H}_T(R)) & \longrightarrow & K_n(R) \to \cdots \\ \downarrow \cong & & \downarrow f & & \downarrow \\ \cdots \to K_{n+1}(R_1[T_1^{-1}]) \oplus K_{n+1}(R_2[T_2^{-1}]) & \xrightarrow{\delta} & K_n(k) \oplus K_n(k) & \to & K_n(R_1) \oplus K_n(R_2) \to \cdots \end{array}$$

where T_i is the image of T in R_i. The left vertical map is an isomorphism, and the bottom sequence is the direct sum of the localization sequences associated to the ring maps $R_i \to R_i[T_i^{-1}]$. Note that $R_i[T_i^{-1}]$ is isomorphic to the quotient field of R_i. The map δ and, therefore, f is surjective [15; 17, Theorem 4.1]. As the long exact sequences associated to f and the right vertical map have the same 'third terms', we obtain short exact sequences for $n \geq 0$

$$0 \to K_{n+1}(k) \oplus B_n \to K_n(\mathcal{H}_T(R)) \to K_n(k) \oplus K_n(k) \to 0 ,$$

which are split exact. If k is algebraic over a finite field, this follows since the left- and the right-hand groups are primary for different primes, and if R_1 and R_2 are k-algebras, then δ and, therefore, f splits [16, Theorem 4.7]. □

To use Theorem 2.2 to make computations, it is necessary to have information about the image of the connecting homomorphism d. We shall use (1.4) to construct a filtration of this image and to compute the filtration quotients. Let $R^{\alpha} \to R^{\alpha-1}$ be a piece of the filtration (1.4). Let B_n denote the birelative K-groups associated (via a Cartesian square similar to (1.3)) to the generalized pullback diagram used to obtain R^{α} from $R^{\alpha-1}$.

Proposition 2.4. *There is a long exact sequence for* $n \geq 0$,

$$\cdots \to K_{n+1}(R^{\alpha}) \to K_{n+1}(R^{\alpha-1}) \to K_{n+1}(k) \oplus B_n \xrightarrow{d_\alpha} K_n(R^{\alpha}) \to \cdots .$$

Proof. Let T be the multiplicative subset of R^{α} generated by a nonzero divisor in $\ker(R^{\alpha} \to k_{\alpha})$. Then it is easily seen that $R^{\alpha}[T^{-1}] \cong R^{\alpha-1}[T^{-1}]$. Using Theorem 1.4(i) the rest of the proof is analogous to that of Theorem 2.2. □

Consider the commutative square of rings

$$\begin{array}{ccc} R^{\alpha} & \longrightarrow & R^{\alpha-1} \\ \downarrow & & \downarrow \\ S^{\alpha} & \longrightarrow & \tilde{R}^{\alpha-1} \end{array} \tag{2.4}$$

where

$$\tilde{R}^{\alpha-1} = \begin{cases} R_{i(\alpha)} \oplus R_{j(\alpha)} & \text{if } i(\alpha) \neq j(\alpha)\,, \\ R_{i(\alpha)} & \text{if } i(\alpha) = j(\alpha)\,, \end{cases}$$

and $S^{\alpha} = \{\tilde{r} \in \tilde{R}^{\alpha-1} \mid f_{i(\alpha)}(\tilde{r}) = g_{j(\alpha)}(\tilde{r})\}$. The vertical maps are the coordinate-wise projections.

Lemma 2.5. *Square* (2.4) *has the excision property.*

Proof. This follows from Theorem 2.2 and Proposition 2.4. □

We are now ready to describe the image of the map d in Theorem 2.2.

Theorem 2.6. *For* $n \geq 1$, $K_n(R)$ *is filtered by the subgroups* $F^a = \ker(K_n(R) \to K_n(R^a))$ *for* $a = 0, \ldots, m$. *Let* $F^{-1} = K_n(R)$. *Then*

$$F^{a-1}/F^a \cong \begin{cases} \operatorname{im}(K_n(R) \to K_n(\Pi R_\beta)) & \text{if } a = 0\,, \\ \operatorname{im}(d_a) & \text{if } 0 < a \leq m \end{cases}$$

(d_a *as in Proposition* 2.4).

Proof. Let $0 < a < m$, the assertion being obvious for $a = 0$, m. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & F^a & \longrightarrow & K_n(R) & \longrightarrow & K_n(R^a) \\
 & & \downarrow & & \| & & \downarrow h_a \\
0 & \longrightarrow & F^{a-1} & \longrightarrow & K_n(R) & \longrightarrow & K_n(R^{a-1})
\end{array}
$$

The rows are exact, and the left vertical map is a monomorphism. By Proposition 2.4, the alleged filtration quotient is isomorphic to $\ker h_a$. A Snake-Lemma argument will finish the proof provided that $K_n(R)$ maps onto $\ker(h_a)$.

With notation as in (1.4) the diagram

$$
K_n(R) \xrightarrow{h_m} K_n(R^{m-1}) \to \cdots \to K_n(R^{a+1}) \xrightarrow{h_{a+1}} K_n(R^a) \xrightarrow{h_a} K_n(R^{a-1})
$$

$$
g^i_m,\ \ldots,\ g^i_{a+1},\ g^i_a,\ g^i_{a-1} \colon \ \longrightarrow K_n(\tilde{R}^i)
$$

commutes for all $i \leq m-1$, where the vertical maps are induced by the coordinate-wise projections. First let $i = a$, and $x \in \ker(h_a)$. Then $g^a_a(x) = 0$, and therefore, by Lemma 2.5, $x = h_{a+1}(x')$ for some x'. If we now choose $i = a+1$, then clearly $g^{a+1}_{a+1}(x') = 0$, so x' is in the image of h_{a+2}. Induction then shows that x is in the image of $K_n(R)$, finishing the proof. □

3. Applications

For a ring A, let $\mathcal{M}od(A)$ denote the exact category of finitely generated A-modules. If A is Noetherian, then $\mathcal{M}od(A)$ is abelian. The groups $K_n(\mathcal{M}od(A))$ are commonly denoted by $G_n(A)$.

Theorem 3.1. *Let R be a Dedekind-like ring satisfying* (2.1) *and the hypotheses of Theorem* 2.3 *for each pullback diagram* (1.1). *Then there is a long exact sequence for $n \geq 1$ (notation as in Definition* 1.1*),*

$$
\cdots \to \bigoplus_{\alpha} (K_n(k_\alpha) \oplus K_{n+1}(k_\alpha) \oplus B_n^{(\alpha)}) \to K_n(R) \to G_n(R) \to \cdots .
$$

Proof. Let T, $\mathcal{H}_T(R)$, and $\mathcal{C}(R)$ be as in the beginning of the proof of Theorem 2.2. Let $\mathcal{K}$ be the category of finitely generated T-torsion R-modules. One obtains a commutative diagram of spaces

$$
\begin{array}{ccccc}
BQ\mathcal{H}_T(R) & \longrightarrow & BQ\mathcal{C}(R) & \longrightarrow & BQ\mathcal{P}^1(T^{-1}R) \\
\downarrow & & \downarrow & & \downarrow \\
BQ\mathcal{K} & \longrightarrow & BQ\mathcal{M}od(R) & \longrightarrow & BQ\mathcal{M}od(T^{-1}R)
\end{array}
\tag{3.1}
$$

Since $T^{-1}R$ is regular (as a localization of the regular ring ΠR_β), the right vertical map is a weak homotopy equivalence. The rows are quasi-fibration sequences by Quillen's Localization Theorems. The middle vertical map induces the desired long exact sequence, and we need to study only the left vertical map since the two maps have weakly equivalent homotopy fibres. An argument along the lines of the proof of Theorem 2.2 allows us to assume that R is complete local as in (2.3). Let $F_i = R_i[T^{-1}]$ be the field of fractions of R_i $(i = 1, 2)$. Diagram (3.1) then induces a map of localization sequences

$$\begin{array}{ccccccc} \cdots \longrightarrow & K_{n+1}(F_1) \oplus K_{n+1}(F_2) & \longrightarrow & K_n(\mathcal{H}_T(R)) & \longrightarrow & K_n(R) & \longrightarrow \cdots \\ & \cong \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & G_{n+1}(F_1) \oplus G_{n+1}(F_2) & \longrightarrow & K_n(\mathcal{H}) & \longrightarrow & G_n(R) & \longrightarrow \cdots \end{array}$$

By Devissage $K_n(\mathcal{H}) \cong K_n(k)$, and the left vertical map is an isomorphism since fields are regular. By Theorem 2.3,

$$K_n(\mathcal{H}_T(R)) \cong K_{n+1}(k) \oplus B_n \oplus K_n(k) \oplus K_n(k),$$

where B_n denotes the obvious birelative K-group. Since Gersten's Conjecture is true for R_1 and R_2 [15, 17], the bottom left map and, therefore, the middle vertical map is surjective. As the middle and left vertical maps in (3.1) have weakly equivalent homotopy fibres, the theorem follows after taking the direct sum over all α. □

Theorem 3.2. *Let R be a Dedekind-like ring with total ring of fractions F, such that the map $K_*(R_\beta) \to K_*(F_\beta)$ is injective for all β, where F_β is the field of fractions of R_β. Then*

(i) *The map $G_*(R) \to K_*(F)$ is injective,*

(ii) *There is a short exact sequence*

$$0 \to \bigoplus_\beta K_n(R_\beta) \to G_n(R) \to \bigoplus_\alpha K_{n-1}(k_\alpha) \to 0.$$

Proof. The total ring of quotients of R is $\Pi_\beta F_\beta$ [9, Lemma 6.4]. An argument similar to that in the proof of Theorem 2.2 allows us to reduce to the complete local case, described by diagram (2.3). Consider the commutative diagram of categories

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{g} & \mathcal{M}od(R_1 \times R_2) & \longrightarrow & \mathcal{M}od(F_1 \times F_2) \\ \downarrow f & & \downarrow & & \downarrow = \\ \mathcal{H} & \longrightarrow & \mathcal{M}od(R) & \longrightarrow & \mathcal{M}od(F_1 \times F_2) \end{array}$$

(where $\mathcal{H}$ and $\mathcal{K}$ denote the obvious categories). The rows induce quasi-fibration sequences by Quillen's Localization Theorems, and the left and middle vertical functors are forgetful functors. A Devissage argument shows that f induces a split epimorphism of K-groups, and g induces the zero map by hypothesis. The theorem now follows easily. □

Remark. In the case of only one residue field which is finite, the map in Theorem 3.2(i) is split and the sequence in (ii) is split exact. This follows from [16, Theorem 4.4] applied to the diagram in the proof of Theorem 3.2.

The next application concerns a question considered in [20]: Let R be a commutative ring and $\mathfrak{m}$ a maximal ideal; when is the map $K_3(R) \to K_3(R/\mathfrak{m})$ surjective? The results in [20] for the most part assume that the map $K_3(R) \to K_3(R_{\mathfrak{m}})$ is surjective. If R is a Dedekind-like group ring, it is possible to determine for which maximal ideals $\mathfrak{m}$ this assumption holds. The notation is as in Example 1.2(ii).

Theorem 3.3. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{Z}C_n$, where C_n is a cyclic group of square-free order n. The canonical map $\sigma: K_3(\mathbb{Z}C_n) \to K_3((\mathbb{Z}C_n)_{\mathfrak{m}})$ is surjective if and only if $(\mathbb{Z}C_n)_{\mathfrak{m}} \cong (\mathbb{Z}C_p)_{\mathfrak{m}'}$ for some maximal ideal $\mathfrak{m}'$ of $\mathbb{Z}C_p$, and $p = 1$ or p is prime, $p \mid n$.*

Proof. Consider the map of MV-sequences

$$\begin{array}{ccccc} K_3(\mathbb{Z}C_n) & \longrightarrow & K_3(\prod R_\beta) & \longrightarrow & \bigoplus_\alpha (K_3(k_\alpha) \oplus B_2^{(\alpha)}) \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ K_3(A) & \xrightarrow{g} & K_3(\mathbb{Z}/[\zeta_d]) \oplus K_3(\mathbb{Z}[\zeta_{dp}]) & \longrightarrow & K_3(k) \oplus B_2 \end{array}$$

where A denotes the obvious ring, the middle and right vertical maps are projections, and f is induced by coordinate-wise projection. If $\mathfrak{m} = \ker(\mathbb{Z}C_n \to k = k_\alpha)$ for some α, then $(\mathbb{Z}C_n)_{\mathfrak{m}} \cong A_{\mathfrak{m}'}$ where $\mathfrak{m}' = \ker(A \to k)$. Since $K_3(\mathbb{Z}[\zeta_l])$ does not change under localization [18, Theorème 3], the MV-sequences of A and $A_{\mathfrak{m}'}$ are isomorphic. Hence, in this case, σ is surjective if and only if f is. If $d = 1$, then $A = \mathbb{Z}C_p$, in which case f is surjective since $\mathbb{Z}C_p$ is a direct summand of $\mathbb{Z}C_n$ for all primes p dividing n. Now assume that $d > 1$. If f were surjective, then $K_3(\mathbb{Z}C_n) \to K_3(\mathbb{Z}[\zeta_d])$ would also be surjective, since $K_3(\mathbb{Z}[\zeta_d])$ is a direct summand of $K_3(A)$. But $K_3(\mathbb{Z}[\zeta_d])$ shows up as $K_3(\mathbb{Z}[\zeta_{d'p'}])$ for some other pair (d', p') such that $d'p' = d$. In particular, this would imply, that the map g in the above diagram is also surjective, if (d, p) is replaced by (d', p'). In the commutative diagram

$$\begin{array}{ccc} K_3(\mathbb{Z}) & \longrightarrow & K_3(\mathbb{Z}[\zeta_{dp}]) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c} \\ K_3(\mathbb{F}_p) & \xrightarrow{b} & K_3(k) \end{array}$$

the image of a is nonzero [2], and b is injective [13, Theorem 8(i)]. Therefore, c cannot be the zero map. Consequently, f is not surjective if $d > 1$.

If $\mathfrak{m}$ is any other maximal ideal, then $K_3((\mathbb{Z}C_n)_\mathfrak{m}) \cong K_3(\mathbb{Z}[\zeta_l])_\mathfrak{m}$, for some l. A similar argument as above shows that σ is surjective if and only if $l = 1$. This finishes the proof. □

Example. Let $C_3 = \langle\sigma\rangle$, $\mathfrak{m} = (2, 1 + \sigma + \sigma^2)$, then $R_\mathfrak{m} \cong \mathbb{Z}[\zeta_3]_{(2)}$. Therefore, $K_3(\mathbb{Z}C_3) \to K_3((\mathbb{Z}C_3)_\mathfrak{m})$ is not surjective, in contrast to the surjectivity of the composition $K_3(\mathbb{Z}C_3) \to K_3(\mathbb{Z}C_3/\mathfrak{m}) \cong K_3(\mathbb{F}_4)$ [20, Theorem, p. 47].

Finally we shall use Theorems 2.2 and 2.6 to make some computations for the rings in Example 1.2. First consider $\mathbb{Z}C_n$. Theorem 2.6 reduces the computation of $K_*(\mathbb{Z}C_n)$ to the study of the sequences of Proposition 2.4. Lack of information about these sequences in most cases leads to rather inconclusive results. As an illustration we determine an upper bound for $K_2(\mathbb{Z}C_{30})$. Here, $\Pi R_\beta = \Pi_{d|30}\mathbb{Z}[\zeta_d]$. By standard number theory, $\mathbb{Z}[\zeta_d]/(p) \cong \mathbb{F}_p[\zeta_d]$ for all pairs (d, p) except for $(15, 30)$. The ideal $2 \cdot \mathbb{Z}[\zeta_{15}]$ splits as the product of two maximal ideals whose associated residue fields are both isomorphic to $\mathbb{F}_{16}$. In this case one obtains two identical pullback diagrams (after projection onto each of the two copies of $\mathbb{F}_{16}$)

$$\begin{array}{ccc} \mathbb{Z}[\zeta_{15}] & \searrow & \\ & & \mathbb{F}_{16} \\ \mathbb{Z}[\zeta_{30}] & \nearrow & \end{array}$$

Theorem 3.4. $K_2(\mathbb{Z}C_{30})$ *has a filtration by subgroups* F^a, $a = 0, \ldots, 13$, *such that*

(i) $$K_2(\mathbb{Z}C_{30})/F^0 \cong \bigoplus_{d|30} K_2(\mathbb{Z}[\zeta_d]) \cong K_2(\mathbb{Z})^2 \oplus K_2(\mathbb{Z}[\zeta_3])^2 \oplus K_2(\mathbb{Z}[\zeta_5])^2 \oplus K_2(\mathbb{Z}[\zeta_{15}])^2\,,$$

(ii) *the quotients* F^{a-1}/F^a, $a > 0$, *are as listed in the last column of Table* 1.

Proof. The groups F^a are defined as in Theorem 2.6. The map $K_2(\mathbb{Z}C_{30}) \to K_2(\Pi_d\mathbb{Z}[\zeta_d])$ is surjective, since K_2 of a finite field is zero, and so are the birelative K_1-groups. Furthermore, the group $K_2(\mathbb{Z}[\zeta_3]) = 0$ [23, pp. 429 and 436]. This proves (i).

The birelative K_2-group $B_2^{(d,p)}$ associated to the pullback diagram (1.2) of the pair (d, p) is isomorphic to the underlying additive group of the residue field [6],

Table 1

(d, dp)	$\mathbb{F}_p(\zeta_d)$	$K_3(\mathbb{F}_p(\zeta_d)) \oplus B_2^{(d,p)}$	filtration quotient
$(1,2)$	$\mathbb{F}_2$	$\mathbb{Z}/3 \oplus \mathbb{Z}/2$	0
$(1,3)$	$\mathbb{F}_3$	$\mathbb{Z}/8 \oplus \mathbb{Z}/3$	0
$(1,5)$	$\mathbb{F}_5$	$\mathbb{Z}/24 \oplus \mathbb{Z}/5$	0
$(2,6)$	$\mathbb{F}_3$	$\mathbb{Z}/8 \oplus \mathbb{Z}/3$	0
$(2,10)$	$\mathbb{F}_5$	$\mathbb{Z}/24 \oplus \mathbb{Z}/5$	0
$(3,6)$	$\mathbb{F}_4$	$\mathbb{Z}/15 \oplus (\mathbb{Z}/2)^2$	0 or $\mathbb{Z}/2$
$(3,15)$	$\mathbb{F}_{25}$	$\mathbb{Z}/624 \oplus (\mathbb{Z}/5)^2$	(*)
$(5,10)$	$\mathbb{F}_{16}$	$\mathbb{Z}/255 \oplus (\mathbb{Z}/2)^4$	(*)
$(5,15)$	$\mathbb{F}_{81}$	$\mathbb{Z}/6560 \oplus (\mathbb{Z}/3)^4$	(*)
$(6,30)$	$\mathbb{F}_{25}$	$\mathbb{Z}/624 \oplus (\mathbb{Z}/5)^2$	(*)
$(10,30)$	$\mathbb{F}_{81}$	$\mathbb{Z}/6560 \oplus (\mathbb{Z}/3)^4$	(*)
$(15,30)$	$\mathbb{F}_{16}$	$\mathbb{Z}/255 \oplus (\mathbb{Z}/2)^4$	(*)
$(15,30)$	$\mathbb{F}_{16}$	$\mathbb{Z}/255 \oplus (\mathbb{Z}/2)^4$	(*)

Here, (*) in in the last column indicates an unknown quotient of the group to the left of it.

i.e. $\mathbb{F}_p[\zeta_d]^+$ in all cases except for $(15, 30)$ where it is $\mathbb{F}_{16}^+$. We need to order the pullback diagrams in some way to construct filtration (1.4). Choosing the ordering as in Table 1 makes it easy to determine the filtration quotients for the first five pairs. For the remaining ones this seems to be very difficult.

By Lemma 2.5 there is a map of long exact sequences for $a < 12$:

$$\begin{array}{ccccc} K_3(R^a) & \longrightarrow & K_3(R^{a-1}) & \xrightarrow{f} & K_3(\mathbb{F}_p(\zeta_d)) \oplus B_2^{(d,p)} \\ \downarrow & & \downarrow & & \downarrow = \\ K_3(S^a) & \longrightarrow & K_3(\mathbb{Z}[\zeta_d]) \oplus K_3(\mathbb{Z}[\zeta_{dp}]) & \xrightarrow{f'} & K_3(\mathbb{F}_p(\zeta_d)) \oplus B_2^{(d,p)} \end{array}$$

(with notation as in diagram (2.4)). For $a \leq 6$ it is easy to see that the middle vertical map maps onto the summand $K_3(\mathbb{Z}[\zeta_{dp}])$. Furthermore, the bottom left map maps onto $K_3(\mathbb{Z}[\zeta_d])$. The map f' is an epimorphism for $a \leq 5$ by [19, Corollary 1.6] and the fact that $K_3(\mathbb{Z}[\zeta_p]) \to K_3(\mathbb{F}_p)$ is surjective for $p = 2, 3, 5$, and for $a = 6$ has a cokernel of order at most 2 [19, Proposition 2.10]. Since the next map in the top sequence is the map d_a of Theorem 2.6, the first six filtration quotients are as indicated in Table 1. □

Now consider the Dedekind-like ring

$$\mathbb{Z}(p) = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n \equiv m \pmod{p}\}, \quad \text{for some prime } p \in \mathbb{Z}.$$

Theorem 3.5. (i) $\tilde{K}_2(\mathbb{Z}(2)) \cong \tilde{K}_2(\mathbb{Z}C_2) \cong K_2(\mathbb{Z})$.

(ii) *There is an exact sequence*

$$\mathbb{Z}/3 \to \tilde{K}_2(\mathbb{Z}(3)) \to K_2(\mathbb{Z}) \to 0 .$$

(iii) For $p \geq 5$, *there is a short exact sequence*

$$0 \to G_p \oplus \mathbb{Z}/p \to \tilde{K}_2(\mathbb{Z}(p)) \to K_2(\mathbb{Z}) \to 0,$$

where G_p *is a cyclic group of order* $(p^2-1)/24$.

Proof. Consider the MV-sequence

$$\cdots \to K_3(\mathbb{Z}) \to K_3(\mathbb{F}_p) \oplus B_2^{(p)} \to \tilde{K}_2(\mathbb{Z}(p)) \to K_2(\mathbb{Z}) \to 0$$

where $K_2(\mathbb{Z}(p)) \cong K_2(\mathbb{Z}) \oplus \tilde{K}_2(\mathbb{Z}(p))$. The zero on the right is $K_2(\mathbb{F}_p) \oplus B_1^{(p)}$. We know that $B_2^{(p)} \cong \mathbb{Z}/p$, $K_3(\mathbb{F}_p) \cong \mathbb{Z}/p^2-1$, and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$. The map $K_3(\mathbb{Z}) \to K_3(\mathbb{F}_p)$ is surjective for $p = 2$, 3, 5 [19], and for $p > 5$ the image has order 24 [2]. This completes the proof. □

The next application concerns the rings $\mathbb{Z}[\sqrt{m}]$, $m \equiv 1 \pmod 8$, m square-free, of Example 1.2(iii). It is well known that $\mathbb{Z}[\sqrt{m}]$ in this case is not integrally closed in $\mathbb{Q}(\sqrt{m})$ and that $\mathbb{Z}[\omega]$, $\omega = \frac{1}{2}(1+\sqrt{m})$ is its integral closure.

Theorem 3.6. *The inclusion* $\mathbb{Z}[\sqrt{m}] \to \mathbb{Z}[\omega]$ *induces an exact sequence*

$$\mathbb{Z}/2 \to K_2(\mathbb{Z}[\sqrt{m}]) \to K_2(\mathbb{Z}[\omega]) \to 0.$$

Proof. This follows immediately from the MV-sequence associated to $\mathbb{Z}[\sqrt{m}] \to \mathbb{Z}[\omega]$, using the fact that $K_3(\mathbb{Z}) \to K_3(\mathbb{F}_2)$ is surjective and $K_2(\mathbb{F}_2) \oplus B_1 = 0$. □

Finally we shall consider the rings of Example 1.2(v), thereby generalizing a result of Dayton and Roberts [4, Theorem 0] about K_2 of n lines in the plane.

Theorem 3.7. *Let* k *be a field of characteristic* $p > 0$. *Let* $f_1, \ldots, f_n \in k[X, Y]$ *be equations of* n *distinct straight lines no three of which pass through a point. Let* $m \geq 1$ *be the number of intersection points, and let* $A = k[X, Y]/f_1 \cdot \ldots \cdot f_n$. *Then for all* $i \geq 1$

$$K_i(A) \cong K_i(k) \oplus (m-n+1)K_{i+1}(k) \oplus mB_i,$$

where B_i *is the ith birelative K-group of the pullback diagram*

$$\begin{array}{ccc} k[X] & \searrow & \\ & & k \\ k[Y] & \nearrow & \end{array}$$

with the obvious maps, and, for any group G, aG *means* a*-fold direct sum.*

Proof. Since the maps in the above pullback diagram induce isomorphisms in K-theory, Theorem 2.1, and therefore Theorem 2.2, is valid without any assumptions about $K_*(k)$. We obtain the MV-sequence

$$\cdots \to \tilde{K}_i(A) \xrightarrow{f} \bigoplus_{\beta=1}^{n-1} K_i(k[u_\beta]) \to m(K_i(k) \oplus B_{i-1}) \to \cdots,$$

where $K_i(A) = K_i(k) \oplus \tilde{K}_i(A)$. It is then straightforward to see that the map f is zero. The theorem follows. □

For $i = 2$, we obtain [4, Theorem 0] in the positive characteristic case, since $B_2 \cong k^+$.

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